Journal of the Nigerian Association of Mathematical Physics
***Volume* 17 (November, 2010), pp 123 - 128**

Polynomial approximation approach to transient heat conduction with internal heat generation

[a]P. O. Olanrewaju, [b]O.T. Arulogun and [c]M.E. Ojewumi
[a]Department of Pure and Applied Mathematics
[b]Department of Computer Science and Engineering
Ladoke Akintola University of Technology, Ogbomoso, Nigeria
[c] Department of Chemical Engineering
Covenant University, Ota. Ogun State

Abstract

This work reports polynomial approximation approach to transient heat conduction in a long slab, long cylinder and sphere with linear internal heat generation. It has been shown that the polynomial approximation method is able to calculate average temperature as a function of time for higher value of Biot numbers. This agrees with Keshavarz and Taheri [19] and also shows that their work becomes a special case of ours. The simplified relations obtained in this study can be used for engineering calculations in many conditions.

List of symbols

A	Surface area	B	Dimensionless number in equation (7)
Bi	Biot Number	F_0	Fourier Number
k	Thermal conductivity	h	Heat transfer coefficient
m	Order of the geometry	r	Coordinate
R	Maximum coordinate	S	Shape factor
t	Time	T	Temperature
V	Volume	x	Dimensionless coordinate
Q	Heat of reaction		

Greek symbols

α	Thermal diffusivity	β	Temperature coefficient parameter
γ	Modified temperature coefficient parameter	λ	Radiation heat parameter

1.0 Introduction

Accurate and comprehensive computational techniques such as finite difference, finite volume, and finite element methods can be applied to solve partial differential equations that model transport phenomena by distributed parameter formulations. However, in the analysis of complex industrial processes or systems, engineers sometimes rather prefer to predict and control the system behaviour using a simpler or simplified model that approximates accurately the original distributed parameter formulation but involves fewer state variables and consequently less equations to be solved and also in the thermohydralic analysis of nuclear reactors, this classical approach is extremely useful and sometimes mandatory when a simplified formulation of transient heat conduction is needed. Model reduction techniques have received increasing attention in recent years, both in the applied mathematics community and in various application areas such as thermal systems, chemical engineering, electronic systems, and building simulation [16,24,20,15,21,33,17,4,5,]. In the analysis of complex thermal systems, the lumped parameter formulation is a powerful engineering tool when a simplified model of the transient heat conduction is sought. As a rule of thumb, the classical lumped parameter approach, where uniform temperature is assumed within the region, is in general restricted to problems with Biot number less than 0.1. In most engineering applications, however, the Biot

[a]Corresponding author: E-mail: oladapo_anu@yahoo.ie, el. +2348138485550

number is much higher [14]. In other words, the moderate to low temperature gradient assumption is not reasonable in such applications, thus more accurate approach should be adopted. Improved lumped models have been developed by different approaches [12,32,11,13,34,36,6,30,31]. Nonlinear boundary conditions have been also investigated [7,1,35]. Cotta and Mikhailov [12] presented a systematic formalism to provide improved lumped parameter formulations for steady and transient heat conduction problems based on Hermite approximations for integrals that define averaged temperatures and heat fluxes. This approach has been shown to be efficient in a variety of practical applications [12, 32, 13, 26, 36, 28]. Recently, Alhama and Zueco [3] applied for the first time a lumped model to nonlinear heat conduction problem of a slab with linearly temperature dependent thermal conductivity.They studied both the cooling and heating processes and presented solutions as a function of the Biot number and a dimensionless parameter representing the heating or cooling process.

Recently, the dynamics of chaotic instabilities in boiling water nuclear reactors have aroused interest. In such studies, the lumped parameter approach has been the only available option in the fuel dynamics model [39] . However, a simple lumped model is only valid for very low Biot numbers. In this preliminary models, solid resistance can be ignored in comparison with fluid resistance, and so the solid has a uniform temperature that is simply a function of time. The criterion for the Biot number is about 0.1, which is applicable just for either small solids or for solids with high thermal conductivity. In other words, the simple lumped model is valid for moderate to low temperature gradients [9,18,23].

Lots of investigations have been done to use or modify the lumped model. The purpose of modified lumped parameter models is to establish simple and more precise relations for higher values of Biot numbers and large temperature gradients. For example, if a model is able to predict average temperature for Biot numbers up to 10, such a model can be used for a much wider range of materials with lower thermal conductivity. Chang and Lahey [8] used one dimensional homogeneous assumption for adiabatic two phase flow, a one node lumped parameter approach for heated wall dynamics, and neutron point kinetics for the consideration of nuclear feedback in a boiling water reactor loop. They found that a boiling channel coupled with a riser could experience chaotic oscillations. Alhama and Campo [2] have studied the application of a simple lumped model for the cooling of a long slab with different Biot numbers in each side. The question this raised was under what circumstances can the transient cooling of a long slab by asymmetric heat conduction be treated with a lumped model? Their results show that a simple lump model is qualified to handle the general distribution model without incurring temperature errors that exceed 5% as long as the combination of the Biot numbers are confined to the area circumscribed by a curvilinear rectangle that has its upper right vertex at 0.1075 for both sides. Cotta and Mikhailov [4] proposed an improved lumped parameter model based on Hermit approximation for integral that defined average temperature and heat flux. Regis et al. [27] have shown that the Cotta model can be used to predict average temperature in a nuclear fuel rod for Biot numbers up to 20. Su and Cotta [37] have presented a higher order lumped parameter formulation for simplified light water reactor thermohydraulic analysis. The Hermit approximation method was also used by Su [38] for transient heat conduction in long slabs with different Biot numbers in each side. His results compared to finite difference solution yield significant improvement of average temperature prediction over the classical lumped model. Also, Sadat proposed another modified lumped model for one dimensional transient heat conduction in a long slab, long cylinder and sphere by using a Singular Perturbation Method [29]. His result is the same as [38] in case of the long slab. Finally, Keshavarz and Taheri [19] examined an improved lumped analysis for transient heat conduction by using the polynomial approximation method. They shown that in comparison to finite difference solution, the improved model is able to calculate average temperature as a function of time for higher value of Biot numbers. The comparison also shows that the presented model has better accuracy when compared with others recently developed models.

In this work, we present polynomial approximation approach to transient heat conduction in the presence of heat generation which is an extension of the recent work of Keshavarz [19] therefore Keshavarz work becomes a special case of ours.

2.0 Mathematical formulations

Unsteady state one dimensional temperature distribution of a long slab, long cylinder or sphere with internal heat generation can be expressed by the following partial differential equation. Heat transfer coefficient is assumed to be constant for all geometries, as illustrated in Fig. 1

$$\frac{\partial T}{\partial t} = \alpha \frac{1}{r^m}\frac{\partial}{\partial r}\left(r^m \frac{\partial T}{\partial r}\right) + Q\left((T - T_\infty)\right) \qquad (2.1)$$

where m = 0 for slab, 1 for cylinder and 2 for sphere.
Boundary conditions are :

[a]Corresponding author: E-mail: oladapo_anu@yahoo.ie, el. +2348138485550

$$\frac{\partial T}{\partial r}=0 \quad at \quad r=0 \tag{2.2}$$

$$-k\frac{\partial T}{\partial r}=h\left(T-T_{\infty}\right) at\, r=R \tag{2.3}$$

And initial condition:

$$T=T_0 \quad at \quad t=0 \tag{2.4}$$

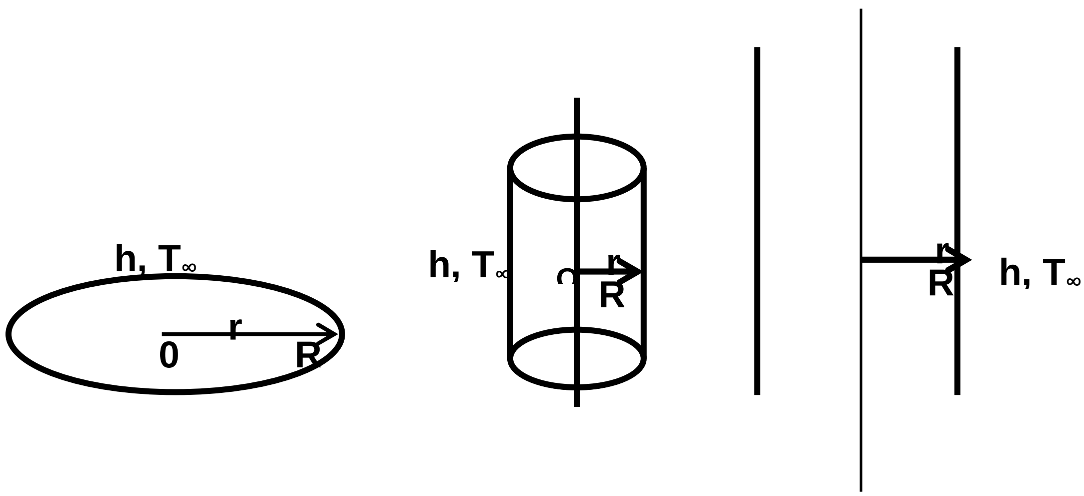

Fig. 1: Geometry of the problem

In the derivation of Eq. (2.1), it is assumed that thermal conductivity is independent of temperature. If not, temperature dependence must be applied, but the same procedure can be followed. For simplicity, Eq. (2.1) and boundary conditions can be rewritten in dimensionless form:

$$\frac{\partial \theta}{\partial \tau}=\frac{1}{x^m}\frac{\partial}{\partial x}\left(x^m\frac{\partial \theta}{\partial x}\right)+\lambda\theta \tag{2.5}$$

$$\frac{\partial \theta}{\partial x}=0\, at\, x=0 \tag{2.6}$$

$$\frac{\partial \theta}{\partial x}=-B_i\theta \quad at \quad x=1 \tag{2.7}$$

$$\theta=1 \; at \; \tau=0 \tag{2.8}$$

It should be noted that for a long slab with the same Biot number in both sides, temperature distribution is the same for each half, and so just one half can be considered. Dimensionless parameters are defined as follows:

$$\theta=\frac{T-T_{\infty}}{T_0-T_{\infty}}, \quad B_i=\frac{hR}{k}, \quad \tau=\frac{\alpha t}{R^2}, \quad x=\frac{r}{R}, \quad \lambda=\frac{QR^2(T_0-T_{\infty})}{\gamma\alpha} \tag{2.9}$$

3. Polynomial approximation method

Polynomial approximation method is one of the simplest, and in some case, one of the most accurate methods to estimate partial differential equation [25]. It involves the selection of the proper polynomial with time dependent-coefficients and to convert a partial differential equation into an integral equation. This integral can be then converted into an ordinary differential equation, where the dependent variable is average temperature and independent variable is time. Following [19], we choose the following polynomial

$$\theta_p=a_0(\tau)+a_1(\tau)x+a_2(\tau)x^2 \tag{3.1}$$

[a]Corresponding author: E-mail: oladapo_anu@yahoo.ie, el. +2348138485550

Note that the polynomial must satisfy all boundary and initial conditions.
Using the first boundary condition, we obtain

$$\theta_p = a_0(\tau) + a_2(\tau)x^2 \quad (3.2)$$

The average temperature for long slab, long cylinder and sphere can be written as

$$\bar{\theta} = \frac{\int_v \theta \, dv}{\int dv} = \frac{\int_0^1 \theta x^m dx}{\int_0^1 x^m dx} = (m+1)\int_0^1 x^m \theta \, dx \quad (3.3)$$

From (3.3), m=0 for slab, 1 and 2 for cylinder and sphere, respectively.
Eq. (3.3) reduces to

$$\bar{\theta} = a_0 + \frac{m+1}{m+3} a_2 \quad (3.4)$$

Integration of both sides of eq. (3.3) yields

$$\int_0^1 x^m \frac{\partial \theta}{\partial \tau} dx = \int_0^1 \frac{\partial}{\partial x}\left(x^m \frac{\partial \theta}{\partial x} \right) dx + \int_0^1 x^m \lambda(\theta) dx \quad (3.5)$$

From eq.(3.3) , we have

$$\frac{\partial \bar{\theta}}{\partial \tau} = (m+1)\frac{\partial \theta}{\partial x}\Big|_{x=1} + \lambda(\bar{\theta}) \quad (3.6)$$

On the other hand, the derivation of eq.(3.1) and applying the second boundary condition leads to

$$\frac{\partial \theta_p}{\partial x}\Big|_{x=1} = 2a_2 \quad (3.7)$$

Which implies

$$2a_2 = -B_i(a_0 + a_2) \quad (3.8)$$

Substituting a_0 from eq.(3.8) into eq.(3.4) yields

$$\frac{d\bar{\theta}_p}{d\tau} - \left(\frac{(m+B_i+3) - B_i(m+3)(m+1)}{m+B_i+3} \right)\bar{\theta} = \lambda \quad (3.9)$$

Let

$$A = \left(-\frac{(m+B_i+3) + B_i(m+3)(m+1)}{m+B_i+3} \right)$$

Then,

$$\bar{\theta}_p = \frac{\lambda}{A}\left(1 - e^{-A\tau}\right) + e^{-A\tau}$$

For $A \succ 0$, $\bar{\theta}_p \to \frac{\lambda}{A}$ as $\tau \to \infty$

i.e $\bar{\theta}_p \to \frac{\lambda(m+B_i+3)}{B_i(m+1)(m+3) - (m+B_i+3)}$ as $\tau \to \infty$

Conclusions

It is clearly seen that when $\lambda = 0$ our result is the same with [19] but we added heat generation term to our model which makes it new compared to [19]. Hence, we can use polynomial approximation approach to solve heat equation since it helps us to generate average temperature for the system.

[a]Corresponding author: E-mail: oladapo_anu@yahoo.ie, el. +2348138485550

Acknowledgments

POO want to thank Professor Makinde for his kind gesture during my visit at Cape Peninsula University of Technology, Cape Town, South Africa.

References

[1] F. Alhama, A. Campo, Electric network representation of the unsteady cooling of a lumped body by nonlinear heat transfer modes, J. Heat Transf. 124 (2002) 988–992.

[2] Alhama F, Campo A (2001) The connection between the distributed and lumped models for asymmetric cooling of long slabs by heat convection.Int Comm Heat Mass Transf 28:127-137

[3] F. Alhama, J. Zueco, Application of a lumped model to solids with linearly temperature-dependent thermal conductivity, Appl. Math. Model. 31 (2007) 302–310.

[4] O. Balima, Y. Favennec, M. Girault, D. Petit, Comparison between the modal identification method and the POD- Galerkin method for model reduction in nonlinear diffusive systems, Int. J. Numer. Methods Eng. 67 (7) (2006) 895–915.

[5] O. Balima, Y. Favennec, D. Petit, Model reduction for heat conduction with radiative boundary conditions using the modal identification method, Numer. Heat Transf. Part B Fund. 52 (2) (2007) 107–130.

[6] O. Bautista, F. Mendez, I. Campos, Transient heat conduction in a solid slab using multiple-scale analysis, Heat Mass Transf. 42 (2005) 150–157.

[7] A. Campo, R. Villasenor, Subregion of validity of the lumped-based model for transient, radiative cooling of spherical bodies to a zero temperature sink, Int. Commun. Heat Mass Transf. 23 (1996) 855–864.

[8] Chang CJ, Lahey RT (1997) Analysis of chaotic instabilities in boiling systems Nucl Eng Des 167:307-334

[9] Corra EJ, Cotta RM (1998) Enhanced lumped-differential formulations of diffusion Problems. Appl Math Modeling 22:137-152

[10] Cotta RM, Mikhailov MD (1997) Heat conduction – lumped analysis, integral transform, symbolic computation. Wiley, Chichester

[11] R.M. Cotta, M.D. Mikhailov, Heat Conduction – Lumped Analysis, Integral Transforms, Symbolic Computation, Ed. John Wiley and Sons, Chichester, England, 1997.

[12] R.M. Cotta, M.N. Ozisik, J. Mennig, Coupled integral-equation approach for solving phase-change problems in a finite slab, J. Franklin Inst. Eng. Appl. Math. 327 (2) (1990) 225–234.

[13] E.J. Correˆa, R.M. Cotteˆa, Enhanced lumped-differential formulations of diffusion problems, Appl. Math. Model. 22 (1998) 137–152.

[14] P.E. Ergatis, P.G. Massouros, G.C. Athanasouli, G.P. Massouros, Time-dependent heat transfer coefficient of a wall, Int. J. Energy Res. 27 (2003) 795–811.

[15] S. Estrada-Flores, A.C. Cleland, D. Cleland, Prediction of the dynamic thermal behaviour of walls for refrigerated rooms using lumped and distributed parameter models, Int. J. Refrig. 24 (2001) 272–284.

[a]Corresponding author: E-mail: oladapo_anu@yahoo.ie, el. +2348138485550

[16] M. Gutierrez, R.C. Degeneff, P.J. McKenny, J.M. Schneider, Linear, lumped, parameter transformer model-reduction technique, IEEE Trans. Power Deliv. 10 (2) (1995) 853–861.

[17] M. Hernandez-Mora, J.E. Gonzalez, M. Velez-Reyes, J.M. Ortiz-Rodriguez, Y. Pang, E. Scott, Dynamic reduced electrothermal model for integrated power electronics modules (IPEM), J. Electron. Pack. 126 (4) (2004) 477–490.

[18] Holman JP (1981) Heat Transfer, 5th edn, McGraw-Hill, New York

[19] P. Keshavarz . M. Taheri (2007) An improved lumped analysis for transient heat conduction by using the polynomial approximation method. Heat Mass Transfer 43:1151-1156

[20] C. Menezo, H. Bouia, J.J. Roux, J. Virgone, Adaptation of the balanced realization to the coupling of reduced order models for the modelling of the thermal behavior of buildings, Math. Comput. Simul. 53 (4–6) (2000) 395–401.

[21] C. Menezo, J.J. Roux, J. Virgone, Modelling heat transfers in building by coupling reduced-order models, Build Environ. 37 (2) (2002) 133–144.

[22] J. Mennig, T. Auerbach, W. Ha¨lg, Two point Hermite approximations for the solution of linear initial value and boundary value problems, Comp. Methods Appl. Mech. Eng. 39 (1983) 199–224.

[23] Necati Ozisik M (1980) Heat conduction, 1st edn, Wiley, New York

[24] A. Neveu, K. El-Khoury, B. Flament, Simulation of nonlinear non-steady-state conduction: expansion on the branch modes, Int. J. Therm. Sci. 38 (4) (1999) 289–304.

[25] A.C. Pontedeiro, J. Su, R.M. Cotta, Improved lumped model for transient heat conduction in a heat generating cylinder with temperature-dependent thermophysical properties, in: Proceedings of 2004 ASME Heat Transfer/Fluids Engineering Summer Conference, Charlotte, North Carolina, USA, July, 11–15, 2004.

[26] C.R. Regis, R.M. Cotta, J. Su, Improved lumped analysis of transient heat conduction in a nuclear fuel rod, Int. Commun. Heat Mass Transf. 27 (2000) 357–366.

[27] Regis CR, Cotta RM, Su J (2001) Improved lumped analysis of transient heat conduction in a nuclear fuel rod. Int Comm Heat Mass Transfer 27:357-366

[28] N.J. Ruperti, C.V. Falkenberg, R.M. Cotta, J. Su, Engineering analysis of ablative thermal protection for atmospheric reentry: improved lumped formulations and symbolic-numerical computation, Heat Transf. Eng. 25 (2004) 101–111.

[29] Sadat H (2005) A general lumped model for transient heat conduction in one dimensional geometries. Appl Therm Eng 25: 567-576

[30] H. Sadat, A general lumped model for transient heat conduction in one-dimensional geometries, Appl. Therm. Eng. 25 (2005) 567–576.

[31] H. Sadat, A second order model for transient heat conduction in a slab with convective boundary conditions, Appl. Therm. Eng. 26 (2006) 962–965.

[32] F. Scofano-Neto, R.M. Cotta, Improved hybrid lumped-differential formation for double-pipe heat exchanger analysis, J. Heat Transf. 115 (1993) 921–927.

[a]Corresponding author: E-mail: oladapo_anu@yahoo.ie, el. +2348138485550

[33] O.A.Z. Sotomayor, S.W. Park, C. Garcia, Model reduction and identification of wastewater treatment plants - a subspace approach, Lat. Am. Appl. Res. 33 (2) (2003) 135–140.

[34] J. Su, Improved lumped models for asymmetric cooling of a long slab by heat convection, Int. Commun. Heat Mass Transf. 28 (2001) 973–983.

[35] J. Su, Improved lumped models for transient radiative cooling of a spherical body, Int. Commun. Heat Mass Transf. 31 (2004) 85–94.

[36] J. Su, R.M. Cotta, Improved lumped parameter formulation for simplified LWR thermohydraulic analysis, Ann. Nucl. Energy 28 (2001) 1019–1031.

[37] Su J, Cotta CM (2001) Improved lumped parameter formulation for simplified LWR thermohydraulic analysis. Ann Nucl Energy 28:1019-1031

[38] Su J (2001) Improved lumped models for asymmetric cooling of a long slab by heat convection. Int Comm Heat Mass Transf 28:973-983

[39] Zanobetti D (1989) Power station simulators, 1st edn. Elsevier, Amsterdam

[a]Corresponding author: E-mail: oladapo_anu@yahoo.ie, el. +2348138485550