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SUSY QM Factorization of General and Confluent Heun's Differential Operators

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Abstract

In this paper, we provide a supersymmetry method of factorization of general Heun's (GH) and confluent Heun's (CH) operators. Supercharges and superpartners defining the underliying superalgebra are explicictly obtained.

Keywords: Heun's equation, superpotentials, partnerpotentials and riccati equation.

1. Introduction

The General Heun's differential equation GHE is a natural extension of the Riemann hypergeometric differential equation which can be written as [3]

$$P_3(z)y''(z) + P_2(z)y'(z) + P_1(z)y(z) = 0, \tag{1.1}$$

where $P_i(z)$ are arbitrary polynomials of degree $i\left(i=3,2,1\right)$ in the complex variable z. Replacing the variable z by the variable x, the above equation reads as

$$x\left(x-1\right)\left(x-a\right)D^2 y(x) + \gamma\left(x-1\right)\left(x-a\right)+\delta x\left(x-a\right)+\in x\left(x-1\right)Dy(x)+\left[\alpha\beta x-q\right]y(x)=0, \tag{1.2}$$

where $D=\frac{d}{dx}$, $\{a,\ \beta, \gamma,\ \delta, \in, a,\ q\}$ $\left(a \neq 0, 1\right)$ are parameters, generally complex and arbitrary, linked by the Fuschian constraint $\alpha+\beta+1=\gamma+\delta+\in$. This equation has four regular singular points at $\{0,1,\ a,\infty\}$, with the exponents of these singularities being respectively, $\{0,\ 1,-\gamma\}$, $\{0,\ 1-\delta\}$, $\{0,\ 1-\in\}$ *and* $\{\alpha,\ \beta\}$. The equation (1.2) through some confluent processes, transforms into other multi-parameter equations, the so-called Confluent Heun's differential equation (CHE) [3]

$$D^2 y + \left(\alpha + \frac{\beta+1}{x} + \frac{\gamma+1}{x-1}\right)Dy + \frac{\left(2\delta+\alpha\left(\beta+\gamma+2\right)\right)\ x+2\eta+\beta+\left(\gamma-\alpha\right)\left(\beta+1\right)}{2x\left(x-1\right)}\ y=0; \tag{1.3}$$

In the recent work [1], the concept of factorization method, supersymmetry quantum mechanics (SUSY QM) and shape invariant techniques have been extended to Sturm-Liouvulle (SL) equations to solve Schrödinger equations. In the present work this concept shall be extended to the general and confluent Heun's differential equation.

2.0 Factorization of GH, CH, Operators

2.1 General Method of Factorization of SL Operators

In this section, we extend to GH and CH operators the general method of factorization of SL operators developed in the work [1] to construct new solvable potentials. For a matter of convenience we first briefly recall the results of [1].

2.2 Brief Review of General Method of Factorization of SL Operators

Following [1], the concept of factorization method was extended to SL equation, using SUSY QM formalism. Consider the one-dimensional second order differential equation.

$$H\Phi = \xi\Phi, \quad \Phi,\ \Phi' \in AC_{loc}\left(]a, b[\right), \tag{2.1}$$

where $$H = -\sigma(x)\frac{d^2}{dx^2} - \tau(x)\frac{d}{dx} + V(x) \cdot \tag{2.2}$$

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ξ is a constant, $\sigma(x), \tau(x)$ and $V(x)$ are real functions defined on the open interval $\left(]a, b[\right) \subseteq R$ and $AC_{loc}\left(a, b\right)$ is the set of local absolute continuous functions given by

$$AC_{loc}(a, b) = \left\{ f \in AC[\alpha_1, \beta_1],\ \forall [\alpha_1, \beta_1] \subset (a, b),\ [\alpha_1, \beta_1]\ compact \right\}, \tag{2.3}$$

$$AC[\alpha_1, \beta_1] = \left\{ f \in C[\alpha_1, \beta_1],\ f(x) = f(a_1) + \int_{\alpha_1}^{x} g(t)dt,\ g \in: L^1[\alpha_1, \beta_1] \right\}. \tag{2.4}$$

The suitable Hilbert space $\mathbb{H} = L^2\left([a, b], \rho(x)dx\right)$ with the inner product defined by means of a non-negative weight function $\rho(x)$ $on\left([a, b]\right)$:

$$\langle u, v \rangle = \int_a^b \overline{u}(x) v(x) \rho(x) dx, \quad u(x), v(x) \in \mathbb{H}, \tag{2.5}$$

where $\overline{u}$ is the complex conjugate of u.

The purpose of this section is to introduce a factorization model with an annihilator operator of the form

$$A = k(a)\left[\frac{d}{dx} + W(x)\right], \tag{2.6}$$

with domain: $$D(A) = \{u \epsilon \mathbb{H}, ku' + kWu \epsilon \mathbb{H}\ \} \tag{2.7}$$

where k and W are continuous functions on $\left(]a, b[\right)$. We infer that $D(A)$ is dense in $\mathbb{H}$ since $H^{1,2}\left(\left[\left(]a, b[\right)\right], \rho(x)dx\right)$ is dense in $\mathbb{H}$ and $H^{1,2}\left(\left(]a, b[\right), \rho(x)dx\right) \subset D(A)$ where $H^{m,n}(\Omega)$ is a Sobolev space of indices $\{m, n\}$. The operator A is closed in $\mathbb{H}$. The adjoint operator A^+ is given by [1]

$$D\left(A^+\right) = \{v \in \mathbb{H} | \exists \overline{v} \in \mathbb{H}\}: \langle Au, v \rangle = \langle Au, \overline{v} \rangle\ \forall u \in D(A),\ A^+ v = \overline{v}. \tag{2.8}$$

The explicit expression of A^+ is given through the following theorem

Theorem 2.1 [4] Suppose the following boundary condition

$$k(x)\rho(x)\, u(x) v(x)\Big|_a^b = 0,\ \forall u \in D(A)\ and\ \ v \in D(A^+), \tag{2.9}$$

is verified. Then the operator A^+ can be written as

$$A^+ = k(x)\left[-\frac{d}{dx} + W(x) + \mu(x)\right], \tag{2.10}$$

where $\mu(x)$ is a real continuous function defined by $\mu(x) = -\frac{d}{dx}\ln\left[k(x)\rho(x)\right]\cdot$

Let H_1 *and* H_2 be the product operators A^+A *and* AA^+, respectively,

$$H_1 = A^+A, \quad H_2 = AA^+,$$

with the corresponding domains

$$D(H_1) = \left\{u \in D(A^+), \upsilon = Au \in D(A^+) \text{ and } A^+\upsilon \in \mathbb{H}\right\},$$

$$D(H_2) = \left\{u \in D(A^+), \upsilon = A^+u \in D(A) \text{ and } A\upsilon \in \mathbb{H}\right\}\cdot \tag{2.12}$$

Remark that

$$H^{1,2}(]a, b[, \rho(x)dx) \subset D(A) \subset D(A^+),$$

$$D(H_1), D(H_2) \supset H^{2,2}(]a, b[, \rho(x)dx)\cdot$$

We infer that $D(H_1)$ *and* $D(H_2)$ are dense in $\mathbb{H}$. Furthermore, the following theorem gives the additional conditions to subject to the functions of k and the potential V so that the operator H factorizes in terms of A *and* A^+.

Theorem 2.2 [4] Suppose that

(i) k *and* μ are related to σ *and* τ as:

$$k^2 = \sigma; \quad k(k' - k\mu) = \tau; \tag{2.13}$$

(ii) the potential function V is related to the W by the Riccati type equation

$$V - \xi_0 = \sigma(W^2 - W) - \tau W\cdot \tag{2.14}$$

Then the operators $H_{1,2}$ are self-adjoint and

$$H_1 = A^+A = H - \xi_0 = -\sigma\frac{d^2}{dx^2} - \tau\frac{d}{dx} + \sigma(W^2 - W) - \tau W,$$

$$H_2 = AA^+ = -\sigma\frac{d^2}{dx^2} - \tau\frac{d}{dx} + \sigma(W^2 + W^1) + (\tau - \sigma')W + k(k\mu)'. \tag{2.15}$$

Let us remark that the condition $k(k' - k\mu) = \tau$ of (2.14) can be deduced from the Pearson equation defined in [4] and the constraint $k^2 = \sigma$. By means of the operators A *and* A^+, we can form a superalgebra as follows;

$$\{Q_i, Q_j\} = Q_iQ_j + Q_jQ_i = H_{ss}\delta_{ij}, \quad [H_{ss}, Q_i] = 0; \; i, j = 1, 2,$$

where $Q_1 = (Q^+ + Q^-)/\sqrt{2}$ *and* $Q_2 = (Q^+ - Q^-)/i\sqrt{2}$

with

$$Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}, \; Q^- = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix}, \; H_{ss} = \begin{pmatrix} A^+A & 0 \\ 0 & AA^+ \end{pmatrix}\cdot \tag{2.16}$$

We can rewrite the operators $H_{1,2}$ as

$$H_1 = A^+A = -\sigma\frac{d^2}{dx^2} - \tau\frac{d}{dx} + V_1, \text{ and } H_2 = AA^+ = -\sigma\frac{d^2}{dx^2} - \tau\frac{d}{dx} + V_2, \tag{2.17}$$

where

$$V_1=\sigma\left(W^2-W'\right)-\tau W,\quad V_2=\sigma\left(W^2+W'\right)-(\tau-\sigma')W+k\left(k\sigma\right)'$$
(2.18)

It clearly appears that SUSY QM is extended to SL operators. We design here the operators H_1, H_2 as SUSY partner. V_1, V_2 are SUSY partner potentials. We shall denote various forms of corresponding superpotentials and patnerpotentials of GHE, CHE, by GHEW_j, CHEW_j and GHEV_j respectively and $j=1,\cdots,k$. The integer k depends on the number of the corresponding solutions of the Heun's equation and its confluence form.

2.3 Factorization of General Heun's (GH) Differential Operator

The second order differential operators corresponding to GHE reads as

$$H^{GHE}=-x(x-1)(x-a)D^2-\big(\gamma(x-1)(x-a)+\delta x(x-a)$$
$$+\in(x-1)x\big)D-(\alpha\beta x-q),$$
(2.19)

having the following factorization characteristics

(1) $\sigma=x(x-1)(x-a),\ k^2=x(x-1)(x-a)$ which implies $k=\pm\sqrt{x(x-1)(x-a)}$.

(2) $\tau=\big(\gamma(x-1)(x-a)+\delta x(x-a)+\in x(x-1)\big),$

(3) $V=-(\alpha\beta x-q),$

(4) $\mu=\dfrac{1/2-\gamma}{x}+\dfrac{1/2-\delta}{x-1}+\dfrac{1/2-\in}{x-a}.$

The operator H factorizes into two first order differential operators $A=k(x)\big(D+W(x)\big)$ and $A^+=k(x)\big(-D+W(x)+\mu\big)$. The operator H, also could be expressed in terms of $H_{1,2}$ as

$$H_1=-\sigma(x)D^2-\tau(x)D+V_1(x),\ H_2=-\sigma(x)D^2-\tau(x)D+V_2(x),$$
(2.20)

$$V_1(x)=\sigma\left(W^2-W'\right)-\tau W,\quad V_2(x)=\sigma\left(W^2+W'\right)-\left(\tau-\sigma'\right)W+k\left(k\alpha\right)'\cdot$$
(2.21)

Letting $V_1=V-\xi_0$ where $\xi_0=0$ *and* $W=-z'(x)/z(x)$ into the Riccati equation of V_1, we obtain the original GHE equation given below

$$x(x-1)(x-a)D^2z+\big(\gamma(x-1)(x-a)+\delta x(x-a)+\in x(x-1)\big)Dz+(\alpha\beta x-q)z=0 \quad (2.22)$$

which admits 192 local solutions [2] corresponding to 192 superpotentials

$$GHEW_j=-\big(\ln z_j(x)\big),\quad j=1,2,\cdots,192,$$

where z_j are the solutions of (2.22). Similarly, the corresponding partner potentials are obtained by substituting the various superpotentials into

$$\begin{aligned}GHEV_{2j} &= x(x-1)(x-a)\big(GHEW_j^2+GHEW_j^1\big)+\big((a-a)(x-1)(\gamma-1)\\ &+x(x-1)(\in-1)+x(x-a)\big)GHEW_j\\ &+\frac{(1/2-\gamma)}{x^2}\big(-3/2(x-1)(x-a)+1/2\big(x(x-1)+x(x-a)\big)\big)\\ &+\frac{(1/2-\in)}{(x-1)^2}\big(1/2\big((x-1)(x-a)+x(x-a)\big)-3/2x(x-1)\big)\\ &+\frac{(1/2-\delta)}{(x-a)^2}\big(1/2\big((x-1)(x-a)+x(x-1)\big)-3/2x(x-a)\big);\\ &\quad j=1,\ 2,\cdots,192.\end{aligned} \quad (2.23)$$

The operator GH factorizes as

$$A = \varepsilon\sqrt{x(x-1)(x-a)}\ \left(D + GHEW_j(x)\right),$$

$$A^{+} = \varepsilon\sqrt{x(x-1)(x-a)}\ \left(-D + GHEW_j(x)\ + \frac{1/2-\gamma}{x} + \frac{1/2-\delta}{x-1} + \frac{1/2-\in}{x-a}\right); \tag{2.24}$$

$$j = 1,\ 2, \cdots, 192;\ \ \in = \pm 1. \tag{2.25}$$

2.4 Factorization of confluent Heun's (CH) Differential Operator

The second order differential operator corresponding to CHE reads as

$$H^{CHE} = \begin{array}{l} -x(x-1)D^2 - \left(\alpha x(x-1) + (\beta+1)(x-1) + (\gamma+1)x\right)D \\ \dfrac{-\left(2\delta + \alpha(\beta+\gamma+2)\right)x + 2\eta + \beta + (\gamma-\alpha)(\beta+1)}{2} \end{array}, \tag{2.26}$$

having the following factorization characteristics

(1) $\sigma = x(x-1),\ k^2 = x(x-1)$ which implies $k = \pm\sqrt{x(x-1)}$,

(2) $\tau = (\beta+1)(x-1) + \alpha x(x-1)\ +\ x(\gamma+1)$,

(3) $V = -\left[\dfrac{\left(2\delta + \alpha(\beta+\gamma+2)\right)x + 2\eta + \beta + (\gamma-\alpha)(\beta+1)}{2}\right]$,

(4) $\mu = \dfrac{1-\beta}{x} + \dfrac{2-\gamma}{x-1} - \alpha\cdot$

The operator H factorizes into two first order differential operators $A = k(x)(D + W(x))$ and $A^{+} = k(x)(-D + W(x) + \mu)$. The operator H, also could be expressed in terms of $H_{1,2}$ as

$$H_1 = -\sigma(x)D^2 - \tau(x)D + V_1(x),\quad H_2 = -\sigma(x)D^2 - \tau(x)D + V_2(x), \tag{2.27}$$

$$V_1(x) = \sigma(W^2 - W^1) - \tau W,\ \ V_2(x) = \sigma(W^2 + W^1) - (\tau - \sigma')W + k(k\alpha)'. \tag{2.28}$$

Letting $V_1 = V - \xi_0$ where $\xi_0 = 0$ *and* $W = -z'(x)/z(x)$ into the Riccati equation of V_1, we obtain the original CHE equation given below

$$D^2 z + \left(\alpha + \frac{\beta+1}{x} + \frac{\gamma+1}{x-1}\right)D\,z$$
$$+ \frac{\left(2\delta + \alpha(\beta+\gamma+2)\right)x + 2\eta + \beta + (\gamma-\alpha)(\beta+1)}{2x(x-1)}\,z = 0\cdot \tag{2.29}$$

The five parameters $\{\alpha,\ \beta, \gamma,\ \delta, \xi\}$ CHE equation (2.29) together with the three parameters $\{p, \sigma,\ \tau\}$ equation in [5] has the following relations $\alpha^2 = 4p^2,\ \beta^2 = 4\left(1 - 2\sigma + \tau^2\right)p,$
$\gamma^2 = 4\gamma,\ \delta = 2\left(1-\sigma\right)p^2 - \alpha$ *and* $\eta = (2\sigma - 2)p2 - \tau - \beta$. At these values of $\{\alpha,\ \beta, \gamma,\ \delta, \xi\}$, the CHE admits Liouvillian solution, for $\sigma^2 \neq \tau^2$,

$$z = \frac{1}{\sqrt{x-1}}\left(\left(W(\mu^{*},\ v,\ 2\lambda x) + \lambda(\tau-\sigma)\ W(\mu^{*}-1,\ v,\ 2\lambda x)\right)\right)C_2$$
$$+\ (\lambda(\tau+\sigma)M(\mu^{*}, v,\ 2\lambda x)\ +\ ((1-\sigma)\lambda) - v)\ M(\mu^{*}-1,\ v,\ 2\lambda x))\,C_1),$$
(2.30)

where $\mu^{*} = \lambda(1-\sigma) + 1/2$ and $v = \sqrt{\tau^2 - 2\sigma + 1}$, C_1 and C_2 are constants and M and W are Whittaker's functions. The corresponding superpotential read as

$$CHEW_j(x)\ = -(\ln z(x))'.$$
(2.31)

The associated partner potential read as

$$
\begin{aligned}
CHEV_2(x) &= x(x-1)(CHEW_j^2 + CHEW_j') \\
&\quad + \left((x-1)(\beta-1) + \alpha x(x-1) + (\gamma-2)\right)CHEW_j \\
&\quad + \left(1/2(2x-1) + \frac{(\beta-1)(x-1)}{x} + \frac{x(\gamma-2)}{x-1}\right).
\end{aligned}
$$

(2.32)

The operator CH factorizes as, for $\varepsilon = \pm 1$,

$$A = \varepsilon\sqrt{x(x-1)}\,(D + CHEW_j(x)),$$

(2.33)

$$A^+ = \varepsilon\sqrt{x(x-1)}\,(-D + CHEW_j(x) + \frac{1-\beta}{x} + \frac{2-\gamma}{x-1} - \alpha).$$

(2.34)

2.5 Underlying Superalgebra

By means of various forms of A and A^+, we check the superalgebra for commutativity. Let us denote the corresponding supercharges by Q_j^+ *and* Q_j^-, $j = 1, \cdots, k,$ where k depends on the number of solutions of the corresponding Heun's equation. By earlier definition the corresponding supercharges are

$$
\begin{aligned}
Q_j^+ &= 1/\sqrt{2}\begin{pmatrix} 0 & k(x)(-D + G(C)HEW_j(x) + \mu(x)) \\ k(x)(D + G(C)HEW_j(x)) & 0 \end{pmatrix} \\
Q_j^- &= 1/\sqrt{2i}\begin{pmatrix} 0 & k(x)(-D + G(C)HEW_j(x) + \mu(x)) \\ -k(x)(D + G(C)HEW_j(x)) & 0 \end{pmatrix}
\end{aligned}
\qquad (2.35)
$$

where $k(x)$ and μ are different for GHE and its confluent. Evaluating $\left\{Q_i^{\varepsilon}, Q_j^{\varepsilon'}\right\} =$ $Q_j^{\varepsilon}Q_j^{\varepsilon'} + Q_j^{\varepsilon'}Q_i^{\varepsilon}$, $\varepsilon = \pm = \varepsilon'$, and commutators $\left[H_{ssj}, Q_i^{\varepsilon}\right]$, $i = 1, 2$, we have

$$\left\{Q_i, Q_j\right\} = \delta_{ij}\begin{pmatrix} A_j^+ A_j & 0 \\ 0 & A_j A_j^+ \end{pmatrix} = \delta_{ij}H_{ss}\,,\ \left[H_{ssj}, Q_i^-\right] i = 1, 2 = H_{ssj}Q_i - Q_i H_{ssj} = 0 \qquad (2.36)$$

where $H_{ssj} = \begin{pmatrix} A_j^+ A_j & 0 \\ 0 & A_j A_j^+ \end{pmatrix}_{j=1,\cdots,k}$. Hence, Q_i^-, Q_i^+ define supercharges of the system.

3.0 Concluding Remarks

In this paper, we have extended the method of SUSY QM factorization of SL operators to those of Heun's differential operators. Solvable potentials were obtained. The factoring operators fulfill superalgebra. It is obivious in these cases that the potentials obtained so far are not shape invariant but of supersymmetry in nature.

References

[1] M. N. Hounkonnou, K. S. Sodoge, and E. S. Azatassou, Factorization of Differential Operator of Sturm Liouville Type, J. Phys A: Math. Gen. **38**, (2005), 371-390.

[2] S. M. Robert, the 192 Solution of the Heun's equation, Mathematics of Computation **76** (268), (2007), 811-843.

[3] A. Ronveaux, Heun's Differential equation (Oxford University press, Oxford, 1995).

[4] K. Sodoga, PhD thesis, IMSP, (2005), Benin.

[5] E. S. Cheb-Terrab., New closed form solutions in terms of ${}_pF_q$ for families of the general, Confluent and Bi-Confluent Heun differential equations, J. Phys. A: Math Gen. **37** (2004) 9923.