Journal of the Nigerian Association of Mathematical Physics
***Volume* 17 (November, 2010), pp 1 - 6**
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Martingales of the Generalized Conditional Expectations on the Generalized Positive Operators of a von Neumann algebra.

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Abstract

We extend the notion of generalized conditional expectation and Martingale onto the set of generalized positive operators (the extended positive part) of a von Neumann algebra.

1.0 Introduction:

The study of conditional expectation and martingale convergence was initiated by [19,20]. Lance [10] obtained the almost sure martingale convergence on von Neumann algebras, he extend the result of [4]. [15] obtained the strong martingale convergence . Using the result of [3] on the convergence of modular operators Δ and modular conjugation J on von Neumann algebras, [9] gave the conditions for a generalized conditional expectation to have strong martingale convergence on von Neumann algebras. [1] gave a condition that is independent of the filtration as was done in the case of Hiai and Tsukada. In this paper we attempt to extend these notions onto the extended positive part of a von Neumann algebra developed by [7].

Preliminaries: We recall here the notions and results on generalized positive operators as discussed in [7].

Definition:

A weight φ on a von Neumann algebras $\mathcal{M}$ is a function $\varphi: \mathcal{M}_+ \to [0, \infty]$ that satisfies,

(i) $\varphi(\lambda x) = \lambda\varphi(x)$ $\quad x \in \mathcal{M}_+,\ \lambda \geq 0$

(ii) $\varphi(x + y) = \varphi(x) + \varphi(y)$ $\quad x, y \in \mathcal{M}_+$

We say that φ is normal if $\varphi(sup x_i) = \sup \varphi(x)$ for any bounded increasing net of positive operators $(x_i)_{i\in I}$. φ is faithful if $\varphi(x^*x) = 0 \ \Rightarrow\ x = 0$,and semifinite if η_φ is $\sigma -$ strongly dense in $\mathcal{M}$.To any weight φ is associated a $\sigma -$ weakly continuous one parameter group of *-automorphism ($\sigma_t^\varphi)_{t\in R}$ on the von Neumann algebras $\mathcal{M}$, called the modular automorphism group.

Definition :

Let $\mathcal{M}$ a von Neumann algebra , and $\mathcal{M}_*^+$ its positive predual, a generalized positive operator affiliated with $\mathcal{M}$ is a map $\hat{x}$: $\mathcal{M}_*^+ \to [0, \infty]$ satisfying,

(1) $\hat{x}(\lambda\phi) = \lambda\hat{x}(\phi)$, $\quad \phi \in \mathcal{M}_*^+$ $,\lambda \geq 0$

(2) $\hat{x}(\phi + \psi) = \hat{x}(\phi) + \hat{x}(\psi)$ $\quad \phi,\ \psi\ \in \mathcal{M}_*^+$

(3) $\hat{x}$ is lower semicontinuous

The set of all such maps is called the extended positive part of $\mathcal{M}$, or the set of generalized positive operators denoted by $\widehat{\mathcal{M}}_+$. They are "weights" on the predual of a von Neumann algebra.

The generalized positive operator are added and multiplied by scalars in a natural way

Definition:

Let $\hat{x}, \hat{y}\ \in \widehat{\mathcal{M}}_+$, $a \in \mathcal{M}$ and $\lambda \geq 0$ we defined $\hat{x} + \hat{y}$, $\lambda\,\hat{x},$ and $a^*\hat{x}a$ by

(1) $(\lambda\hat{x})(\phi) = \lambda\hat{x}\,(\phi)$ $\quad \phi \in \mathcal{M}_*^+$

(2) $(\hat{x} + \hat{y}\,)(\phi) = \hat{x}\,(\phi) + \hat{y}(\phi)$ $\quad ,\phi \in \mathcal{M}_*^+$

(3) $(a^*\hat{x}a\,)(\phi) = \hat{x}\,(a\phi a^*\,)$ $\quad ,\ \phi \in \mathcal{M}_*^+$

Remark; $a\phi a^*(x) = \phi(a^*xa)$, $x \in \mathcal{M}$ (*)
hence we have $a\phi a^*(1) = \phi(a^*1a) = \phi(a^*a)$, $1 \in \mathcal{M}$ (**)

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Definition:

If $(\hat{x}_i)_{i\in I}$ is increasing net of elements in $\widehat{\mathcal{M}}_+$ then $\hat{x}(\phi) = sup_i\, \hat{x}_i(\phi)$, $\phi \in \mathcal{M}_*^+$ defines an element in $\widehat{\mathcal{M}}_+$. In particular if $(\hat{x}_j)_{j\in J}$ is a family of elements in $\widehat{\mathcal{M}}_+$,then $\hat{x}(\phi) = \sum_{j\in J} \hat{x}_j(\phi) \in \widehat{\mathcal{M}}_+$.

The relationship between operators in von Neumann algebra and its positive part is given in corollary 1.6 of Haagerup paper [7].

Corollary:

Any $\hat{x} \in \widehat{\mathcal{M}}_+$ is the pointwise limit of an increasing sequence of bounded operators in $\mathcal{M}_+$.

Definition:

If $\hat{x} \in \widehat{\mathcal{M}}_+$, $\phi \in \mathcal{M}_*^+$ and $(x_n)_{n\in\mathbb{N}}$ in $\mathcal{M}$, with $x_n \nearrow \hat{x}$ then $\hat{x}(\phi) = lim_n\, \phi(x_n)$.

Definition: Let $\hat{x} \in \widehat{\mathcal{M}}_+$, a weight on $\widehat{\mathcal{M}}_+$ is given by

$$\varphi_{\hat{x}}(\hat{a}) = \lim_n \varphi_{\hat{x}}(a_n) = \lim_{m,n} \tau(x_m.a_n) = \lim_{m,n} \tau(x_m^{1/2}\, a_n x_m^{1/2}) \quad , \quad \hat{a} \in \widehat{\mathcal{M}}_+$$

We have a theorem from Haagerup [7] which state the form of the spectral resolution for the generalized positive operator.

Theorem: Let $\mathcal{M}$ be a von Neumann algebra. Each $\hat{x} \in \widehat{\mathcal{M}}_+$ has a unique spectral resolution of the form

$$\hat{x}(\phi) = \int_0^\infty \lambda d\phi(e_\lambda) + \infty(p) \quad , \quad \phi \in \mathcal{M}_*^+$$

Where $(e_\lambda)_{\lambda\in[0,\infty[}$ is an increasing family of projections in $\mathcal{M}$ such that $\lambda \to e_\lambda$ is strongly continuous from right ,and $\lim_{\lambda\to\infty} e_\lambda = 1 - p$. moreover e_0 =0 iff $\hat{x}(\phi) > 0$ for any $\phi \in \mathcal{M}_*^+ \setminus \{0\}$ and $\varphi_{\hat{x}}$ is faithfulP=0 iff $\{\phi \in \mathcal{M}_*^+ : \hat{x}(\phi) < 0\}$ is dense in $\mathcal{M}_*^+$ and $\varphi_{\hat{x}}$ is semifinite.

Remark; if $\hat{x}(\phi) < 0$ for any $\phi \in \mathcal{M}_*^+$ then a there exist positive bounded operator $\mathrm{K} \in \mathcal{M}_+$, such that $\hat{x}(\phi) = \phi(\mathrm{K})$, $\phi \in \mathcal{M}_*^+$ and

$$\hat{x}(\omega_\xi) = \begin{cases} \left\|\mathrm{K}^{1/2}\xi\right\|^2 & \xi \in D(\mathrm{K}^{1/2}) \\ \infty \quad otherwise & \end{cases}$$

2.0 Generalized conditional expectation:

Let $\mathcal{M}$ be a semifinite von Neumann algebra and $\mathcal{N}$ its von Neumann subalgebra . Then there exist a conditional expectation E from $\mathcal{M}$ onto $\mathcal{N}$ which is a projection of norm one,having the property $i)\, E(axb) = aE(x)b$, $ii)\, E(x^*)E(x) \le E(x^*x)$, $iii)\, E(x^*) = E(x)^*$, $iv)\, E$ is order preserving (Tomiyama[16,17]) . The conditional expectation $\in (x) = \pi_{\mathcal{N}}^{-1}(E\pi_{\mathcal{M}}(x)\, E)$, in Takesaki [13] exist only when $\mathcal{N}$ is globally invariant under the modular automorphism group σ_t^φ associated with the faithful normal weight φ. Generalized conditional expectation of Accardi and Cecchini [2] defined by $\mathcal{E}(a) = \pi^{-1}\left(j_{\mathcal{U}_0}\, P\, j_{\mathcal{U}}\, \pi(a) j_{\mathcal{U}}\, j_{\mathcal{U}_0} P\right)$ always exist but it is not a projection of norm one neither does it enjoy the useful property $\mathcal{E}(axb) = a\mathcal{E}(x)b,\ a, b \in \mathcal{N}\ and\ x \in \mathcal{M}$. In Goldstein [5] he extends the conditional expectation E to the extended positive part of a von Neumann algebra. Here we extends the generalized conditional expectation $\mathcal{E}$ to the generalized positive operators. We denote our extended generalized conditional expectation by $\hat{\mathcal{E}}$.

We showed that $\hat{\mathcal{E}}$ is invariant with respect to a given normal weight on $\widehat{\mathcal{M}}_+$ .To show the possibility of extending the generalized conditional onto the generalized positive operators , we follow the argument of the proof in [5], which of course is the same even for a generalized conditional expectation $\mathcal{E}$.

Theorem 1: The $\mathcal{E}$ restricted to $\mathcal{M}_+$ extends uniquely to a map $\hat{\mathcal{E}}$ of $\widehat{\mathcal{M}}_+$ onto $\widehat{\mathcal{N}}_+$ which is positive, additive, order-preserving and normal and satisfies , $(\hat{\mathcal{E}}\hat{x})(\phi) = \hat{x}(\phi \circ \mathcal{E})$

Proof: Following Goldstein [5] we have;

$Let\ \hat{x} \in \widehat{\mathcal{M}}_+ , x_n \in \mathcal{M}_+\ \ and\ \ x_n \nearrow \hat{x}\ \ since\ \ \mathcal{E}\ \ is\ positive, \mathcal{E}x_n \nearrow \hat{y}\ \ \ for\ some\ \hat{y} \in \widehat{\mathcal{N}}_+$

$Put\ \ \hat{\mathcal{E}}\hat{x} = \hat{y}\ , if\ \ \ z_n \in \mathcal{M}_+ ,\ z_n \nearrow \hat{x}$, then for each $\phi \in \mathcal{M}_*$.

$\lim_n \phi(x_n) = \lim_n \phi(z_n)$ i.e $x_n - z_n \to 0$, $\sigma - weakly$

Where $\mathcal{E}x_n - \mathcal{E}z_n \to 0$, $\sigma \to weakly$, and thus $\lim_n \phi(x_n) - \lim_n \phi(\mathcal{E}z_n) \to 0$ $\sigma - weakly$

Implies $\mathcal{E}z_n \nearrow \hat{\mathcal{E}}\,\hat{x}$

We have $\left(\hat{\mathcal{E}}\,\hat{x}\right)(\phi) = \lim_n \phi(\mathcal{E}x_n) = \hat{x}(\phi \circ \mathcal{E})$

Hence, $\left(\hat{\mathcal{E}}\,\hat{x}\right)(\phi) = \hat{x}(\phi \circ \mathcal{E})$. It is obvious that $\hat{\mathcal{E}}$ is positive , additive and also normal.

To show that $\hat{\mathcal{E}}$ is invariant with respect to a faithful normal weight we have the following,

Theorem 2: Let $\mathcal{M}$ be a semifinite von Neumann algebra and $\mathcal{N}$ its von Neumann subalgebra and $\widehat{\mathcal{M}}_+$ and $\widehat{\mathcal{N}}_+$ be their respective extended positive part such that $\widehat{\mathcal{N}}_+ \subset \widehat{\mathcal{M}}_+$, then

$\varphi_{\hat{x}} = \varphi_{\hat{x}} \circ \hat{\mathcal{E}}$.

Proof : Let $\hat{x} \in \widehat{\mathcal{M}}_+$ we define a weight on $\widehat{\mathcal{M}}_+$ by

$$\varphi_{\hat{x}}(\hat{a}) = \lim_n \varphi_{\hat{x}}(a_n) = \lim_{m,n} \tau(x_m . a_n) = \lim_{m,n} \tau(x_m^{1/2} a_n x_m^{1/2}) \quad , \quad \hat{a} \in \widehat{\mathcal{M}}_+$$

Let $\hat{\mathcal{E}}: \widehat{\mathcal{M}}_+ \to \widehat{\mathcal{N}}_+$ be our extended generalized conditional expectation on $\widehat{\mathcal{M}}_+$ onto $\widehat{\mathcal{N}}_+$ if $\hat{a} \in \widehat{\mathcal{M}}_+$, $\hat{x} \in \widehat{\mathcal{N}}_+$ then, $x_m \nearrow \hat{x}$ and $a_n \nearrow \hat{a}$ with $x_m \in \mathcal{N}_+$ and $a_n \in \mathcal{M}_+$

$$\varphi_{\hat{x}}(\hat{a}) = \hat{\tau}(\hat{x}.\hat{a}) = \lim_{m,n} \tau(x_m . \mathcal{E}(a_n)) = \lim_{m,n} \tau(x_m^{1/2} \mathcal{E}(a_n) x_m^{1/2})$$
$$= \lim_m \tau(x_m^{1/2} \lim_n \phi(\mathcal{E}a_n) x_m^{1/2}) = \lim_m \tau(x_m^{1/2} \hat{\mathcal{E}}(\hat{a}) x_m^{1/2})$$
$$= \lim_m \tau(\hat{\mathcal{E}}(\hat{a}) \left(x_m^{1/2} \phi x_m^{1/2}\right)) = \lim_m \tau\left(\hat{\mathcal{E}}(\hat{a})\left(\phi\left(x_m^{1/2}(1)x_m^{1/2}\right)\right)\right)$$
$$= \lim_m \tau\left(\hat{\mathcal{E}}(\hat{a})\phi(x_m)\right) = \tau\left(\hat{\mathcal{E}}(\hat{a}) \lim_m \phi(x_m)\right) = \hat{\tau}\left(\hat{\mathcal{E}}(\hat{a})\hat{x}\right)$$
$$\varphi_{\hat{x}}(\hat{a}) = \hat{\tau}\left(\hat{\mathcal{E}}(\hat{a})\hat{x}\right) = \varphi_{\hat{x}}\left(\hat{\mathcal{E}}\,\hat{a}\right)$$

Hence $\varphi_{\hat{x}} = \varphi_{\hat{x}} \circ \hat{\mathcal{E}}$

Here we have used the relation given in remark (**) and the assumption that the increasing sequences are self-adjoint (i.e densely defined on $\mathcal{N}_+$ and $\mathcal{M}_+$).

Theorem 3: If $\varphi_{\hat{x}}$ is a weight on $\widehat{\mathcal{M}}_+$ and $\hat{\mathcal{E}}$, $\mathcal{E}$ are the generalized conditional expectations on $\widehat{\mathcal{M}}_+$ and $\mathcal{M}_+$ respectively then $\varphi_{\hat{x}}\left(\hat{\mathcal{E}}\,\hat{a}\right) = \varphi_{\hat{x}}(\hat{a}(\phi \circ \mathcal{E}))$

Proof: $\varphi_{\hat{x}}\left(\hat{\mathcal{E}}\,\hat{a}\right) = \varphi_{\hat{x}}(\lim_n \phi(\mathcal{E}a_n))$, where $a_n \in \mathcal{M}_+, a_n \nearrow \hat{a}$

$$\varphi_{\hat{x}}\left(\hat{\mathcal{E}}\,\hat{a}\right) = \tau\left(x_m^{1/2} \lim_n \phi(\mathcal{E}a_n) x_m^{1/2}\right) = \lim_n \tau\left(x_m^{1/2} \phi(\mathcal{E}a_n) x_m^{1/2}\right)$$
$$\varphi_{\hat{x}}\left(\hat{\mathcal{E}}\,\hat{a}\right) = \lim_n \tau\left(x_m^{1/2} (\phi \circ \mathcal{E}) a_n) x_m^{1/2}\right) = \tau\left(x_m^{1/2} \lim_n (\phi \circ \mathcal{E}) a_n x_m^{1/2}\right)$$
$$\varphi_{\hat{x}}\left(\hat{\mathcal{E}}\,\hat{a}\right) = \tau\left(x_m^{1/2}\, \hat{a}(\phi \circ \mathcal{E}) x_m^{1/2}\right) = \hat{\tau}\left(\hat{a}(x_m^{1/2} (\phi \circ \mathcal{E}) x_m^{1/2})\right)$$
$$\varphi_{\hat{x}}\left(\hat{\mathcal{E}}\,\hat{a}\right) = \hat{\tau}(\hat{a}((\phi \circ \mathcal{E}) x_m) = \varphi_{\hat{x}}(\hat{a}(\phi \circ \mathcal{E}))$$
$$\varphi_{\hat{x}}\left(\hat{\mathcal{E}}\,\hat{a}\right) = \varphi_{\hat{x}}(\hat{a}(\phi \circ \mathcal{E}))$$

3.0 Martingales:

Let $\widehat{\mathcal{M}}_+$ be the extended positive part of a semifinite von Neumann algebra $\mathcal{M}$, with a normal weight $\varphi_{\hat{x}}(\cdot) = \lim_m \tau(x_m \cdot)$ .We have the following notations,

$n_{\hat{\varphi}} = \left\{\hat{x} \in \widehat{\mathcal{M}}_+ : \hat{\varphi}_{\hat{x}}(\cdot) = \tau(x_m \cdot) < \infty\right\}$, $(\hat{m}_{\hat{\varphi}})_+ = (\hat{n}_{\hat{\varphi}})_+^* (\hat{n}_{\hat{\varphi}})_+$ .Let $(\hat{\mathcal{H}}_{\hat{\varphi}}^+, \hat{\pi}_{\hat{\varphi}})$ be the G.N.S representation of $\widehat{\mathcal{M}}_+$ induced by $\varphi_{\hat{x}}$, and $\eta_{\hat{\varphi}}: (\hat{n}_{\hat{\varphi}})_+ \to \hat{\mathcal{H}}_{\hat{\varphi}}^+$ is the canonical injection map. Then $\left(\hat{\mathcal{U}}_{\hat{\varphi}}\right)_+ = \left\{\hat{x}_{\hat{\varphi}} : \hat{x}_{\hat{\varphi}} \in (\hat{n}_{\hat{\varphi}})_+^* \cap (\hat{n}_{\hat{\varphi}})_+\right\}$ is the extended positive part achieved left Hilbert algebra and $\hat{\pi}_{\hat{\varphi}}(\widehat{\mathcal{M}}_+)$ is its extended left positive part von Neumann algebra. We have the following, modular operator $\hat{\Delta}_{\hat{\varphi}}$, modular conjugation $\hat{J}_{\hat{\varphi}}$ and the modular automorphism $\hat{\sigma}_t^{(\hat{\varphi})}$ are associated with $\varphi_{\hat{x}}$ .We fix an increasing net $(\mathcal{N}_\alpha)_+^\wedge$ of closed subsets of the extended positive part of a von Neumann algebra $\widehat{\mathcal{M}}_+$. For each $\alpha \in [0,\infty]$ let $\hat{\psi}_\alpha$ be a normal weight on $(\mathcal{N}_\alpha)_+^\wedge$ such that the restriction of the weight $\varphi_{\hat{x}}$ to $(\mathcal{N}_\alpha)_+^\wedge$ is given by $\hat{\psi}_\alpha = \hat{\varphi}_{/(\mathcal{N}_\alpha)_+^\wedge}$ and $(\mathcal{N}_\alpha)_+^\wedge$ satisfy the following conditions

1. $(\mathcal{N}_\alpha)_+^\wedge \subseteq (\mathcal{N}_\beta)_+^\wedge \quad \alpha \leq \beta$
2. $\widehat{\mathcal{M}}_+ = \left(\bigcup_{\alpha\in[0,\infty]} (\mathcal{N}_\alpha)_+^\wedge \right)''$
3. $(\mathcal{N}_\alpha)_+^\wedge = \bigcap_{t>\alpha} (\mathcal{N}_t)_+^\wedge$

For each $\alpha \in [0,\infty]$ and $\hat{\psi}_\alpha$ on $(\mathcal{N}_\alpha)_+^\wedge$ we take $(\hat{n}_\alpha)_+ = n_{\hat{\varphi}} \cap (\mathcal{N}_\alpha)_+^\wedge$ where $n_{\hat{\varphi}} = \{\hat{x} \in \widehat{\mathcal{M}}_+ : \hat{\varphi}_{\hat{x}}(\cdot) = \tau(x_m \cdot) < \infty\}$ and $(\hat{m}_\alpha)_+ = (\hat{n}_\alpha)_+^*(\hat{n}_\alpha)_+$.

The G.N.S representation of $(\mathcal{N}_\alpha)_+^\wedge$ is given by $(\widehat{\mathcal{H}}_\alpha^+, \hat{\pi}_\alpha)$ and left Hilbert algebra is given by $(\hat{\mathcal{U}}_\alpha)_+ = \{\hat{x}_\alpha : \hat{x}_\alpha \in (\hat{n}_\alpha)_+^* \cap (\hat{n}_\alpha)_+\}$, the modular operator $\hat{\Delta}_\alpha$, modular conjugation $\hat{J}_\alpha$ and the modular automorphism $\hat{\sigma}_t^{(\alpha)}$ are associated with $\hat{\psi}_\alpha$. Then $\widehat{\mathcal{H}}_\alpha^+$ is an increasing net of family of subspaces of $\widehat{\mathcal{H}}^+$. Let $\hat{P}_\alpha$ be the orthogonal projection of $\widehat{\mathcal{H}}^+$ onto $\widehat{\mathcal{H}}_\alpha^+$ with $\widehat{\mathcal{H}}^+ = \cup \widehat{\mathcal{H}}_\alpha^+$, then $\hat{P}_\alpha \in \hat{\pi}_\alpha(\mathcal{N}_\alpha)_+^\wedge)'$ .Then the generalized conditional expectation $\hat{\mathcal{E}}_\alpha : \widehat{\mathcal{M}}_+ \to (\mathcal{N}_\alpha)_+^\wedge$ is given by $\hat{\mathcal{E}}_\alpha(\hat{x})(\phi) = \hat{\pi}_\alpha^{-1}\big(\hat{J}_\alpha \hat{P}_\alpha \hat{J}_{\varphi_{\hat{x}}} \hat{\pi}_{\varphi_{\hat{x}}}(\hat{x})\ \hat{J}_{\varphi_{\hat{x}}}\ \hat{J}_\alpha\big)$

Where $\hat{P}_\alpha = \lim_n \phi(P_{n\alpha})$, $\hat{J}_\alpha = \lim_n \phi(J_{n\alpha})$, $\hat{\pi}_\alpha^{-1} = \lim_n \phi(\pi_{n\alpha}^{-1})$.

Hence our generalized conditional expectation $\widehat{\mathcal{M}}_+$ *onto* $(\mathcal{N}_\alpha)_+^\wedge$ is given by

$\hat{\mathcal{E}}_\alpha(\hat{x})(\phi) = \lim_n \phi\left(\pi_{n\alpha}^{-1}\big(J_{n\alpha} P_{n\alpha}\, J_{\varphi_{\hat{x}}}\, \pi_{\varphi_{\hat{x}}}(x_n)\, J_{\varphi_{\hat{x}}}\ J_{n\alpha}\big)\right)$.

The sequence $(\hat{\mathcal{E}}_\alpha)$ is called a martingale if, whenever $\alpha \leq \beta$, $\quad \hat{\mathcal{E}}_\beta \hat{\mathcal{E}}_\alpha = \hat{\mathcal{E}}_\alpha \hat{\mathcal{E}}_\beta = \hat{\mathcal{E}}_\alpha$.

If we let $\alpha \to \infty$,then we have $\hat{\psi}_\alpha \to \hat{\psi}_\infty$, hence we have the following map

$\hat{\mathcal{E}}_\infty : \widehat{\mathcal{M}}_+ \to (\mathcal{N}_\infty)_+^\wedge$. Before we state the Hiai-Tsukada type strong martingale convergence conditions for an increasing filtrations. We will give a detailed proof of lemma (1) in [9] .

Lemma 1: If $\hat{x} \in (\hat{n}_{\hat{\varphi}})_+$, then $\hat{\mathcal{E}}(\hat{x}) \in \hat{n}_{\hat{\psi}}$ and $\eta_{\hat{\varphi}}\left(\hat{\mathcal{E}}(\hat{x})\right) = \hat{J}_{\hat{\psi}} P \hat{J}_{\hat{\varphi}}\, \eta_{\hat{\varphi}}(\hat{x})$.

Proof: Let $\hat{x} \in (\hat{n}_{\hat{\varphi}})_+$,then $\hat{\mathcal{E}}(\hat{x}) \in \hat{n}_{\hat{\psi}}$, choose a net $\{b_j\}$ in $m_{\hat{\varphi}} \cap (\mathcal{N})_+^\wedge$ with $b_j \nearrow 1$

We have $\eta_{\hat{\varphi}}\left(\hat{\mathcal{E}}(\hat{x})\right) = lim_j\ \hat{J}_{\hat{\psi}} \hat{\pi}_{\hat{\psi}}(b_j) \hat{J}_{\hat{\psi}}\ \eta_{\hat{\varphi}}\left(\hat{\mathcal{E}}(\hat{x})\right) = lim_j\ \left(\hat{\pi}_{\hat{\psi}}(b_j)\right)'\eta_{\hat{\varphi}}\left(\hat{\mathcal{E}}(\hat{x})\right)$

$= lim_j\, \hat{\pi}_{\hat{\psi}}\left(\hat{\mathcal{E}}(\hat{x})\right)\eta_{\hat{\varphi}}(b_j) = lim_j\, \hat{\pi}_{\hat{\psi}}\left(\hat{\mathcal{E}}(\hat{x})\right)\hat{\pi}_{\hat{\varphi}}(b_j)\Omega = lim_j\, \hat{\pi}_{\hat{\psi}}\left(\hat{\mathcal{E}}(\hat{x})\right)\hat{\pi}_{\hat{\varphi}}(b_j)\eta_{\hat{\varphi}}(1)$

$= lim_j\, \hat{\pi}_{\hat{\psi}}\left(\hat{\mathcal{E}}(\hat{x})\right)\hat{J}_{\hat{\varphi}}\left(\hat{\pi}_{\hat{\varphi}}(b_j)\right)'\hat{J}_{\hat{\varphi}}\eta_{\hat{\varphi}}(1) = lim_j\, \hat{\pi}_{\hat{\psi}}\left(\hat{\mathcal{E}}(\hat{x})\right)\hat{J}_{\hat{\varphi}}\left(\hat{\pi}_{\hat{\psi}}(b_j)\right)'\hat{J}_{\hat{\varphi}}\,\hat{J}_{\hat{\varphi}}\,\Omega$

$= lim_j\, \hat{\pi}_{\hat{\psi}}\left(\hat{\mathcal{E}}(\hat{x})\right)\hat{J}_{\hat{\varphi}}\left(\hat{\pi}_{\hat{\varphi}}(b_j)\right)'\ \Omega\ = lim_j\, \hat{\pi}_{\hat{\psi}}\left(\hat{\mathcal{E}}(\hat{x})\right)\ \hat{\pi}_{\hat{\varphi}}(b_j)\,\hat{J}_{\hat{\varphi}}\ \Omega\ = lim_j\, \hat{\pi}_{\hat{\psi}}\left(\hat{\mathcal{E}}(\hat{x})\right)\ \hat{\pi}_{\hat{\varphi}}(b_j)\Omega$

$= lim_j\, \hat{\pi}_{\hat{\psi}}\left(\hat{\mathcal{E}}(\hat{x})\right)\hat{\pi}_{\hat{\varphi}}(b_j)\eta_{\hat{\varphi}}(1)\ = lim_j\, \hat{\pi}_{\hat{\psi}}\left(\hat{\mathcal{E}}(\hat{x})\right)\eta_{\hat{\varphi}}(b_j)$

$= lim_j\ \left(\hat{J}_{\hat{\psi}}\hat{P}\hat{J}_{\varphi}\hat{\pi}_{\varphi}(\hat{x})\,\hat{J}_{\varphi}\ \hat{J}_{\hat{\psi}}\hat{J}_{\hat{\psi}}\,\eta_{\hat{\varphi}}(b_j)\right) = lim_j\ \left(\hat{J}_{\hat{\psi}}\hat{P}\hat{J}_{\hat{\varphi}}\hat{\pi}_{\hat{\varphi}}(\hat{x})\,\hat{J}_{\hat{\varphi}}\,\eta_{\hat{\varphi}}(b_j)\right) = lim_j\ \left(\hat{J}_{\hat{\psi}}\hat{P}\left(\hat{\pi}_{\hat{\varphi}}(x)\right)'\ \eta_{\hat{\varphi}}(b_j)\right)$

$= lim_j\ \left(\hat{J}_{\hat{\psi}}\hat{P}\hat{\pi}_{\hat{\varphi}}(b_j)\,\eta_{\hat{\varphi}}(x)\right) = \ lim_j\ \left(\hat{J}_{\hat{\psi}}\hat{P}\ \hat{J}_{\hat{\varphi}}\left(\hat{\pi}_{\hat{\varphi}}(b_j)\right)'\hat{J}_{\hat{\varphi}}\ \eta_{\hat{\varphi}}(x)\right)$

$= \ lim_j\ \left(\hat{J}_{\hat{\psi}}\hat{P}\ \hat{J}_{\hat{\varphi}}\left(\hat{\pi}_{\hat{\varphi}}(b_j)\right)'\hat{J}_{\hat{\varphi}}\ \hat{\pi}_{\hat{\varphi}}(x)\eta_{\hat{\varphi}}(1)\right) \quad = \ lim_j\ \left(\hat{J}_{\hat{\psi}}\hat{P}\ \hat{J}_{\hat{\varphi}}\left(\hat{\pi}_{\hat{\varphi}}(b_j)\right)'\hat{J}_{\hat{\varphi}}\ \hat{\pi}_{\hat{\varphi}}(x)\hat{J}_{\hat{\varphi}}\ \Omega\right)$

$= \ lim_j\ \left(\hat{J}_{\hat{\psi}}\hat{P}\ \hat{J}_{\hat{\varphi}}\left(\hat{\pi}_{\hat{\varphi}}(b_j)\right)'\left(\hat{\pi}_{\hat{\varphi}}(x)\right)'\Omega\right)\ = lim_j\, \hat{J}_{\hat{\psi}}\hat{P}\ \hat{J}_{\hat{\varphi}}\left(\hat{\pi}_{\hat{\varphi}}(b_j x)\right)'\ \Omega$

$= lim_j\, \hat{J}_{\hat{\psi}}\hat{P}\ \hat{J}_{\hat{\varphi}}\left(\hat{\pi}_{\hat{\varphi}}(b_j x)\right)'\eta_{\hat{\varphi}}(1) = \ lim_j\, \hat{J}_{\hat{\psi}}\hat{P}\ \hat{J}_{\hat{\varphi}}\ \hat{\pi}_{\hat{\varphi}}(1)\,\eta_{\hat{\varphi}}$

$\eta_{\hat{\varphi}}\left(\hat{\mathcal{E}}(\hat{x})\right)\ = lim_j\, \hat{J}_{\hat{\psi}}\hat{P}\ \hat{J}_{\hat{\varphi}}\eta_{\hat{\varphi}}(b_j x) = \ \hat{J}_{\hat{\psi}}\hat{P}\hat{J}_{\hat{\varphi}}\ \eta_{\hat{\varphi}}(x)$

We now state, the Hiai- Tsukada type strong martingale convergence for the generalized conditional expectations on the generalized positive operators (the increasing case) in our setting.

Theorem 4: The following conditions are equivalent for the increasing case

(i) $\cup_\alpha \left(\hat{\mathcal{U}}_\alpha\right)_+ (\subset \hat{\mathcal{U}}_\infty)$ is a core of $\Delta_\infty^{1/2}$

(ii) $s - \lim_\alpha \hat{\mathcal{E}}_\alpha(\hat{x}) = \hat{\mathcal{E}}_\infty(\hat{x})$ for every $\hat{x} \in \widehat{\mathcal{M}}_+$

(iii) $\left\| \eta_{\hat{\varphi}}\left(\hat{\mathcal{E}}_\alpha(\hat{x})\right) - \eta_{\hat{\varphi}}\left(\hat{\mathcal{E}}_\infty(\hat{x})\right)\right\| \to 0$ for every $\hat{x} \in n_{\hat{\varphi}}$

Proof: (i)$\Rightarrow (ii)$

Let $\hat{x} \in \widehat{\mathcal{M}}_+$ and $\hat{\xi} \in \widehat{\mathcal{H}}^+$ with $\xi_n \nearrow \hat{\xi} \in \widehat{\mathcal{H}}^+$ and $x_n \in \mathcal{M}_+$

$$\left\|\hat{\pi}_\infty\left(\hat{\mathcal{E}}_\alpha(\hat{x})\right)\hat{\xi} - \hat{\pi}_\infty\left(\hat{\mathcal{E}}_\infty(\hat{x})\right)\hat{\xi}\right\| = lim_n\phi \left\|\pi_\infty\left(\mathcal{E}_{n,\alpha}(x_n)\right)\xi_n - \pi_\infty\left(\mathcal{E}_{n,\infty}(x_n)\right)\xi_n\right\|$$

$$= lim_n\emptyset \left\|\pi_\infty(\varepsilon_{n,\alpha})P_\alpha\xi_n - \pi_\infty\left(\mathcal{E}_{n,\alpha}(x_n)\right)\xi_n - \pi_\infty\left(\mathcal{E}_{n,\alpha}(x_n)\right)P_\alpha\xi_n - \pi_\infty\left(\mathcal{E}_{n,\alpha}(x_n)\right)\xi_n\right\|$$

$$\leq lim_n\emptyset \left\|\pi_\infty(\varepsilon_{n,\alpha})P_\alpha\xi_n - \pi_\infty\left(\mathcal{E}_{n,\alpha}(x_n)\right)\xi_n\right\| + lim_n\emptyset \left\|\pi_\infty(\varepsilon_{n,\alpha})P_\alpha\xi_n - \pi_\infty\left(\mathcal{E}_{n,\alpha}(x_n)\right)\xi_n\right\|$$

$$+ lim_n\emptyset \left\|\pi_\infty(\varepsilon_{n,\alpha})P_\alpha\xi_n - \pi_\infty\left(\mathcal{E}_{n,\alpha}(x_n)\right)\xi_n\right\|$$

$$\leq lim_n\emptyset\left\|\pi_\infty(\varepsilon_{n,\alpha})(P_\alpha\xi_n - \xi_n)\right\| + lim_n\emptyset \left\|\pi_\infty(\varepsilon_{n,\alpha})P_\alpha\xi_n - \pi_\infty\left(\mathcal{E}_{n,\alpha}(x_n)\right)\xi_n\right\|$$

Now since π_∞ is isometric and $\mathcal{E}_{n,\alpha} \to \mathcal{E}_{n,\infty}$ as $\alpha \to \infty$ from [9] then ,we have $x_n \in \mathcal{M}_+$ $\mathcal{E}_{n,\infty}(x_n) = x_n$ and also if $P_\alpha \to P_\infty$ as $\alpha \to \infty$,we have $P_\infty\ (\xi_n) = \xi_n$,

hence

$$\leq \phi\ \|(x_n)\|\|(P_\alpha\xi_n - \xi_n)\|$$
$$+ lim_n\emptyset\left\| J_{n,\alpha}P_{n,\alpha}J_{n,\varphi}\pi_{n,\varphi}(x_n)J_{n,\varphi}J_{n,\alpha}P_{n,\alpha}\xi_n - J_{n,\infty}P_{n,\infty}J_{n,\varphi}\pi_{n,\varphi}(x_n)J_{n,\varphi}J_{n,\infty}P_{n,\infty}\xi_n\right\|$$

Since s-$lim_\alpha\left(lim_n\emptyset(J_{n,\alpha}P_{n,\alpha})\right) = \left(lim_n\emptyset(J_{n,\infty}P_{n,\infty})\right)$

and s-$lim_\alpha\left(lim_n\emptyset(P_{n,\alpha})\right) = \left(lim_n\emptyset(P_{n,\infty})\right)$

we have,

$$\leq lim_n\emptyset\ \|(x_n)\|\|(\xi_n - \xi_n)\|$$
$$+ lim_n\emptyset\left\| J_{n,\infty}P_{n,\infty}J_{n,\varphi}\pi_{n,\varphi}(x_n)J_{n,\varphi}J_{n,\infty}\xi_n - J_{n,\infty}P_{n,\infty}J_{n,\varphi}\pi_{n,\varphi}(x_n)J_{n,\varphi}J_{n,\infty}P_{n,\infty}\xi_n\right\|$$
$$\leq lim_n\emptyset\ \|(x_n)\|\|(\xi_n - \xi_n)\|$$
$$+ lim_n\emptyset\left\| J_{n,\infty}P_{n,\infty}J_{n,\varphi}\pi_{n,\varphi}(x_n)J_{n,\varphi}J_{n,\infty}\xi_n - J_{n,\infty}P_{n,\infty}J_{n,\varphi}\pi_{n,\varphi}(x_n)J_{n,\varphi}J_{n,\infty}P_{n,\infty}\xi_n\right\| \to 0$$

Therefore, $\left\|\hat{\pi}_\infty\left(\hat{\mathcal{E}}_\alpha(\hat{x})\right)\hat{\xi} - \hat{\pi}_\infty\left(\hat{\mathcal{E}}_\infty(\hat{x})\right)\hat{\xi}\right\| \to 0$, hence $s - lim_\alpha\hat{\varepsilon}_\alpha(\hat{x}) = \hat{\varepsilon}_\infty(\hat{x})$, for every $\hat{x} \in \widehat{\mathcal{M}}_+$

For $(iii) \Rightarrow (i)$

Suppose that , s-$lim_\alpha\left(lim_n\emptyset(J_{n,\alpha}P_{n,\alpha})\right) = \left(lim_n\emptyset(J_{n,\infty}P_{n,\infty})\right)$,

hence , s-$lim_\alpha\left(lim_n\emptyset(J_{n,\alpha}\ P_{n,\alpha}\ J_{n,\infty}\ P_{n,\infty})\right) = \left(lim_n\emptyset(J_{n,\infty}\ P_{n,\infty})\right)$, take the generalized conditional expectation as follows ,

$$\hat{\mathcal{E}}_{n,\infty}: (\mathcal{N}_\infty)_+^\wedge \to (\mathcal{N}_\alpha)_+^\wedge \quad \text{and } \hat{\mathcal{E}}_{\infty,\infty}: (\mathcal{N}_\infty)_+^\wedge \to (\mathcal{N}_\infty)_+^\wedge$$

If $\hat{x} \in (\hat{n}_\infty)_+^* \cap (\hat{n}_\infty)_+$ and for each α, $\hat{\mathcal{E}}_{\infty\alpha}(\hat{x}) \in (\hat{n}_\alpha)_+^* \cap (\hat{n}_\alpha)_+$

Now since s-$lim_\alpha\left(lim_n\emptyset(J_{n,\alpha}P_{n,\alpha})\right) = \left(lim_n\emptyset(J_{n,\infty})\right)$

then $\left\|\eta_{\hat{\varphi}}\left(\hat{\varepsilon}_{\infty,\alpha}(\hat{x})\right) - \eta_{\hat{\varphi}}(\hat{x})\right\| = lim_n\phi\left\|\left(J_{n,\alpha}P_{n,\alpha}J_{n,\infty}\eta_{\hat{\varphi}}(x_n)\right) - \eta_\varphi(x_n)\right\| \to 0$

hence,

$$\left\|\Delta_\infty^{1/2}\left(\eta_{\hat{\varphi}}\left(\hat{\varepsilon}_{\infty,\alpha}(\hat{x})\right)\right) - \Delta_\infty^{1/2}\left(\eta_{\hat{\varphi}}\left(\hat{\varepsilon}_{\infty,\infty}(\hat{x})\right)\right)\right\|$$
$$= \left\|\eta_{\hat{\varphi}}(\hat{\sigma}_{-i/2}^{\hat{\varphi}}\left(\hat{\varepsilon}_{\infty,\alpha}(\hat{x})\right) - \eta_{\hat{\varphi}}(\hat{\sigma}_{-i/2}^{\hat{\varphi}}\left(\hat{\varepsilon}_{\infty,\infty}(\hat{x})\right)\right\|$$
$$= \left\|\hat{\pi}_\infty(\hat{\sigma}_{-i/2}^{\hat{\varphi}}\left(\hat{\varepsilon}_{\infty,\alpha}(\hat{x})\right)\Omega - \hat{\pi}_\infty(\hat{\sigma}_{-i/2}^{\hat{\varphi}}\left(\hat{\varepsilon}_{\infty,\infty}(\hat{x})\right)\Omega\right\|$$
$$\leq \left\|\hat{\pi}_\infty(\hat{\sigma}_{-i/2}^{\hat{\varphi}}\left(\hat{\varepsilon}_{\infty,\alpha}(\hat{x})\right)\Omega - \left(\hat{\varepsilon}_{\infty,\infty}(\hat{x})\right)\Omega\right\|$$
$$\leq \left\|\hat{\sigma}_{-i/2}^{\hat{\varphi}}\right\|\ \left\|\left(\hat{\varepsilon}_{\infty,\alpha}(\hat{x})\right) - \left(\hat{\varepsilon}_{\infty,\infty}(\hat{x})\right)\right\| \to 0$$

where is taken $\hat{\Omega}$ is identify with Ω, thus $(iii) \Rightarrow (i)$ is proved. For $(ii) \Rightarrow (iii)$ the argument is easily adapted from [9].

Summary:

The trace functional in the construction of Haagerup L_p space was implemented by a generalized positive operator, likewise a kind of generalized conditional expectation called operator valued weights is defined using the notions of generalized positive operator .We can also enlarged the stochastic base in noncommutative stochastic integrals, if we replace $\mathcal{N}_\alpha$ with $(\mathcal{N}_\alpha)_+^\wedge$ then we will have an enlarged stochastic base since $\mathcal{N}_\alpha \subset (\mathcal{N}_\alpha)_+^\wedge$. Hence a stochastic process ${X_\alpha}^\wedge$ adapted to an expected filtration $(\mathcal{N}_\alpha)_+^\wedge$ becomes a martingale if ${X_\alpha}^\wedge \in D(\hat{\varepsilon}_\beta)$ and $\hat{\varepsilon}_\beta$ ${X_\alpha}^\wedge = {X_\beta}^\wedge$ for $\beta < \alpha$, and our stochastic integral define with respect to this martingale will extend those defined on filtration $\mathcal{N}_\alpha$ of von Neumann algebras.

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