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Analytic derivation of the wave profile and phase speed of sixth order Stokes waves in deep water

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Abstract

The wave profile for steady, surface gravity Stokes waves in deep water is investigated. The expression for wave profile for sixth order is derived analytically by substituting the Taylor series approximations for the variables in the free boundary conditions. The wave potential is represented by Fourier series and the coefficients in the series are written as perturbation expansions in terms of a parameter which increases with wave height. The expressions are numerically studied and analyzed graphically. The sixth order phase speed increases with increasing wave steepness δ. It is observed that the wave profile and the phase speed for the sixth order are higher than those of lower orders.

1.0 Introduction

The theoretical understanding of water waves started with the work of Airy, Stokes and their contemporaries in the nineteenth century, [16] and [22]. While linearization about the rest state provided the first insights into dynamics of water waves and led to the development of the linear theory, it was observed that actual water wave characteristics deviate significantly from the linear theory predictions. This motivated an extensive study of the nonlinear wave theory [3].

Waves on the surface of the ocean with periods of 3 to 25 seconds are primarily generated by winds and are the prominent features of the sea surface of the world. These are otherwise known as surface gravity waves. Other wave motions that exist on the ocean include internal waves, tides etc. ([16], [19]).

Whichever section of ocean zone considered, deep water, intermediate or shallow water in extreme conditions, the nonlinearity in the wave kinematics is large and has a strong influence on the design parameters [15]. The knowledge of these waves and the energy they generate are essential for the design of coastal and deep water structures since they are the major factors that influence the geometry of beaches, water ways, shore protection measures, hydraulic, and other civil and military coastal structures [11]. Consequently, estimates of wave conditions are needed in almost all coastal engineering studies [1].

Since waves are one of the most complex phenomena in nature, it is not quite simple to achieve a full understanding of their fundamental character and behaviour. Engineers build various maritime structures, breakwaters and quay walls for ports and harbours, seawalls and jetties for shore protection; platforms and rigs for the exploitation of oil beneath the seabed. These are some examples of maritime structures. These structures must perform their functions in the natural environment being subjected to the hostile effects of winds, water wave currents, earthquakes, etc. To ensure their designated performance, there is need to carry out a comprehensive investigation in order to understand the environmental conditions. The investigation must be as accurate as possible so that the effects of the environment on the structures can be assessed rationally. Hence, the need for the study of nonlinear waves and other ocean wave's phenomena cannot be over-emphasized. The influence of long nonlinear waves on seabed, offshore structures, and local ecosystem in certain parts of the coastal areas apparently is much larger than expected from the linear wave theory.

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Waves carry energy along with them. Wave energy is being transported along the sea surface. The amount of energy at sea each point on the wave train carries is directly related to the amplitude of the wave oscillation at the same point and water particle velocity beneath the wave profile.

Water waves are influenced at every point along their propagation path by the depth of the sea. In deep water, waves are able to move freely without regard to the geometry of the submerged terrain and the speed is an increasing function of wave length. As the depth decreases, the influence of the sea bottom topography becomes significant, causing the wave trains to slow down.

Stokes wave is a steady periodic wave, propagating under gravity with constant speed in the surface of an infinitely deep irrotational flow. The free surface is determined by Laplace's equation, kinematics and dynamic boundary conditions. The latter states that pressure in the flow at the surface is constant [21].

The characteristics of Stokes waves generally in the ocean as applicable to deep and shallow waters have been intensively studied in the last one and half century. Stokes in 1847 and some of his contemporaries such as Michell in 1893 carried out some studies in the nineteenth century. Since then, there have been tremendous achievements in the study of Stokes wave types in ocean engineering. The detailed analysis of the mathematical and physical description of the phenomena is readily available in such publications as: [2], [5], [10], [19], [21], [22] etc.

Stokes waves properties had been exploited by a number of theorists in numerical and analytical study of certain geophysical processes. [7] calculated the force and couple associated with Stokes waves on vertical piercing cylinder in both deep water and water of finite depth. This work explained the effect on oil rigs of the propagating ocean waves.

Fenton [4] obtained the solution of Stokes waves to fifth order using numerical approach. [12] obtained the solution of Stokes waves in the water of finite depth in form of solitary waves, which propagate into the adjoining estuary as bores. [14] obtained the form of fifth order approximation using the analytical approach to ensure that the theoretical approach is very close to the observed wave forms in the ocean.

Further, [13] obtained the effects of wave steepness on the potential and kinetic energies of Stokes waves.

In this study, Stokes sixth order theory is derived by substituting Taylor series approximations for the variables in the free boundary conditions; the order of solution depends on the number of Taylor series terms included.

For practical problems, an application-oriented method which attempts to obtain accurate solutions even for high waves are based essentially on numerical methods, and thus not presented in analytical forms. In problems where the waves are not very high, it is usually more reasonable to use approximate analytical forms, such as cnodial theory for shallow and intermediate water or Stokes theory for deeper water.

The essential feature of Stokes theory for periodic steady waves is that the coefficients in these series can be written as perturbation expansions in terms of a parameter which increases with wave height. Stokes used wave steepness factor $\delta = ak$, as the leading term in a Fourier series, in which the wave number, $k = 2\Pi/L$; L = wavelength, and a is the wave amplitude [4].

Fenton [4] obtained the expression for the free surface profile to fifth order as

$$
\begin{aligned}
k\eta(x) &= kd + \varepsilon \cos kx + \varepsilon^2 B_{22} \cos 2kx + \varepsilon^3 B_{31}(\cos kx - \cos 3kx) \\
&+ \varepsilon^4 (B_{42} \cos 2kx + B_{44} \cos 4kx) + \varepsilon^5 [(-B_{53} + B_{55}) \cos kx + B_{53} \cos 3kx \\
&+ B_{55} \cos 5kx] + 0(\varepsilon^6)
\end{aligned}
\tag{1.1}
$$

where $\varepsilon = ka$

$$B_{22} = \frac{1}{2}, B_{31} = \frac{-3}{8}, B_{42} = \frac{1}{3}, B_{44} = \frac{1}{3}, B_{53} = \frac{99}{128}, B_{55} = \frac{125}{384}$$

in deep water as $kd \to \infty$

$$B_{22} = -3, B_{31} = -6, B_{42} = -9, B_{44} = -12, B_{53} = -12, B_{55} = -12$$

in shallow water as $kd \to 0$ where d is the mean depth of the water.

By considering the deep water limit for equation (1.1), the expression for $\eta(x)$, the height of the free surface above the mean sea level in deep water becomes

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$$k\eta(x) = \varepsilon \cos kx + \frac{1}{2}\varepsilon^2 \cos 2kx + \frac{3}{8}\varepsilon^3(\cos 3kx\text{-}\cos kx) + \frac{1}{3}\varepsilon^4(\cos 2kx + \cos 4kx) +$$
$$\frac{1}{384}\varepsilon^5(-422\cos kx + 297\cos 3kx + 125\cos 5kx) + 0(\varepsilon^6) \quad (1.2)$$

Using Airy's linear theory, the wave speed is determined from the relation $c = w/k$,

For the Stokes fifth order theory , c is expressible as:

$$c = \left[\frac{g}{k}\ (1 + a^2c_1 + a^4c_2)\tanh kd\right]^{\frac{1}{2}} \quad (1.3)$$

where d is the distance from the seabed to the still water level (SWL).

If the Airy wave theory is treated as the first order theory, the wave profile and velocity of the Stokes wave theory can be expressed as follows:

$$\eta = \sum_{n=1}^{m} f_{mn} \cos n\theta \quad (1.4)$$

$$u = \sum_{n=1}^{m} F_{mn} \cosh nkz \cos n\theta \quad (1.5)$$

where the subscript n expresses the mode order, i.e. n = 1,2,3,5 denotes the first order, the second order, the third order and the fifth order theories, respectively [17] where: η is wave profile, u is horizontal velocity of water particles.

Jamaloddin [8] adopted the method of solution for the Stokes' fifth order theory in the form

$$\eta = \frac{1}{k}\sum_{n=1}^{5} F_n \cos n(kx - wt) \quad (1.6)$$

Where η is the instantaneous vertical displacement of sea surface from the still water level (SWL).

The waves derived from linearized equations based on the small-amplitude assumptions were sinusoidal, assuming that the water depth is constant or infinitely large. However, the waves of the sea are often not of small amplitude.

This study is to further investigate the analytic form of the wave profile of sixth order Stokes waves and to compare it with those of lower order. This is expected to throw light on the limiting wave height in deep water.

2.0 Sixth order Stokes waves

Review of earlier development

The fluid medium is assumed to be irrotational and incompressible .

Let ϕ and ψ be the velocity potential and stream function respectively.

Following [10], the following apply

$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial z^2} = 0 \quad (2.1)$$

$$\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial z^2} = 0 \quad (2.2)$$

To solve eqn (2.2), the boundary conditions are:

$$\psi = 0 \quad \text{for} \quad z = \eta \quad (2.3)$$

$$\psi = k_1 \quad \text{for} \quad z = -h \quad (2.4)$$

$$p = k_2 \quad \text{for} \quad z = \eta \quad (2.5)$$

$\eta = \eta(x,t)$ is the wave profile

Dynamic boundary condition is

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$$g\eta + \frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] = k_3 \text{ for } z = \eta \quad (2.6)$$

General solution for eqn (2.2) is

$$\Psi = c_1 z + c_2 + \left(c_3 \text{coskx} + c_4 \text{sinkx}\right)\left(c_5 e^{kz} + c_6 e^{-kz}\right) \quad (2.7)$$

c_i (i = 1, 2, 3, 4, 5, 6) are constants

Since the fluid depth is assumed to be infinite

$$c_6 = 0, \text{ and } c_1 = -c$$

$$\therefore \Psi = -cz + c_2 + \left(c_3 \text{coskx} + c_4 \text{sinkx}\right) c_5 e^{kz} \quad (2.8)$$

A possible form of stream function is obtained if $c_4 = 0$ and $c_2 = 0$

$$\Psi = -cz + c_3 c_5 \text{coskx} e^{kz} \quad (2.9)$$

(divide through by c)

$$\frac{\Psi}{c} = -z + \frac{c_3 c_5}{c} \cos kx e^{kz} \quad (2.10)$$

$$\text{let } \beta = \frac{c_3 c_5}{c}$$

$$\therefore \frac{\Psi}{c} = -z + \beta e^{kz} \cos kx \quad (2.11)$$

But when $\Psi = 0, \ z = \eta$

$$-\eta + \beta e^{k\eta} \text{coskx} = 0 \quad (2.12)$$

$$\eta = \beta e^{k\eta} \text{coskx} \quad (2.13)$$

By perturbation methods involving $\eta(x,t)$, the following were obtained:

First order approximation

$$\eta(x) = -a\text{coskx} \quad (2.14)$$

Second order approximation

$$\eta(x) = -a\text{coskx} + \frac{1}{2} ka^2 \text{cos2kx} \quad (2.15)$$

Third order approximation

$$\eta(x) = -a\text{coskx} + \frac{1}{2} ka^2 \text{cos2kx} - \frac{3}{8} k^2 a^3 \text{cos3kx} \quad (2.16)$$

Fourth order approximation

$$\eta(x) = - a \text{ coskx} + (\frac{1}{2} k a^2 + \frac{11}{6} k^3 a^4) \text{ cos2kx}$$

$$- \frac{3}{8} a^3 k^2 \text{ cos3kx} + \frac{1}{3} a^4 k^3 \text{ cos4kx} \quad (2.17)$$

Fifth Order approximation

$$\eta = -a \cos kx + \left(\frac{1}{2} a^2 k + \frac{11}{6} a^4 k^3\right) \cos 2kx - \left(\frac{3}{8} a^3 k^2 + \frac{235}{96} a^5 k^4\right) \cos 3kx$$

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$$+\frac{1}{3}k^3 a^4 \cos 4kx - \frac{31}{96}a^5 k^4 \cos 5kx.$$

Analytic derivation of 6th order

The fifth order solution of Stokes waves is of the form

$$\frac{\psi}{c} = -z + \beta e^{kz}\cos kx + \gamma e^{2kz}\cos 2kx + \alpha e^{3kz}\cos 3kx \qquad (2.18)$$

$$(\psi = 0 \text{ at } z = \eta)$$

Including the next term in the Fourier expansion for ψ and adjusting the coefficient,we have

$$\frac{\psi}{c} = -z + \beta e^{kz}\cos kx + \gamma e^{2kz}\cos 2kx + \alpha e^{3kz}\cos 3kx$$
$$+ \xi e^{4kz}\cos 4kx \qquad (2.19)$$

By assumption, $\alpha = o(\beta^5)$

Dynamic boundary condition is

$$g\eta + \frac{1}{2}\left[\left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial z}\right)^2\right] = k_3 \text{ at } z = \eta \qquad (2.20)$$

$$\psi = -c\left[z - \beta e^{kz}\cos kx - \gamma e^{2kz}\cos 2kx - \alpha e^{3kz}\cos 3kx - \xi e^{4kz}\cos 4kx\right]$$

$$\frac{\partial\psi}{\partial x} = -c\left[k\beta e^{kz}\sin kx + 2k\gamma e^{2kz}\sin 2kx + 3k\alpha e^{3kz}\sin 3kx + 4\xi e^{4kz}\cos 4kx\right] \qquad (2.21)$$

$$\left(\frac{\partial\psi}{\partial x}\right)^2 = c^2\left[k\beta e^{kz}\sin kx + 2k\gamma e^{2kz}\sin 2kx + 3k\alpha e^{3kz}\sin 3kx + 4\xi e^{4kz}\cos 4kx\right]^2$$

$$\frac{1}{c^2}\left(\frac{\partial\psi}{\partial x}\right)^2 = k^2\beta^2 e^{2kz}\sin^2 kx + 4k^2\gamma^2 e^{4kz}\sin^2 2kx + 9k^2\alpha^2 e^{6kz}\sin^2 3kx$$
$$+16k^2\xi^2 e^{8kz}\sin^2 4kx + 4k^2\beta\gamma e^{3kz}\sin kx\sin 2kx + 6k^2\beta\alpha e^{4kz}\sin kx\sin 3kx$$
$$+ 12k^2\gamma\alpha e^{5kz}\sin 2kx\sin 3kx + 16k^2\gamma\xi e^{6kz}\sin 4kx\sin 2kx$$
$$+ 24k^2\alpha\xi e^{7kz}\sin 4kx\sin 3kx$$

(2.22)

Similarly,

$$\frac{\partial\psi}{\partial z} = c\left[-1 + k\beta e^{kz}\cos kx + 2k\gamma e^{2kz}\cos 2kx + 3k\alpha e^{3kz}\cos 3kx + 4k\xi e^{4kz}\cos 4kx\right]$$

(2.23)

$$\left(\frac{\partial\psi}{\partial z}\right)^2 = c^2\left[-1 + k\beta e^{kz}\cos kx + 2k\gamma e^{2kz}\cos 2kx + 3k\alpha e^{3kz}\cos 3kx + 4k\xi e^{4kz}\cos 4kx\right]^2$$

$$\frac{1}{c^2}\left(\frac{\partial\psi}{\partial z}\right)^2 = 1 + k^2\beta^2 e^{2kz}\cos^2 kx + 4k^2\gamma^2 e^{4kz}\cos^2 2kx + 9k^2\alpha^2 e^{6kz}\cos^2 3kx$$

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$+ 16k^2\xi^2 \; e^{8kz} \; \cos^2 4kx \; + \; 4k^2\beta\,\gamma\, e^{3kz} \; \text{coskxcos2kx} \; + \; 6k^2\beta\,\alpha\, e^{4kz} \; \text{coskxcos3kx} \; +$

$+8k^2\beta\,\xi\, e^{5kz} \; \text{coskxcos4kx} + \; 12k^2\gamma\,\alpha\, e^{5kz}\text{cos2kxcos3kx} \; +16k^2\gamma\,\xi\, e^{6kz}\text{cos2kxcos4kx}$

$+ 24k^2\alpha\,\xi\, e^{7kz}\text{cos3kxcos4kx} - 2k\beta\, e^{kz} \; \text{coskx} \; - \; 4k\,\gamma\, e^{2kz} \; \text{cos2kx}$

$- \; 6k\alpha\, e^{3kz} \; \text{cos3kx} - 8k\xi\, e^{4kz} \; \text{cos4kx}$

(2.24)

Adding eqns (2.22) and (2.24)

$$\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2 = c^2[\, 1 + k^2\beta^2 e^{2kz} + 4k^2 \gamma^2 e^{4kz} + 9k^2\alpha^2 e^{6kx} +16k^2\xi^2 e^{8kx}$$

$$+ \; 4k^2\beta\,\gamma\, e^{3kz} \; \text{coskx} \; + \; 6k^2\beta\,\alpha\, e^{4kz} \; \text{cos2kx} \; + \; 8k^2\beta\,\xi\, e^{5kz} \; \text{cos3kx}$$

$$+ \; 12k^2\gamma\,\alpha\, e^{5kz}\text{coskx} \; + \; 16k^2\gamma\,\xi\, e^{6kz}\text{cos2kx} \; + \; 24k^2 \; \alpha\xi\, e^{7kz}\text{coskx}$$

$$- \; 2k\beta\, e^{kz} \; \text{coskx} \; - \; 4k\,\gamma\, e^{2kz} \; \text{cos2kx} \; - \; 6k\alpha\, e^{3kz} \; \text{cos3kx}$$

$$- \; 8k\xi\, e^{4kz} \; \text{cos4kx}$$

(2.25)

$$g\eta \; + \; \frac{1}{2}\left[\left(\frac{\partial \psi}{\partial x}\right)^2 + \left(\frac{\partial \psi}{\partial z}\right)^2\right] = k_3 \quad \text{now becomes}$$

$$\frac{2g\eta}{c^2} + 1 + k^2\beta^2 e^{2kz} + 4k^2 \gamma^2 e^{4kz} + 9k^2\alpha^2 e^{6kz} +16k^2\xi^2 e^{8kx}$$

$$+ 4k^2\beta\,\gamma\, e^{3kz} \; \text{coskx} \; + \; 6k^2\beta\,\alpha\, e^{4kz} \; \text{cos2kx} \; + \; 8k^2\beta\,\xi\, e^{5kz} \; \text{cos3kx}$$

$$+ \; 12k^2\gamma\,\alpha\, e^{5kz}\text{coskx} \; + \; 16k^2\gamma\,\xi\, e^{6kz}\text{cos2kx} \; + \; 24k^2 \; \alpha\xi\, e^{7kz}\text{coskx} - 2k\beta\, e^{kz} \; \text{coskx}$$

$$- \; 4k\,\gamma\, e^{2kz} \; \text{cos2kx} \; - \; 6k\alpha\, e^{3kz} \; \text{cos3kx} - \; 8k\xi\, e^{4kz} \; \text{cos4kx} \; = \; k_4 \qquad (2.26)$$

(at $z = \eta$)

$$\frac{2g\eta}{c^2} + k^2\beta^2 e^{2kz} + 4k^2 \gamma^2 e^{4kz} + 9k^2\alpha^2 e^{6kz} +16k^2\xi^2 e^{8kx}$$

$$+4k^2\beta\,\gamma\, e^{3kz} \; \text{coskx} \; + \; 6k^2\beta\,\alpha\, e^{4kz} \; \text{cos2kx} \; + \; 8k^2\beta\,\xi\, e^{5kz} \; \text{cos3kx}$$

$$+ \; 12k^2\gamma\,\alpha\, e^{5kz}\text{coskx} \; + \; 16k^2\gamma\,\xi\, e^{6kz}\text{cos2kx} \; + \; 24k^2 \; \alpha\xi\, e^{7kz}\text{coskx} - 2k\beta\, e^{kz} \; \text{coskx}$$

$$- \; 4k\,\gamma\, e^{2kz} \; \text{cos2kx} \; - \; 6k\alpha\, e^{3kz} \; \text{cos3kx} - \; 8k\xi\, e^{4kz} \; \text{cos4kx} \; = \; k_5 \qquad (2.27)$$

then putting $z = \eta$ when $\psi = 0$ in eqn (2.19))

$$\eta = \beta\, e^{k\eta} \; \text{coskx} \; + \; \gamma\, e^{2k\eta} \; \text{cos2kx} \; + \; \alpha\, e^{3k\eta} \; \text{cos3kx} + \; \xi\, e^{4k\eta} \; \text{cos4kx} \qquad (2.28)$$

$$\frac{2g\eta}{c^2} + k^2\beta^2 e^{2k\eta} + 4k^2 \gamma^2 e^{4k\eta} + 9k^2\alpha^2 e^{6k\eta} + 16k^2\xi^2 e^{8k\eta} + 4k^2\beta\,\gamma\, e^{3k\eta} \; \text{coskx}$$

$$+ \; 6k^2\beta\,\alpha\, e^{4k\eta} \; \text{cos2kx} \; + \; 12k^2\gamma\,\alpha\, e^{5k\eta}\text{coskx} \; + \; 16k^2\gamma\,\xi\, e^{6k\eta}\text{coskx} \; + \; 24k^2 \; \alpha\xi\, e^{7k\eta}\text{coskx}$$

$$- \; 2k\left[\beta\, e^{k\eta} \; \text{coskx} \; + \; \gamma\, e^{2k\eta} \; \text{cos2kx} \; + \; \alpha\, e^{3k\eta} \; \text{cos3kx} + \xi\, e^{4k\eta} \; \text{cos4kx} + \right]$$

$$- \; 2k\,\gamma\, e^{2k\eta} \; \text{cos2kx} \; - \; 4k\alpha\, e^{3k\eta} \; \text{cos3kx} - \; 6k\xi\, e^{3k\eta} \; \text{cos4kx} \; = \; k_5 \qquad (2.29)$$

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Substituting for $\eta = \beta e^{k\eta}\cos kx + \gamma e^{2k\eta}\cos 2kx + \alpha e^{3k\eta}\cos 3kx + \xi e^{4k\eta}\cos 4kx$

we have

$$\frac{2g\eta}{c^2} + k^2\beta^2 e^{2k\eta} + 4k^2\gamma^2 e^{4k\eta} + 9k^2\alpha^2 e^{6k\eta} + 16k^2\xi^2 e^{6k\eta} + 4k^2\beta\gamma e^{3k\eta}\cos kx$$

$$+ 6k^2\beta\alpha e^{4k\eta}\cos 2kx + 8k^2\beta\xi e^{5k\eta}\cos 3kx + 12k^2\gamma\alpha e^{5k\eta}\cos kx$$

$$+ 16k^2\gamma\xi e^{6k\eta}\cos 2kx + 24k^2\alpha\xi e^{7k\eta}\cos kx - 2k\eta$$

$$- 2k\gamma e^{2k\eta}\cos 2kx - 4k\alpha e^{3k\eta}\cos 3kx - 6k\xi e^{4k\eta}\cos 4kx = k_5 \quad (2.30)$$

$$\frac{2g\eta}{c^2} + k^2\beta^2 e^{2k\eta} - 2k\eta - 2k\gamma e^{2k\eta}\cos 2kx - 4k\alpha e^{3k\eta}\cos 3kx - 6k\xi e^{4k\eta}\cos 4kx \approx 0 \quad (2.31)$$

(neglecting $o(\beta^6)$ and above) . Recall from (2.28)

$$\eta = \beta e^{k\eta}\cos kx + \gamma e^{2k\eta}\cos 2kx + \alpha e^{3k\eta}\cos 3kx + \xi e^{4k\eta}\cos 4kx$$

Making coskx the subject of the formula ,

$$\beta e^{k\eta}\cos kx = \eta - \gamma e^{2k\eta}\cos 2kx - \alpha e^{3k\eta}\cos 3kx - \xi e^{4k\eta}\cos 4kx \quad (2.32)$$

$$\cos kx = \beta^{-1} e^{-k\eta}(\eta - \gamma e^{2k\eta}\cos 2kx - \alpha e^{3k\eta}\cos 3kx - \xi e^{4k\eta}\cos 4kx) \quad (2.33)$$

$$\cos 2kx = 2\cos^2 kx - 1 \quad (2.34)$$

$$\cos 3kx = 4\cos^3 kx - 3\cos kx \quad (2.35)$$

$$\cos 4kx = 8\cos^4 kx - 8\cos^2 kx +$$

$$\cos 2kx = 2\left[\beta^{-1} e^{-k\eta}(\eta - \gamma e^{2k\eta}\cos 2kx - \alpha e^{3k\eta}\cos 3kx - \xi e^{4k\eta}\cos 4kx\right]^2 - 1$$

$$= 2\beta^{-2} e^{-2k\eta}(\eta^2 + \gamma^2 e^{4k\eta}\cos^2 2kx + \alpha^2 e^{6k\eta}\cos^2 3kx + \xi^2 e^{8k\eta}\cos^2 4kx$$

$$- 2\eta\gamma e^{2k\eta}\cos 2kx - 2\eta\alpha e^{3k\eta}\cos 3kx - 2\eta\xi e^{4k\eta}\cos 4kx + ...) - 1$$

(2.36)

$$= 2\beta^{-2}\eta^2 e^{-2k\eta} + 2\beta^{-2}\gamma^2 e^{2k\eta}\cos^2 2kx + 2\beta^{-2}\alpha^2 e^{4k\eta}\cos^2 3kx$$

$$+ 2\beta^{-2}\xi^2 e^{6k\eta}\cos^2 4kx - 4\beta^{-2}\gamma\eta\cos 2kx - 4\beta^{-2}\alpha\eta e^{k\eta}\cos 3kx - 4\beta^{-2}\xi\eta e^{2k\eta}\cos 4kx - 1$$

(2.37)

$$\Rightarrow -2k\gamma e^{2k\eta}\cos 2kx = -2k\gamma e^{2k\eta}(2\beta^{-2}\eta^2 e^{-2k\eta} + 2\beta^{-2}\gamma^2 e^{2k\eta}\cos^2 2kx$$

$$+ 2\beta^{-2}\alpha^2 e^{4k\eta}\cos^2 3kx + 2\beta^{-2}\xi^2 e^{6k\eta}\cos^2 4kx - 4\beta^{-2}\gamma\eta\cos 2kx$$

$$- 4\beta^{-2}\alpha\eta e^{k\eta}\cos 3kx - 4\beta^{-2}\xi\eta e^{2k\eta}\cos 4kx - 1)$$

$$= -4k\beta^{-2}\gamma\eta^2 - 4k\beta^{-2}\gamma^3 e^{4k\eta}\cos^2 2kx -$$

(2.38)

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$$4k\beta^{-2}\gamma\,\alpha^2\ e^{6k\eta}\cos^2 3kx - 4k\beta^{-2}\gamma\,\xi^2\ e^{8k\eta}\cos^2 4kx + 8k\beta^{-2}\ \gamma^2\eta\ e^{2k\eta}\cos 2kx$$
$$+\ 8k\beta^{-2}\ \gamma\,\alpha\eta\ e^{3k\eta}\cos 3kx + 8k\beta^{-2}\ \gamma\,\xi\eta\ e^{3k\eta}\cos 4kx + 2k\,\gamma\,e^{2k\eta}$$

(2.39)

$$= -4k\beta^{-2}\gamma\,\eta^2 \quad + \quad + \ 2k\gamma\,e^{2k\eta} \tag{2.40}$$

(neglecting terms of $o(\beta^6)$ and above)

$$\cos 2kx = 2\cos^2 kx - 1 \quad = \quad 2\beta^{-2}e^{-2k\eta}\eta^2 \ -1$$
(2.41)

$$\cos 3kx = 4\cos^3 kx - 3\cos kx \ = \ 4\beta^{-3}e^{-3k\eta}\eta^3 \ - 3\beta^{-1}e^{-k\eta}\eta$$
(2.42)

$$\cos 4kx = 8\cos^4 kx - 8\cos^2 kx + 1 \ = \ 8\beta^{-4}e^{-4k\eta}\eta^4 \ - 8\beta^{-2}e^{-2k\eta}\eta^2 + 1$$
(2.43)

Also, from eqn (2.31), $\dfrac{2g\eta}{c^2} + k^2\beta^2\ e^{2k\eta} - 2k\eta - 2k\,\gamma\,e^{2k\eta}\ \cos 2kx \ - \ 4k\alpha\ e^{3k\eta}\ \cos 3kx$

$- \ 6k\xi\ e^{4k\eta}\ \cos 4kx \ \approx \ 0$

Substituting the values obtained above for cos2kx , cos3kx and cos4kx gives

$$\frac{2g\eta}{c^2} + k^2\beta^2\ e^{2k\eta} - 2k\eta - 4k\beta^{-2}\ \gamma\,\eta^2 - 16\,k\,\alpha\beta^{-3}\eta^3 - 48k\xi\beta^{-4}\eta^4 + k^2\beta^2 e^{2k\eta} \approx 0$$
(2.44)

Applying Taylor's series for $e^{2k\eta}$

$$\frac{2g\eta}{c^2} + k^2\beta^2\left[1 + 2k\eta + \frac{4k^2\ \eta^2}{2} + \frac{8k^3\ \eta^3}{6} + \frac{16k^4\ \eta^4}{24} + o(\eta^5)\right]$$
$$- \ 2k\eta - 4k\beta^{-2}\ \gamma\,\eta^2 - 16\,k\,\alpha\beta^{-3}\eta^3 - 48k\xi\beta^{-4}\eta^4 \ \approx \ 0 \tag{2.45}$$

$$\frac{2g\eta}{c^2} + k^2\beta^2 + 2k^3\beta^2\eta + 2k^4\beta^2\eta^2 + \frac{4k^5\beta^2\ \eta^3}{3} + \frac{2k^6\beta^2\ \eta^4}{3}$$
$$- \ 2k\eta - 4k\beta^{-2}\ \gamma\,\eta^2 - 16\,k\,\alpha\beta^{-3}\eta^3 - 48k\xi\beta^{-4}\eta^4 \ \approx \ 0 \tag{2.46}$$

$$\left(\frac{2g}{c^2} + 2k^3\beta^2 - 2k\right)\eta + \left(2k^4\beta^2 - 4k\beta^{-2}\ \gamma\right)\eta^2 + \left(\frac{4k^5\beta^2}{3} - 16\,k\,\alpha\beta^{-3}\right)\eta^3$$
$$+ \left(\frac{2k^6\beta^2}{3} - 48k\,\xi\beta^{-4}\right)\eta^4 + k^2\beta^2 = 0$$

(2.47)

Equating coefficients of powers of η ; η : $\dfrac{2g}{c^2} + 2k^3\beta^2 - 2k \ = \ 0$ (2.48)

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$$\eta^2 : \quad 2k^4\beta^2 - 4k\beta^{-2}\gamma = 0 \tag{2.49}$$

$$\Rightarrow \quad \gamma = \frac{1}{2}\beta^4 k^3 \tag{2.50}$$

$$\eta^3 : \quad \frac{8k^5\beta^2}{6} - 16\,k\,\alpha\beta^{-3} = 0 \tag{2.51}$$

$$\Rightarrow \quad \alpha = \frac{1}{12}\beta^5 k^4 \tag{2.52}$$

$$\eta^4 : \quad \frac{2k^6\beta^2}{3} - 48k\,\xi\beta^{-4} = 0 \tag{2.53}$$

$$\Rightarrow \quad \xi = \frac{1}{72}\beta^6 k^5 \tag{2.54}$$

$$\frac{2g}{c^2} = 2k - 2\beta^2 k^3 = 2k(1 - \beta^2 k^2) \tag{2.55}$$

$$g = c^2 k(1 - \beta^2 k^2) \tag{2.56}$$

$$\frac{g}{k}\,\frac{1}{(1 - \beta^2 k^2)} = c^2$$

$$c^2 = \frac{g}{k}\left(1 + \beta^2 k^2 + \frac{(-1)(-2)}{2!}\beta^4 k^4 + \frac{(-1)(-2)(-3)}{3!}\beta^6 k^6 + \ldots\right) \tag{2.57}$$

$$c^2 = \frac{g}{k}\left(1 + \beta^2 k^2 + \beta^4 k^4 + \beta^6 k^6 \ldots\right) \text{ sixth order phase velocity} \tag{2.58}$$

From eqn (2.28) ; $\eta = \beta e^{k\eta}\cos kx + \gamma e^{2k\eta}\cos 2kx + \alpha e^{3k\eta}\cos 3kx + \xi e^{4k\eta}\cos 4kx$

Recall: $\gamma = \frac{1}{2}\beta^4 k^3$, $\alpha = \frac{1}{12}\beta^5 k^4$, $\xi = \frac{1}{72}\beta^6 k^5$

$$\eta = \beta e^{k\eta}\cos kx + \frac{1}{2}\beta^4 k^3 e^{2k\eta}\cos 2kx + \frac{1}{12}\beta^5 k^4 e^{3k\eta}\cos 3kx + \frac{1}{72} e^{4k\eta}\cos 4kx \tag{2.59}$$

let $\eta = \eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5 + \ldots$ (2.60)

Utilising Taylor's series for $e^{k\eta}$, $e^{2k\eta}$, $e^{3k\eta}$, $e^{4k\eta}$ and substituting in eqn (2.59), gives

$$\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5 + \eta_5\beta^6$$

$$= \beta\left(1 + k\eta + \frac{1}{2}k^2\eta^2 + \frac{k^3\eta^3}{6} + \frac{k^4\eta^4}{24} + \ldots\right)\cos kx$$

$$+ \frac{1}{2}\beta^4 k^3\left(1 + 2k\eta + \frac{4k^2\eta^2}{2} + \frac{8k^3\eta^3}{6} + \frac{16k^4\eta^4}{24}\ldots\right)\cos 2kx$$

$$+ \frac{1}{12}\beta^5 k^4\left(1 + 3k\eta + \frac{9k^2\eta^2}{2} + \frac{27k^3\eta^3}{6} + \frac{81k^4\eta^4}{24}\ldots\right)\cos 3kx$$

$$+ \frac{1}{72}\beta^6 k^5\left(1 + 4k\eta + \frac{16k^2\eta^2}{2} + \frac{64k^3\eta^3}{6} + \frac{256k^4\eta^4}{24}\ldots\right)\cos 4kx \tag{2.61}$$

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$$= \beta [1 + k(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5 + \eta_5\beta^6)$$

$$+ \frac{k^2}{2}(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3 \beta^4 + \eta_4\beta^5 + \eta_5\beta^6)^2$$

$$+ \frac{k^3}{6}(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5 + \eta_5 \beta^6)^3$$

$$+ \frac{k^4}{24}(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3 \beta^4 + \eta_4\beta^5 + \eta_5 \beta^6)^4$$

$$+ \frac{k^5}{120}(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5 + \eta_5 \beta^6)^5] \cos kx$$

$$+ \frac{1}{2} \beta^4 k^3 [1 + 2k (\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5 + \eta_5 \beta^6)$$

$$+ 2k^2 (\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5 + \eta_5 \beta^6)^2$$

$$+ \frac{4k^3}{3}(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5 + \eta_5 \beta^6)^3$$

$$+ \frac{2k^4}{3}(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5 + \eta_5 \beta^6)^4$$

$$+ \frac{4k^5}{15}(\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4\beta^5 + \eta_5 \beta^6)^5] \cos 2kx$$

$$+ \frac{1}{12} \beta^5 k^4 [1 + 3k (\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4 \beta^5 + \eta_5 \beta^6)$$

$$+ \frac{9k^2}{2} (\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4 \beta^5 + \eta_5 \beta^6)^2$$

$$+ \frac{1}{72} \beta^6 k^5 [1 + 4k (\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4 \beta^5 + \eta_5 \beta^6)$$

$$+ 8k^2 (\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4 \beta^5 + \eta_5 \beta^6)^2$$

$$+ \frac{32}{3} k^3 (\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4 \beta^5 + \eta_5 \beta^6)^3$$

$$+ \frac{32}{3} k^3 (\eta_0\beta + \eta_1\beta^2 + \eta_2\beta^3 + \eta_3\beta^4 + \eta_4 \beta^5 + \eta_5 \beta^6)^4] \cos 4kx \qquad (2.62)$$

Equating coefficients of β, β^2, β^3, β^4 and β^5 we have

$$\beta^3 : \quad \eta_2 = (k\eta_1 + \frac{k^2}{2} \eta^2{}_0) \cos kx \qquad (2.65)$$

$$= (k(k \cos^2 kx) + \frac{k^2}{2} \cos^2 kx) \cos kx$$

$$\eta_2 = \frac{3}{2} k^2 \cos^3 kx \qquad (2.66)$$

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$$\beta^4: \quad \eta_3 = (k\eta_2 + k^2\eta_0\eta_1 + \frac{k^3}{6}\eta_0^3)\cos kx + \frac{1}{2}k^3\cos 2kx \qquad (2.67)$$

$$\eta_3 = k\eta_2\cos kx + k^2\eta_0\eta_1\cos kx + \frac{k^3}{6}\eta_0^3\cos kx + \frac{1}{2}k^3\cos 2kx$$

$$= \frac{3}{2}k\cos kx\,(k^2\cos^3 kx) + k^2(\cos kx)(k\cos^2 kx)(\cos kx) + \frac{k^3}{6}(\cos kx)^3\cos kx + \frac{1}{2}k^3\cos 2kx$$
(2.68)

$$= \frac{3}{2}k^3\cos^4 kx + k^3\cos^4 kx + \frac{k^3}{6}\cos^4 kx + \frac{1}{2}k^3\cos 2kx \qquad (2.69)$$

$$\eta_3 = \frac{8}{3}k^3\cos^4 kx + \frac{1}{2}k^3\cos 2kx \qquad (2.70)$$

$$\beta^5: \quad \eta_4 = (k\eta_3 + \frac{k^2}{2}\eta_1^2 + k^2\eta_0\eta_2 + \frac{k^3}{2}\eta_0^2\eta_1)\cos kx + k^4\eta_0\cos 2kx + \frac{k^4}{12}\cos 3kx \qquad (2.71)$$

$$\eta_4 = \frac{31}{6}k^4\cos^5 kx + \frac{1}{24}k^4\cos^4 kx + \frac{5}{6}k^4\cos 3kx + \frac{3}{4}k^4\cos kx \qquad (2.72)$$

$$\beta^6: \eta_5 = (k\eta_4 + k^2\eta_1\eta_2 + \frac{k^3}{2}\eta_0^2\eta_2 + \frac{k^4}{8}\eta_0^3\eta_1 + \frac{k^4}{120}\eta_0^5)\cos kx$$

$$+(k^4\eta_1 + k^5\eta_0^2)\cos 2kx + \frac{1}{4}k^5\eta_0\cos 3kx + \frac{1}{72}k^5\cos 4kx \qquad (2.73)$$

$$\eta_5 = \frac{906}{120}k^5\cos^6 kx + \frac{13}{24}k^5\cos 4kx + \frac{13}{24}k^5\cos 2kx + \frac{1}{24}k^5\cos^5 kx + 2k^5\cos^3 kx$$

$$- k^5\cos^2 kx + 2k^5\cos^3 kx - k^5\cos^2 kx + \frac{1}{75}k^5\cos 4kx + \frac{3}{4}k^5\cos^2 kx \qquad (2.74)$$

From eqn (2.60), $\eta = \beta\eta_0 + \beta^2\eta_1 + \beta^3\eta_2 + \beta^4\eta_3 + \beta^5\eta_4 + \beta^5\eta_5$,

Substituting for $\eta_0, \eta_1, \eta_2, \eta_3, \eta_4$ and η_5

$$\eta = \beta\cos kx + \beta^2 k\cos^2 kx + \frac{3}{2}\beta^3 k^2\cos^3 kx + \beta^4(\frac{8}{3}k^3\cos^4 kx + \frac{1}{2}k^3\cos 2kx)$$

$$+ \beta^5(\frac{31}{6}k^4\cos^5 kx + \frac{1}{24}k^4\cos^4 kx + \frac{5}{6}k^4\cos 3kx + \frac{3}{4}k^4\cos kx)$$

$$+ \beta^6(\frac{906}{120}k^5\cos^6 kx + \frac{13}{24}k^5\cos 4kx + \frac{13}{24}k^5\cos 2kx + \frac{1}{24}k^5\cos^5 kx + 2k^5\cos^3 kx$$

$$- k^5\cos^2 kx + 2k^5\cos^3 kx - k^5\cos^2 kx + \frac{1}{75}k^5\cos 4kx + \frac{3}{4}k^5\cos^2 kx) \qquad (2.75)$$

Substituting for the identities, we have : $\eta = (\beta + \frac{9}{8}\beta^3 k^2 + \frac{191}{48}\beta^5 k^4 + \frac{581}{192}\beta^3 k^2)\cos kx$

$$+ (\frac{1}{2}\beta^2 k + \frac{11}{6}\beta^4 k^3 + \frac{1}{48}\beta^5 k^4 + \frac{1327}{384}\beta^6 k^5)\cos 2kx + (\frac{3}{8}\beta^3 k^2 + \frac{235}{96}\beta^5 k^4 + \frac{389}{384}\beta^6 k^5)\cos 3kx$$

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$$+ (\frac{1}{3}\beta^4 k^3 + \frac{1}{192}\beta^5 k^4 + \frac{9459}{4800}\beta^6 k^5)\cos 4kx + (\frac{31}{96}\beta^5 k^4 + \frac{1}{384}\beta^3 k^2)\cos 5kx + \frac{151}{640}\beta^6 k^5 \cos 6kx$$

$$+ (\frac{1}{2}\beta^2 k + \beta^4 k^3 + \frac{1}{64}\beta^5 k^4 - \frac{5}{8}\beta^6 k^5) \qquad (2.76)$$

let $a = (\beta + \frac{9}{8}\beta^3 k^2 + \frac{191}{48}\beta^5 k^4 + \frac{581}{192}\beta^6 k^5)$ (2.77)

let $\beta = c_1 a + c_2 a^2 + c_3 a^3 + c_4 a^4 + c_5 a^5 + c_6 a^6$ (2.78)

$$a = c_1 a + c_2 a^2 + c_3 a^3 + c_4 a^4 + c_5 a^5 + c_6 a^6 + \frac{9}{8}k^2(c_1 a + c_2 a^2 + c_3 a^3 + c_4 a^4 + c_5 a^5 + c_6 a^6)^3$$

$$+ \frac{191}{48}k^4(c_1 a + c_2 a^2 + c_3 a^3 + c_4 a^4 + c_5 a^5 + c_6 a^6)^5 + \ldots$$

(2.79) $+ \frac{581}{192}k^5(c_1 a + c_2 a^2 + c_3 a^3 + c_4 a^4 + c_5 a^5 + c_6 a^6)^6$

$\beta \cong a$ and $\beta^2 \cong a^2$

$$\eta = (a + \frac{9}{8}a^3 k^2 + \frac{191}{48}a^5 k^4 + \frac{581}{192}a^6 k^5)\cos kx + (\frac{1}{2}a^2 k + \frac{11}{6}a^4 k^3 + \frac{1}{48}a^5 k^4 + \frac{1327}{384}a^6 k^5)\cos 2kx$$

$$+(\frac{3}{8}a^3 k^2 + \frac{235}{96}a^5 k^4 + \frac{389}{384}a^6 k^5)\cos 3kx + (\frac{1}{3}a^4 k^3 + \frac{1}{192}a^5 k^4 + \frac{9459}{4800}a^6 k^5)\cos 4kx$$

$$+ (\frac{31}{96}a^5 k^4 + \frac{1}{384}a^3 k^2)\cos 5kx + \frac{151}{640}a^6 k^5 \cos 6kx + (\frac{1}{2}a^2 k + a^4 k^3 + \frac{1}{64}a^5 k^4 - \frac{5}{8}a^6 k^5)$$

(2.80)

Shifting the axes vertically , the additive constant becomes zero. Also, translating the axes laterally by π, the signs of terms containing odd multiples of kx are reversed,we then have

$$\eta = -(a + \frac{9}{8}a^3 k^2 + \frac{191}{48}a^5 k^4 + \frac{581}{192}a^6 k^5)\cos kx + (\frac{1}{2}a^2 k + \frac{11}{6}a^4 k^3 + \frac{1}{48}a^5 k^4 + \frac{1327}{384}a^6 k^5)\cos 2kx$$

$$-(\frac{3}{8}a^3 k^2 + \frac{235}{96}a^5 k^4 + \frac{389}{384}a^6 k^5)\cos 3kx + (\frac{1}{3}a^4 k^3 + \frac{1}{192}a^5 k^4 + \frac{9459}{4800}a^6 k^5)\cos 4kx$$

$$- (\frac{31}{96}a^5 k^4 + \frac{1}{384}a^3 k^2)\cos 5kx + \frac{151}{640}a^6 k^5 \cos 6kx$$

(2.81)

The term $(\frac{1}{2}a^2 k + a^4 k^3 + \frac{1}{64}a^5 k^4 - \frac{5}{8}a^6 k^5)$ is ignored to agree with the observed wave form by suitable choice of origin.

$$\eta = -a\cos kx + (\frac{1}{2}a^2 k + \frac{11}{6}a^4 k^3)\cos 2kx - (\frac{3}{8}a^3 k^2 + \frac{235}{96}a^5 k^4)\cos 3kx$$

$$+ \frac{1}{3}a^4 k^3 \cos 4kx - \frac{31}{96}a^5 k^4 \cos 5kx + \frac{151}{640}a^6 k^5 \cos 6kx$$

(2.82)

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while the term $-\frac{9}{8}a^3 k^2 - \frac{191}{48}a^5 k^4 - \frac{581}{192}a^6 k^5 = 0$

because, a linear wave term cannot contain terms involving the product of wave amplitude.

Eqn (2.82) is the desired sixth order Stokes waves profile. It is a superposition of the first to fifth order in addition to extra term having wave amplitude six times that of the linear solution.

Eliminating the additive constant by an appropriate vertical translation of the coordinates, the profile of the Stokes wave contains components of different wavelengths propagating at the velocity c. It is no longer sinusoidal.

Following Kinsman (1965), the expressions for third and fourth order Phase speed are as follows:

$$c^2 = \frac{g}{k}(1 + \beta^2 k^2) \quad ; \text{ Third order phase speed} \qquad (2.83)$$

$$c^2 = \frac{g}{k}(1 + \beta^2 k^2 + \frac{1}{2}\beta^4 k^4) \quad ; \text{ Fourth order phase speed} \qquad (2.84)$$

The phase speed for fifth order Stokes waves as obtained by Oyetunde & Okeke (2004) is:

$$c^2 = \frac{g}{k}(1 + \beta^2 k^2 + \beta^4 k^4) \quad ; \text{ Fifth order phase speed}$$

(2.85)

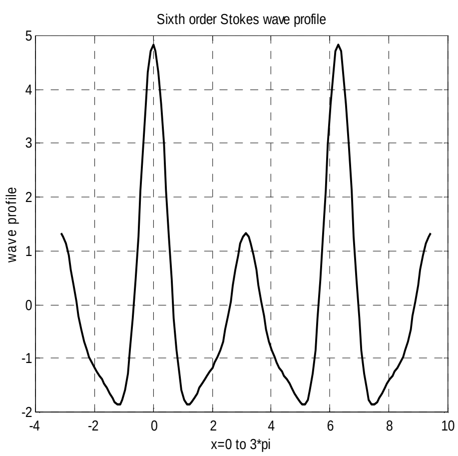

Fig. 1 : Sixth order Stokes waves profile.

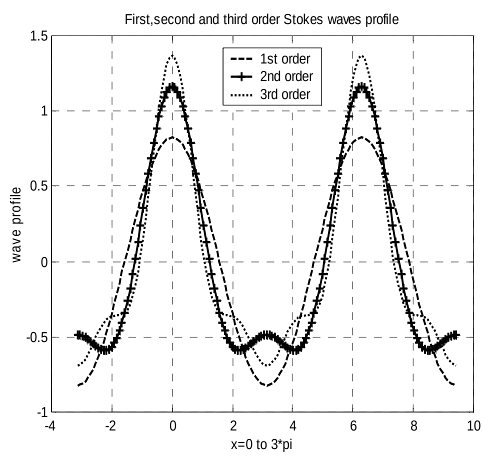

Fig. 2: First, second and third order Stokes waves profile

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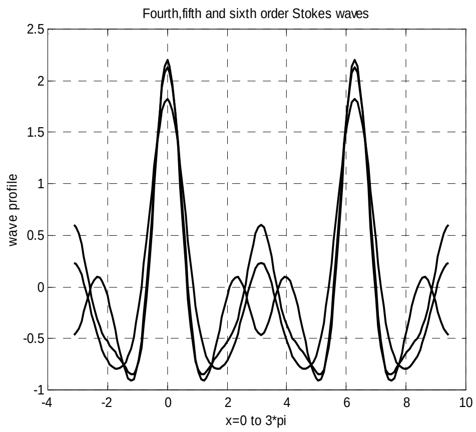

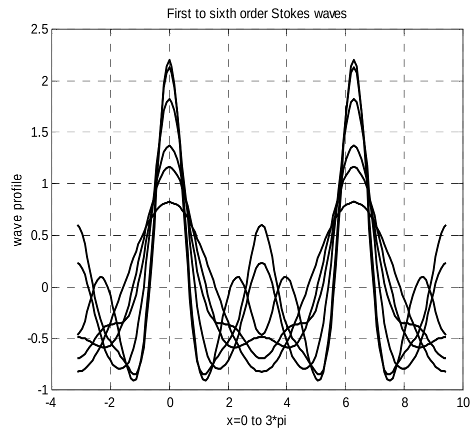

Fig. 3: Fourth, fifth and sixth order Stokes waves profile. Fig. 4: First to sixth order Stokes waves profile

From (2.58) $c^2 = \frac{g}{k}(1 + \beta^2 k^2 + \beta^4 k^4 + \beta^6 k^6)$ sixth order phase speed

Following Kinsman (1965), the phase speed in deep water can be written in terms of wave steepness δ for third to sixth orders as follows:

$$c = \left[\frac{g}{k}(1 + \pi^2\delta^2)\right]^{\frac{1}{2}} \quad \text{; Third order phase speed} \tag{2.86}$$

$$c = \left[\frac{g}{k}(1 + \pi^2\delta^2 + \frac{1}{2}\pi^4\delta^4)\right]^{\frac{1}{2}} \quad \text{; Fourth order phase speed} \tag{2.87}$$

$$c = \left[\frac{g}{k}(1 + \pi^2\delta^2 + \pi^4\delta^4\right]^{\frac{1}{2}} \quad \text{; Fifth order phase speed} \tag{2.88}$$

$$c = \left[\frac{g}{k}(1 + \pi^2\delta^2 + \pi^4\delta^4 + \pi^6\delta^6)\right]^{\frac{1}{2}} \quad \text{; Sixth order phase speed} \tag{2.89}$$

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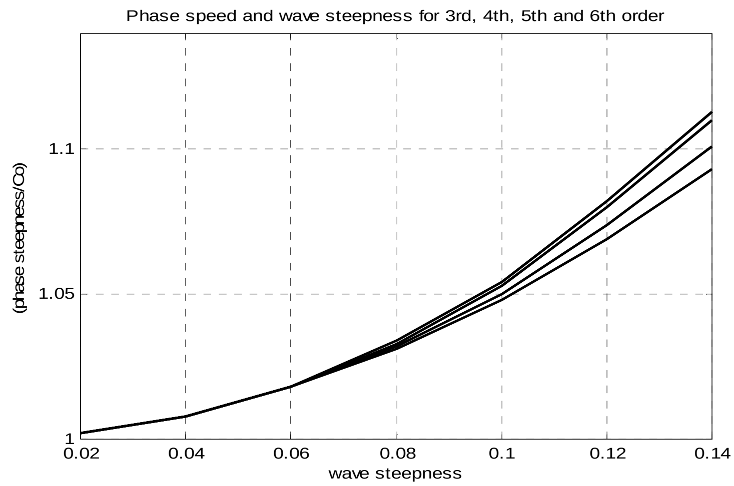

Fig. 5: phase speed and wave steepness for 3rd to 6th order

$Co = \sqrt{gh_o}$, $h_o =$ typical depth of shallow water

It can be observed that the phase speed increases as the wave steepness increases from third order to sixth order. However the maximum wave steepness is 0.61

Findings and conclusion

As the order increases, the wave profile increases and the phase speed equally increases. However, the higher the order, the closer the solution to the real wave profile in the physical form in oceanography. From the analytical derivation of the sixth order Stokes wave, there is no much difference between the theoretical wave form of the fifth order and sixth order wave profile as shown in the graph. Similarly, there might not be much difference between the fifth and the sixth order in the application to the observable physical wave form.

However, the approximate solutions at higher order Stokes might necessarily converge (breaking amplitude) as observed from the analytic derivation of sixth order when compared with lower orders. Numerical form of the solutions might be of better accuracy at higher orders.

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