Journal of the Nigerian Association of Mathematical Physics
***Volume* 17 (November, 2010), pp 137 - 144**
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On The Extensive Form Of N-Person Cooperative Games

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Abstract

This paper is concerned with improving our conceptual understanding of the extensive form of N-person cooperative games. The extensive form of N-person cooperative game is such that the game is played repeatedly for very much number of times, such that in the long run, the chances of being favoured and not being favoured are equally likely. Using the same examples as used by [3], it was observed that the Nucleolus method turned out to be better with standard error of 0.1334 and 0.2887 and coefficient of variation of 7.7% and 9.38% than the Shapley value method with standard error of 0.1498 and 0.3442 and coefficient of variation of 8.65% and 11.19% respectively. We also observe that in all, the standard error and coefficients of variation using both methods are lower in the extensive form of N-person cooperative games than in the normal form of N-person cooperative games.

Keywords Extensive form game, Normal form game, characteristic function, Coalition, Imputation, Player, Payoff, Strategy and Core

General Background

1.0 Introduction:

While there is an extensive literature on the theory of infinitely repeated games, empirical evidence on how "the shadow of the future" affects behaviour is difficult to predict and inconclusive. The tension between player's incentives that encourages opportunistic behaviour and the profit that comes from cooperation is a central feature of human interaction. Games can be described formally at various levels of details. A coalition (cooperative) game is a high-level description, specifying only what payoffs each potential player or group can obtain by the cooperation of its members. Cooperative game theory investigates such coalition games with respect to the relative amounts of power held by various players, or how a successful coalition should divide its proceeds. This is most naturally applied to situations arising in political sciences international relations or in business, where the concepts of power or profits are most important

This paper focuses primarily on the extensive form of N-person cooperative game. The extensive form, also called a game tree, is a more detailed form of game. It is a complete description of how the game is played over time. This includes the order in which players take actions, the information that players have at the time they must take those actions, and the time at which any uncertainty in the situation is resolved. For more details of this, see for example; [1], [5], [9], and [11].

The philosophy of N -person cooperative game in the extensive form is based on the fact that in practice, many players can play the game at the same time and at the end of the game, players gain or suffer losses as the case may be. This type of game includes such games like gambling with a die, business firms engaged in the production of similar products (competition), and/or sharing the profit accruing to their business through cooperation- statistical joint effects. A "joint effect" is an effect that is the joint result of two or more factors here called players.

To see how this statistical joint effect can be resolved, we shall employ the Shapley value method introduced by Lloyd [10] and the Nucleolus method introduced by [8] for comparison. The basis for comparison is the fact that given the Core for N-person cooperative game in the extensive form with characteristic function, V, one point of the

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Core is more efficient for allocation (fairer allocation) than other points in the Core. However, the search for the efficient allocation based on the above two solution concepts differs. The standard error, coefficient of variation shall be used to choose the better ("best") optimal allocation. "Best" in this context, refers to the method that gives the minimum standard error and coefficient of variation.

Preliminaries

2.0 The Core: Suppose an imputation, X, is being proposed as a division of the total amount due the players denoted by V (N). If there exists a coalition, S, whose total return from X is less than what that coalition can achieve acting by itself, that is, if $\sum_{i \in s} X_i < V(S)$, then there will be a tendency for the coalition, S, to form a group and reject the proposed X because such a coalition could guarantee each of its members more than they would receive from X. Such imputation has an inherent instability. As a solution concept, the Core presents a set of imputations without distinguishing one point of the imputation as being preferable to another.

2.2 Characteristic Function: The pair, G (N,V), gives the coalitional form of an N-person game, where N is the number of players and V is a real-valued function, called the characteristic function of the game, defined on the set, S, of all coalitions (subsets of N), and satisfying

(i) $V(\phi) = 0$ and

(ii) If S and T are disjoint coalitions, $(S \cap T) = \phi$, then $V(S) + V(T) \leq V(S \cup T)$

2.3 The Shapley Value: [10] presented a solution concept to N-person cooperative games called the Shapley value. He argued that given the Core of a game, one imputation in the Core, is a more preferable imputation to other imputations in the Core. This, he achieved by computing the value of the i^{th} marginal contribution of the player into the game using the formula

$$X_i = \sum_{\substack{S \in N \\ i \in S}} P_N(S)\left[V(S \cup i) - V(S)\right] \quad (1)$$

where $P_N(S) = \frac{|S|!(N - |S| - 1)!}{N!}$ and other symbols of equation (1) have the same meanings as highlighted earlier in this work. For any characteristic function, Loiyd Shapley showed that there is a unique reward vector,

$X = (X_1, X_2, \ldots, X_N)$, satisfying the following axioms:

i) Efficiency: $\sum_{i \in N} \phi_i(V) = V(N)$;

ii) Symmetry: if i and j are such that $V(S \cup \{i\}) - V(S \cup \{j\})$ for every coalition, S, not containing i and j, then $\phi_i(V) = \phi_j(V)$;

iii) Dummy Axiom: if i is such that $V(S) = V(S \cup \{i\})$ for every coalition, S, not containing i, then $\phi_i(V) = 0$;

iv) Additivity: if U and V are characteristic functions, then $\phi_i(U + V) = \phi_i(U) + \phi_i(V)$, where $\phi_i(V)$ and $\phi_i(U)$ are functions that assign a value to each of the players in the game.

2.4 The Nucleolus: Another interesting value function for a N-person cooperative game is found in the Nucleolus, a concept introduced by Schmeidler (1969): see, for example,[7] and [2]. Instead of applying the marginal contribution of the i^{th} player due to Shapley to compute for the value of the game, we look at a given characteristic function, V, and attempt to find an imputation, $X = (X_1, X_2, X_3)$, that minimizes the worst inequity. That is, we ask each coalition how dissatisfied it is with the proposed imputation or allocation, X, and we attempt to minimize the maximum dissatisfaction: see, for example, [4] and Carter and Walker (www.http/ideas.repc.org/s/wuk/cantec.html-5k).

3.0 Methods and Applications

3.1 Some Applicable Examples

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Example 1 (Ferguson [6]): Consider the three–person game with players 1, 2 and 3, each with two pure strategies and payoff tables.
If Player 1 chooses strategy a, we have table 1

Table 1

Player 2		Player 3: a	Player 3: b
	a	$(a,a,a)=(0,3,1)$	$(a,a,b)=(2,1,1)$
	b	$(a,b,a)=(4,2,3)$	$(a,b,b)=(1,0,0)$

If Player 1 chooses strategy b, we have table 2.

Table 2

Player 2		Player 3: a	Player 3: b
	a	$(b,a,a)=(1,0,0)$	$(b,a,b)=(1,0,1)$
	b	$(a,b,a)=(0,0,1)$	$(b,b,b)=(0,1,1)$

Here, we construct the winnings of player 1 playing against players 2 and 3. This is contained in table 3.

Table 3

Player 1		Players 2 and 3: (a,a)	(a,b)	(b,a)	(b,b)
	a	0	2	4	1
	b	1	1	0	0

Putting this in the extensive form, we have the tree diagram in fig 1

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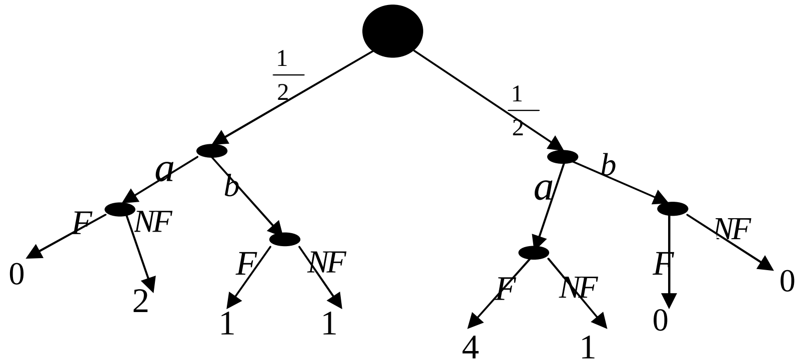
Fig 1. The Tree Diagram for player 1



Where F and NF means favoured and not favoured respectively. Using the long run expected value criteria, we have the expected payoff as follows.

$aa \;\; F : \frac{1}{2}(0)+\frac{1}{2}(4)=2$; $ab \;\; F : \frac{1}{2}(0)+\frac{1}{2}(0)=0$; $ba \;\; F : \frac{1}{2}(1)+\frac{1}{2}(4)=\frac{5}{2}$; $bb \;\; F : \frac{1}{2}(1)+\frac{1}{2}(0)=\frac{1}{2}$;

$aa \;\; NF : \frac{1}{2}(2)+\frac{1}{2}(1)=\frac{3}{2}$; $ab \;\; NF : \frac{1}{2}(2)+\frac{1}{2}(0)=1$; $ba \;\; NF : \frac{1}{2}(1)+\frac{1}{2}(1)=1$; $aa \;\; NF : \frac{1}{2}(1)+\frac{1}{2}(1)=1$

Using the long run expected payoff, we have table 4.

Table 4

	F	NF	Min
$a\ a$	2	$\frac{3}{2}$	$\frac{3}{2}$
$a\ b$	0	1	0
b a	$\frac{5}{2}$	1	1
$b\ b$	$\frac{1}{2}$	1	1
Max	$\frac{5}{2}$	$\frac{3}{2}$	

$\max\min = \frac{3}{2}$

$\min\max = \frac{3}{2}$

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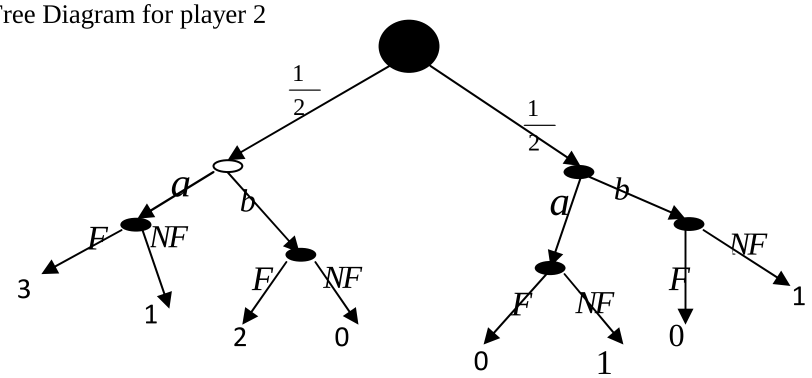
$\therefore$ min max = max min $= \frac{3}{2}$ hence, $v(1) = \frac{3}{2}$

Again, we calculate the winnings of player 2 playing against players 1 and 3

Table 5

		Players 1 and 3			
Player 2		(a,a)	(a,b)	(b,a)	(b,b)
	a	3	1	0	1
	b	2	0	0	1

Fig 2. The Tree Diagram for player 2



Similarly, F and NF means favoured and not favoured respectively. Using the tree diagram in fig 2, we calculate the long run expected payoff for player 2

$aa\ \ F: \frac{1}{2}(3)+\frac{1}{2}(0)=\frac{3}{2}$; $ab\ \ F: \frac{1}{2}(3)+\frac{1}{2}(0)=\frac{3}{2}$; $ba\ \ F: \frac{1}{2}(2)+\frac{1}{2}(0)=1$; $bb\ \ F: \frac{1}{2}(2)+\frac{1}{2}(0)=1$;

$aa\ \ NF: \frac{1}{2}(1)+\frac{1}{2}(1)=1$; $ab\ \ F: \frac{1}{2}(1)+\frac{1}{2}(1)=1$; $ba\ \ F: \frac{1}{2}(0)+\frac{1}{2}(1)=\frac{1}{2}$; $bb\ \ F: \frac{1}{2}(0)+\frac{1}{2}(1)=\frac{1}{2}$

Using the long run expected payoff, we have table 6.

Table 6

	F	NF	Min
$a\ a$	3/2	1	1
$a\ b$	3/2	1	1
$b\ a$	1	1/2	½
$b\ b$	1	1/2 min max $=1$	½
Max	5/2		

max min $=1$

hence, $v(2)=1$

Using the same procedure, we have that

$v(3)=1,\ v(1,2)=\frac{5}{2},\ v(1,3)=\frac{7}{4},\ v(2,3)=2,\ and\ v(1,2,3)=9$ is the value of the game and is found as the largest value in the payoff table

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Hence the characteristics function is given by

$$V(\phi)=0,\quad V(1)=3/2,\quad V(2)=1,\quad V(3)=1,\quad V(1,2)=5/2,\quad V(1,3)=7/2,\quad V(2,3)=2\quad \text{and } V(1,2,3)=9.$$

3.2 Computation Using the Shapley Value Method

Using the Shapley method we have that

$$X_i=\sum_{\substack{S\in N\\ i\in S}} P_N(S)\left[V(S\cup i)-V(S)\right]$$

where $P_N(S)=\dfrac{|S|!(N-|S|-1)!}{N!}$

The order of entry of the players and their individual marginal contribution is as contained in table 7

Table 7

Order of Entry	Players 1	2	3	Total
1, 2, 3	$\frac{3}{2}$	1	$\frac{13}{2}$	9
1, 3, 2	$\frac{3}{2}$	$\frac{29}{4}$	$\frac{1}{4}$	9
2, 1, 3	$\frac{3}{2}$	1	$\frac{13}{2}$	9
2, 3, 1	7	1	1	9
3, 1, 2	$\frac{3}{4}$	$\frac{29}{4}$	1	9
3, 2, 1	7	1	1	9
Average	$\frac{77}{4}$	$\frac{37}{2}$	$\frac{65}{4}$	9

Since each has a probability of 1/6 ie the possible order of being in the game, we have that the Shapley value of the game for the players is given by

$$x_i=(x_1,x_2,x_3)=\left(\frac{1}{6}\left(\frac{77}{4},\frac{37}{2},\frac{65}{4}\right)\right)=(3.21,3.08,2.71)\cdot$$

The value of the game using Nucleolus method is given by

$$x_i=(x_1,x_2,x_3)=\left(\frac{10}{3},\frac{17}{6},\frac{17}{6}\right)=(3.33,2.83,2.83)\cdot$$

Having got the value of the game using the two methods, we compute their standard error and the coefficient of variation. The standard error (S.E) is given by $\dfrac{s}{\sqrt{n}}$.

The variance using the Shapley method is $S^2=\dfrac{3.21^2+3.08^2+2.71^2-3\times 3^2}{3-1}=.0673$

$\therefore$ Standard error $=\sqrt{\dfrac{s^2}{3}}=\sqrt{\dfrac{.0673}{3}}=0.1498$

Coefficient of variation using the shapely method is $\dfrac{s}{\bar{x}}\times 100=\dfrac{.2594}{3}\times 100=8.65\%$

Similarly, the standard error and coefficient of variation using the Nucleolus method are 0.1334 and 7.7%

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Example 2 (Ferguson [6]): Consider the three–person game with players 1, 2 and 3, each with two pure strategies and with payoff matrices.
Using similar construction as we did in Example 1, we obtain the characteristic function as follows

$$\begin{array}{lll} & V(1)=3, & V(1,2)=2, \\ V(\phi)=0, & V(2)=2, & V(1,3)=4, \text{ and } V(1,2,3)=16. \\ & V(3)=5/2, & V(2,3)=3, \end{array}$$

The Shapley value of the game is given by (65/12, 57/12, 35/6) = (5.42, 4.75.53, 5.83) while the Nucleolus of the game is given by (35/6, 29/6, 16/3)= 5.83, 4.83, 5.33).
The Standard error and coefficient of variation using the shapely med and the Nucleolus are 0.3442, 11.19% and 0.2887, 9.38% respectively

If Player 1 chooses strategy a, we have table 9.

3.3 Computation Using the Nucleolus

Similarly, we compute the value of the game using the Nucleolus method as contained in table 8.

Table 8

The Coalitions (s)	The Value of the Coalitions $v(s)$	The excess of the Coalitions $e=v(s)-\sum_i x_i$	The vector of excesses with the first Imputation 3, 3, 3	The Nucleolus $\frac{10}{3},\frac{17}{6},\frac{17}{6}$
x_1	$\frac{3}{2}$	$\frac{3}{2}-x_1$	$\frac{3}{2}$	$\frac{-11}{6}$
x_2	1	$1-x_2$	–2	$\frac{-11}{6}$
x_3	1	$1-x_3$	–2	$\frac{-11}{6}$
x_1x_2	$\frac{5}{2}$	$\frac{5}{2}-x_1-x_2$	$\frac{-7}{2}$	$\frac{-11}{3}$
x_1x_3	$\frac{7}{4}$	$\frac{7}{4}-x_1-x_3$	$\frac{-17}{4}$	$\frac{-53}{12}$
x_2x_3	2	$2-x_2-x_3$	–4	$\frac{-11}{6}$

Table 9

		Player 3	
		a	b
Player 2	a	(a, a, a) = (1, 2, 1)	(a, a, b) = (3, 0, 1)
	b	(a, b, a) = (-1, 6, -3)	(a, b, b) = (3, 2, 1)

If Player 1 chooses strategy b, we have table 10.

Table 10

		Player 3	
		a	b

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Player 2	a b	(b, a, a) = (-1, 2, 4)	(b, a, b) = (1, 0, 3)
		(b, b, a) = (7, 5, 4)	(b, b, b) = (3, 2, 1)

3.4 Comparison of the Shapley value and the Nucleolus in the normal form

Table 11

Example 1			Example 2	
No of players	The Shapley Value	The Nucleolus	The Shapley Value	The Nucleolus
x_1	$\frac{79}{24} = 3.29$	$\frac{37}{12} = 3.08$	$\frac{107}{18} =5.94$	$\frac{41}{9} = 4.56$
x_2	$\frac{67}{24} = 2.79$	$\frac{31}{12} = 2.58$	$\frac{199}{36} = 5.53$	$\frac{56}{9} = 6.22$
x_3	$\frac{35}{12} = 2.92$	$\frac{10}{3} = 3.33$	$\frac{163}{36} = 4.53$	$\frac{47}{9} = 5.22$
Total	9	9	16	16
Mean	3	3	5.33	5.33
Standard Error	0.1497	0.1963	0.76	0.87
Coefficient of Variation	8.6%	11.3%	14.37	16.32%

3.5 Comparison of the Shapley value and the Nucleolus in the Extensive Form

Table 12

Example 1			Example 2	
No of players	The Shapley Value	The Nucleolus	The Shapley Value	The Nucleolus
x_1	$\frac{77}{24}$ =3.21	$\frac{10}{3} = 3.33$	$\frac{65}{12}$ =5.42	$\frac{35}{6} = 5.83$
x_2	$\frac{37}{12}$ =3.08	$\frac{17}{6} = 2.83$	$\frac{57}{12} = 4.75$	$\frac{29}{6} = 4.83$
x_3	$\frac{65}{24}$ =2.71	$\frac{17}{6} = 2.83$	$\frac{35}{6} = 5.83$	$\frac{16}{3} = 5.33$
Total	9	9	16	16
Mean	3	3	5.33	5.33
Standard Error	0.1498	0.1334	0..3442	0.2887
Coefficient of Variation	8.65%	7.7%	11.19	9.38%

4.0 Conclusion

This paper is concerned with improving our conceptual understanding of the <u>**extensive**</u> form of N-person cooperative games. The extensive form of N-person cooperative game is such that the game is played over and over again such that in the long run, the chances of being favoured and not being favoured are equally likely. Using the examples as used by [3], it was observed that the Nucleolus method turned to be better (Best) with standard error of 0.1334 and 0.2887 and coefficient of variation of 7.7% and 9.38% than the Shapley value method with standard error of 0.1498 and 0.3442 and coefficient of variation of 8.65% and 11.19% respectively. We also observe that in all, the standard error and coefficients of variation using both methods were lower in the **extensive** form of N-person cooperative games than in the **normal** form of N-person cooperative games.

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