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Mathematical Modelling of the Gradual Aging of Systems using the weibull hazard function

***Osemwenkhae E. Joseph and Osagie S. Aigbe**
Department of Mathematics,
University of Benin, Benin City

Abstract

This paper proposes a mathematical model for obtaining the shape and scale parameters, and the implication of these parameters in obtaining the aging coefficient of any system which is gradually aging, using the two-parameter weibull hazard function. The shape and scale parameters that determines the aging coefficient of any system is realizable especially when the failure times of the system is modelled by the weibull distribution. It concludes that the nature of the aging property depends on the aging coefficient of the system, which makes the system to gradually age with time.

Keywords: Weibull distribution, aging coefficient, shape parameter, scale parameter, hazard function.

1. Introduction

Systems are collection of simple and/or complex components arranged to a specific design to achieve desired functions with acceptable performance and reliability. The type of components, their qualities, quantities and manner of arrangement (series or parallel) within the system have direct effect on the performance, reliability and lifespan of the system. Also, the system could be made up of combination of parallel and series-arranged components. The arrangement of the components will have effect on the reliability and aging of the system.

Many parametric or non-symmetric (exponential distributions) statistical models have been used to model the aging property of systems. These models sometimes, called life distribution models are the weibull, log-normal, gamma, exponential, modified extreme-value, truncated normal, burnbaum-saunders distributions, etc - see [2], [3] and [9]. The weibull distribution (named after [14]) has remained the most preferred parametric life distribution among these distributions in modelling the aging property of most systems due to its flexibility and versatility in determining the three phases of the failure rates of systems and conducting failure rate predictions and analyses with small failure time data. For references, see [5], [7], [8], [12], [13],[14], etc.

The Weibull distribution has three important statistical properties used in reliability and survival analyses of systems, given as

Weibull pdf, $$f(t) = \left(\frac{\alpha}{\beta}\right)\left(\frac{t}{\beta}\right)^{\alpha-1} \exp-\left(\frac{t}{\beta}\right)^{\alpha}, \quad t \geq 0 \qquad (1.1)$$

Weibull cdf, $$F(t) = 1 - \exp-\left(\frac{t}{\beta}\right)^{\alpha}, \quad t \geq 0 \qquad (1.2)$$

Weibull hazard function, $$Z(t) = \left(\frac{\alpha}{\beta}\right)\left(\frac{t}{\beta}\right)^{\alpha-1}, \quad t \geq 0 \qquad (1.3)$$

where α is the shape parameter, β is the scale parameter and t is the failure times of the system.

2. Aging Property of Systems

Age is a natural and inherent property of any system and is caused by internal and/or external factors experienced by the system. The aging property of the system is determined by the behaviour of the failure rate of the system which can be viewed as one of the following three cases.

(a) If the failure rate is increasing, the system is effectively viewed as aging and this implies that $\alpha > 1$.

Corresponding author: E-mail josemwenkhae@yahoo.com, Tel. +2348037378121

(b) If the failure rate is constant, the system is effectively viewed as non-aging, implying that $\alpha = 1$.
(c) If the failure rate is decreasing, then the system is effectively viewed as anti-aging and here, $\alpha < 1$.

The combination of these cases leads to the typical bathtub-shaped curve (infant mortality, constant and wear-out failure phases) of any system; a property modelled by the weibull distribution.

Also, the scale parameter β, has its role in the monotonicity of the failure rate. If $\beta < 1$, then the aging rate of the system will be increasing fastly and if $\beta > 1$, the aging rate of the system, will be increasing slowly. These situations will exist if the shape parameters, $\alpha > 1$ ($\beta < \alpha$ for $\beta < 1$ or $\beta > \alpha$ for $\beta > 1$) – see [10] and [11]. This leads to the proposition of the aging coefficient of a system that is discussed herein.

3. Methods of Analysis of Gradual Aging of Systems

Generally, systems exhibit gradual positive aging or increasing failure rates due to fatigue or corrosion which leads to wear-out of the components within the systems. Quantitatively, this is as a result of the aging coefficient of the weibull hazard function, which is defined as the ratio of the shape parameter to the scale parameter. As such, the aging coefficient is given by:

$$L = \frac{\alpha}{\beta}, \ t > 0 \tag{3.1}$$

If L is close to zero the system will experience gradual aging (aging slowly). As L continues to increase and close to 0.1, the gradual aging of the system reduces – see [1] and [9]. In obtaining the true value of the aging coefficient of the system that determine its age nature, there is need to obtain the estimates of α and β in a more mathematical form. Consequently, the aim of this paper is to use:

(i) mathematical method to obtain the estimate of α,
(ii) mathematical method to obtain the estimate of β, and
(iii) Apply (i) and (ii) to obtain the aging coefficient defined in (3.1)

Interestingly, there are varieties of parameter estimation methods at the disposal of a researcher to use. They are:

(a) The graphical methods which include the probability plotting and hazard plotting techniques.
(b) The analytical methods which includes the Method of Moments (MOM), Maximum Likelihood Estimation (MLE) method and the Least Square Method (LSM).

Each of these methods has their own shortcomings. The graphical methods have high probability of errors and are inaccurate to estimate the parameters. The analytical methods, on the other hand, are tedious in estimating the parameters as well as requiring more computing time and aids – see [4], [6] and [7].

In this work, we propose a mathematical modelling approach, which will make use of (i) the MLE method to obtain the estimate of β and (ii) the LSM to obtain the estimate of α. We proceed as follows to obtain the estimate of β.

From (1.1),

$$f(t) = \left(\frac{\alpha}{\beta}\right)\left(\frac{t}{\beta}\right)^{\alpha-1} \exp-\left(\frac{t}{\beta}\right)^{\alpha}$$

The likelihood function of (1.1) is

$$L(\alpha,\beta) = \prod_{i=1}^{n}\left[\left(\frac{\alpha}{\beta}\right)\left(\frac{t_i}{\beta}\right)^{\alpha-1} \exp-\left(\frac{t_i}{\beta}\right)^{\alpha}\right]$$

$$= \left(\frac{\alpha}{\beta}\right)^{n}\left[\sum_{i=1}^{n}\left(\frac{t_i}{\beta}\right)^{\alpha-1} \exp-\left(\frac{t_i}{\beta}\right)^{\alpha}\right] \tag{3.2}$$

The log-likelihood function of (3.2) gives;

$$\wedge = \ln(\alpha,\beta) = n\ln\alpha - n\ln\beta + \sum_{i=1}^{n}\left[\left(\frac{t_i}{\beta}\right)^{\alpha-1} \exp-\left(\frac{t_i}{\beta}\right)^{\alpha}\right] \tag{3.3}$$

Corresponding author: E-mail josemwenkhae@yahoo.com, Tel. +2348037378121

Taking partial derivatives of (3.3) with respect to α and β, and equating each of the derivative to zero, we obtain;

$$\hat{\alpha} = \frac{n}{\sum_{i=1}^{n}\left[\ln\left(\frac{t_i}{\hat{\beta}}\right)\left(\left(\frac{t_i}{\hat{\beta}}\right)^{\hat{\alpha}} - 1\right)\right]} \tag{3.4}$$

$$\hat{\beta} = \left[\frac{1}{n}\sum_{i=1}^{n} t_i^{\hat{\alpha}}\right]^{\frac{1}{\hat{\alpha}}} \tag{3.5}$$

From (3.4) and (3.5), we need to obtain the explicit estimate of either α or β so as to obtain the estimate of the other parameter. We shall use the least square method to obtain the estimate of α explicitly.

Hence, from (1.2):

$$F(t) = 1 - \exp - \left(\frac{t}{\beta}\right)^{\alpha}$$

$$1 - F(t) = \exp - \left(\frac{t}{\beta}\right)^{\alpha}$$

$$\ln\left[1 - F(t)\right] = -\left(\frac{t}{\beta}\right)^{\alpha}$$

$$\ln\left[-\ln\left[1 - F(t)\right]\right] = -\alpha \ln \beta + \alpha \ln t \tag{3.6}$$

Applying the least square method on (3.6), we obtain

$$\hat{\alpha} = \frac{\sum_{i=1}^{n}(\ln t_i)\left(\ln\left[-\ln\left(1 - F(t_i)\right)\right]\right) - \frac{1}{n}\left[\sum_{i=1}^{n}(\ln t_i)\sum_{i=1}^{n}\left(\ln - \ln\left[1 - F(t_i)\right]\right)\right]}{\sum_{i=1}^{n}(\ln t_i)^2 - \frac{1}{n}\left[\sum_{i=1}^{n}(\ln t_i)\right]^2} \tag{3.7}$$

where n is the number of failure time data.

Equations (3.5) and (3.7) gives the explicit mathematical formulae for the values of the estimates of α and β from which the aging coefficient L is obtained. Computation of (3.5) and (3.7) is easy, more precise and involves less function evaluation unlike the graphical and the analytical methods, which have high probability of errors, inaccuracy and tediousness in the estimation of its parameters.

4. Simulation/Example.

Suppose the failure times of a new power generator (measured in hours) are given as:
18145.34, 23722.54, 32500.02, 22916.76, 25356.02, 19902.13, 24667.61, 29123.33, 32192.75 and 26398.59. From (3.5) and (3.7), we obtain the shape and scale parameter respectively as,

$$\hat{\alpha} = \frac{-48.4811 - \left[(101.3014)^{(-4.9523)}/10\right]}{1026.5216 - (101.3014)^2/10} = 5.2017 \text{ and } \hat{\beta} = 27054.65$$

From (3.1), the aging coefficient of the system is

$$L = \frac{\hat{\alpha}}{\hat{\beta}} = \frac{5.2017}{27054.65} = 0.0001923.$$

Discussion and Conclusion

Corresponding author: E-mail josemwenkhae@yahoo.com, Tel. +2348037378121

From the values of the estimates of α and L in the above example, the generator is aging with respect to time (i.e. positive aging) and the nature of aging of the generator is gradual, since the aging coefficient L is greater than $\varepsilon>0$ (ε is a small value),provided that $\beta>>\alpha$. Hence, the generator will experience gradual aging with respect to time.

Conclusively, in this work, we have seen that a simpler model for obtaining the shape and scale parameters that determines the aging coefficient of any system is realizable especially when the failure times of the system is modelled by the Weibull distribution.

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Corresponding author: E-mail josemwenkhae@yahoo.com, Tel. +2348037378121