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Normalizer Of Subloops

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Abstract

In [1] Albert proved that the centre Z of a loop (G,.) is isomorphic to the centre Z_M of the multiplication group M(G) of G. In this paper, we prove that the normalizer N(H) of subloop H in G is in one-to-one correspondence with the normalizer N(M(H)) of the subgroup M(H) in M(G).

Keywords: Normalizer, Loop, Subgroup, Multiplication group

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1.0 Introduction

Let (G,.) be a groupoid. If to every element x ε G is associated the mapping L(x) and R(x) called the left and right translation respectively both of which are bijective. We then call (G,.) together with the translations a quasigroup.

L(x): y → xy and R(x): y → yx are therefore permutations on G and can be considered as elements of the symmetric group S(G) of all permutation group S(G) of elements of G. The set of all L(x) and their inverse L$(x)^{-1}$ for x ε G generate a subgroup M_L(G) of symmetric group S(G) called left multiplication group of (G,.). Similarly the set of all R(x) and the inverses R$(x)^{-1}$ for x ε G generate a subgroup $M_R(G)$ of the symmetric group S(G) called the right multiplication group of (G,.). The group generated by all the left and right multiplications of (G,.) is called the multiplication group M(G). Multiplication group is also simportant from geometric point of view because the left and right multiplication in quasigroup can be considered as translations in geometries associated with quasigroup. This is extensively discussed in [3] by Chein .O.,Pflugfelder and Smith.J.D.

In [5] Pflugfelder defined an inner mapping of a loop as follows:

Definition 1. 1 Let (G,.) be a loop with identity e and M(G) a multiplication group of G. Let α ε M(G) such that α(e) = e. α is called an inner mapping of (G,.). Collection of all inner mappings do form a group called inner mapping group I of G .

In [5] Pflugfelder used a technique introduced by Bruck [3] to show that the inner mapping I of a loop (G,.) is generated by R(x,y), L(y,x) and T(x) where R(x,y) = R(x)R(y)R$(xy)^{-1}$

L(y,x) = L(y)L(x)L$(xy)^{-1}$

T(x) = R(x)L$(x)^{-1}$(

2.0 Normalizer

Definition 2..1 Let (G,.) be a loop and H a subloop, x ,y ε G belong to the normalizer N(H) of G if (i) xH= Hx

(ii) (Hx)y = (xy)H

(iii) x(yH)= (xy)H

i .e .N(H) = { x, y ε G| x H=Hx, (Hx)y = H(xy) , x(yH) = (xy)H }

The group M(H) generated by all the left and right multiplications of H is a subgroup of M(G). We wish to find the normalizer N(M(H)) in the group M(G) and show that it is in one-to one correspondence with the normalizer N(H) of H in G.

Recall that the inner mapping I of the loop (G,.) is generated by R(x,y), L(y,x) and T(x).

Theorem 2.2 I is the normalizer of M(H) in M(G) if and only if HI = H

Proof : Let I be the normalizer of M(H) in M(G), we then show HI = H.

Since I is generated by R(x,y), L(y,x) and T(x) , it is enough to show that HR(x,y) = H, HL(y,x) = H ,and HT(x) = H. Recall that R(x,y) = R(x)R(y)R$(xy)^{-1}$
L(y,x) = L(y)L(x)L$(xy)^{-1}$ and T(x) = R(x)L$(x)^{-1}$.
HR(x,y) = { h_iR(x,y)| h_i ε H}
h_iR(x,y) = h_iR(x)R(y)R$(xy)^{-1}$ = ((h_ix)y)R$(xy)^{-1}$) and since I is the normalizer, h_iR(x,y) = $(h_i x)yR(xy)^{-1} = h_j(xy)R(xy)^{-1}$ *for some* $h_j \varepsilon H$. *Thus*

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$h_iR(x,y) = h_j$R(xy)R$(xy)^{-1}$ = h_j. Hence HR(x,y) = H .
Similarly HL(y,x) = { h_iL(y,x)| h_i ε H}
h_iL(y,x) = h_iL(y)L(x)L$(xy)^{-1}$ = x(yh_i)L$(xy)^{-1}$ = xyh_j L$(xy)^{-1}$ for some h_j ε H
= h_jL(xy)L$(xy)^{-1}$ = h_j . Hence HL(y,x) = H .
HT(x) = {h_i T(x)| h_i ε H}
h_iT(x) = h_iR(x)L$(x)^{-1}$ = h_i x L$(x)^{-1}$ = xh_jL$(x)^{-1}$ for some h_j ε H
= h_j L(x) L$(x)^{-1}$ = h_j.
Hence h_iT(x) = h_j which implies HT(x) = H.
Combining the above results, namely : HR(x,y) = H , HL(y,x) = H and HT(x) = H we have HI =H.
Conversely suppose HI = H we need to show that I, which is generated by R(x,y), L(y,x) and T(x) is the normalizer of M(H) in M(G). Since HI = H, then HR(x,y) = H , HL(y,x) = H and HT(x) = H.
HR(x,y) = H implies HR(x)R(y)R$(xy)^{-1}$ = H then HR(x)R(y) = HR(xy). Thus
(Hx)y = H(xy).
Similarly HL(y,x) = H implies HL(y)L(x)L$(xy)^{-1}$ = H. Thus HL(y)L(x) = HL(xy) hence x(yH) = (xy)H.
Also HT(x) = H implies HR(x)L$(x)^{-1}$ = H then HR(x) = HL(x). Thus Hx = xH
Combining the three results, namely HR(x,y) = H, HL(y,x) = H and HT(x) =H and since R(x,y) , L(y,x) and T(x) generate I , it then follows that I is the normalizer of M(H) in M(G).

Corollary 2.2 : Let (G,.) be a loop and H a subloop of G. The normalizer N(H) of H in G is in one-to-one correspondence with the normalizer N(M(H)) 0f M(H) in M(G).

Proof: (G,.) is a loop and H is a subloop of (G,.). Let x,y ε G be a normalizer of H in G i. e x, y ε N(H) then by 2.1 (i) Hx = xH
(ii) (Hx)y = H(xy)
(iii) x(yH) = (xy)H
(i) implies HT(x) = H (ii) implies HR(x,y) = H and (iii) implies HL(y,x) = H.

Since I is generated by R(x,y) , L(y,x) and T(x), it then follows that I is the normalizer of M(H) in M(G) . i.e. N(M(H)) = I.

Conversely suppose I is the normalizer of M(H) in M(G). Since I is generated by T(x) , R(x,y) and L(y,x) then HT(x) = H , HR(x,y) = H and HL(y,x) = H .
Since HT(x) = H then HR(x)L$(x)^{-1}$ = H which implies HR(x) = HL(x)
i.e. Hx = xH . Also since HL(y,x) = H, then HL(y)L(x)L$(xy)^{-1}$ = H
HL(y)L(x) = HL(xy)
(yH)L(x) = HL(xy)
x(yH) = (xy)H.
Similarly HR(x,y) = H then HR(x)R(y)R$(xy)^{-1}$ = H
HR(x)R(y) = HR(xy)
(Hx)y = H(xy)
Thus combining the three results : Hx = xH , x(yH) = (xy)H and (Hx)y = H(xy) then this implies x,y ε H is a normalizer of H in G.

Conclusion This therefore shows that there is one-to-one correspondence between the normalizer N(H) of H in G and the normalizer N(M(H)) of M(H) in M(G). It further shows that the normalizer N(H) of a subioop H in G is expressed in terms of elements of the multiplication group M(G).These elements are the generators of the inner mapping group I of G.

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