Journal of the Nigerian Association of Mathematical Physics
***Volume* 16 (May, 2010), pp 553 - 562**

Multivariate Pareto Minification Processes

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Abstract

Autoregressive (AR) and autoregressive moving average (ARMA) processes with multivariate exponential (ME) distribution are presented and discussed. The theory of positive dependence is used to show that in many cases, multivariate exponential autoregressive (MEAR) and multivariate autoregressive moving average (MEARMA) models consist of associated random variables. Also, we present special cases of the multivariate exponential autoregressive process in which the multivariate process is stationary and has well-known multivariate exponential distribution.

Keywords: Marshall-Olkin multivariate Pareto distribution; Autoregressive minification processes of order 1 and k; Stationary marginal distribution; Joint survival function; Characterizations.

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1.0 Introduction

Pareto distributions have wide applications in different field such as the field of income and wealth modeling as well as failure times, birth rates, mortality rates and reliability models.

The multivariate Pareto distributions have important applications in modeling problems involving distributions of incomes when incomes exceed a certain limit. For example, income from several sources, it is not at all clear that we will be able to confidently visualize marginal features of the distribution. Rather we might be able to speculate that for given levels of income sources 2,3,4,…,k the income from source will have a Pareto like distribution with parameters perhaps depending on the level of income from the other sources, see [1].

The univariate Pareto distribution was first introduced by [13]. Some special bivariate Pareto distribution with homogeneous scale parameters were studied by [4]. Yeh [19] had studied the characterization of multivariate Pareto (III), MP $^{(n)}$ (III) already discussed by [2], through the geometric minimization procedures. All these results were extended to the Multivariate Pareto (MP$^{(n)}$) distribution by [20]. Characterizations of multivariate Pareto distribution, using the minimum of two independent and identically distributed random vectors, were also studied by [20].

The proofs of these characteristics (given in [20]) were based on the general and the particular solutions of the Euler's functional equations of $n \geq 1$variables.

The work of [16] was the first one on first order autoregressive processes with minification structure. Subsequently, many authors developed autoregressive minification processes having various marginal distributions. [5] introduced additive first order autoregressive processes with bivariate exponential and geometric stationary marginal distributions and studied their properties. [7] studied an additive first order autoregressive bivariate exponential process. [3] presented autoregressive logistic processes. [17] studied multivariate minification processes. [15] introduced and studied stationary bivariate minification processes in detail.

A minification process of the first order is given by

$X_n = R \min\left(X_{n-1}, \varepsilon_n\right), \quad n \geq 1,$ where $R > 1$ and $\left\{\varepsilon_n, n \geq 1\right\}$ is an innovation process of

independent and identically distributed random variables. [9] define a first order autoregressive minification process as a sequence having the general structure

$$X_n = \begin{cases} kX_{n-1} & \text{with probability } P \\ k\min\left(X_{n-1}, \in_n\right) & \text{with probability } 1-P \end{cases}$$

where $\left\{\in_n\right\}$ is an innovation process of independent and identically distributed random variables chosen such that $\left\{X_n\right\}$ is a stationary Markov process with a specified marginal distribution function $F_X\left(X\right)$.

Another form of minification process is the one with structure

$$X_n = \begin{cases} k\in_{n-1} & \text{with probability } P \\ k\min\left(X_{n-1}, \in_n\right) & \text{with probability } 1-P. \end{cases}$$

The nature of the structure of $\left\{X_n\right\}$ above is what makes it to be called minification process.

Thomas and Jose [17] introduced and studied the univariate Marshall-Olkin Pareto processes. [18] developed a new family of distributions that were earlier studied by [11] which is similar to those of [14]. In [18], also, AR(1) and AR (k) times series models useful in generating first order and k^{th} order autoregressive minification processes having a specified stationary bivariate marginal distribution are presented and studied. In that paper, they developed the Marshall-Olkin bivariate Pareto distribution as a generalization of the bivariate Pareto distribution of [4]. In that paper, bivariate Pareto AR (1) model and its generalization to AR (k) model with MO-BP stationary distribution were constructed. Those models developed are analogous to the model studied by [8] where the role of addition is replaced by minimization. Recently, [12] presented and studied a bivariate minification process with Marshall-Olkin exponential distribution. In that paper, they went ahead to estimate the unknown parameters and also studied the asymptotic properties of the estimated parameters. The Marshall and Olkin bivariate Pareto distribution has many different applications in the reliability theory and the field of income. This family is a positively quadrant dependent. It is proved that this family tends to bivariate exponential Marshall and Olkin type with nonlinear transformation.

In this paper, we present a Marshall-Olkin multivariate Pareto distribution as an extension of bivariate Pareto distribution earlier introduced by [18]. It is similar to the one introduced by [11] and also similar to those introduced by [14]. In section 2.0, we introduce the modified multivariate Pareto distribution (MO-MP) as an extension of bivariate Pareto distribution of [18]. In section 3.0, some characterizations properties of Marshall-Olkin multivariate Pareto distribution (MO-MP) are introduced and studied. We construct a multivariate Pareto AR (1) model having MO-MP stationary distribution in section 4.0. In section 5.0, we generalize it to the k^{th} order autoregressive model. The models developed here are analogous to the models introduced by [18] which in turn are analogous to the ones presented by [8] where the minimization takes the role of addition.

2.0 Marshall-Olkin multivariate Pareto distribution

Now we consider the multivariate Pareto distribution and distributions related to it, which are generally used for modeling socio-economic data.

Definition 2.1: The random vector $\underline{X} = \left(X_1, X_2 \ldots, X_n\right)$ is said to have a n-variate Pareto distribution with parameters, $p \in \left(0,1\right)$, $\underline{\beta} = \left(\beta_1, \beta_2, \ldots, \beta_n\right) > \underline{0}$ and the scale parameter

$\underline{\sigma} = (\sigma_1, \sigma_2, ..., \sigma_n) > \underline{0}$, and $\underline{X}$ is denoted by $\underline{X}$: $MP^{(n)}(\underline{\sigma}, \underline{\beta}, p)$, if its survival function is of the form

$$\bar{F}_{\underline{X}}(\underline{X}) = \frac{1}{1+\psi(x_1, x_2, ..., x_n)} \qquad (2.1)$$

such that

$$\psi(x_1, x_2, ..., x_n) = \frac{1}{p}\psi\left(p^{\frac{1}{\beta_1}}x_1, p^{\frac{1}{\beta_2}}x_2, ..., p^{\frac{1}{\beta_n}}x_n\right) \qquad (2.2)$$

The equation is true for all $x_i,\ i = 1, 2, ..., n$ and particular $p, \beta_i\ i = 1, 2, ..., n$ where $0 < p < 1;\ \beta_i > 0,\ \forall i.$

Also, $\psi(x_1, x_2, ..., x_n)$ is a monotonically increasing function in $\underline{X}$ satisfying $\lim_{x_1 \to 0}\lim_{x_2 \to 0}...\lim_{x_n \to 0}\psi(x_1, x_2, ...x_n) = 0$ and $\lim_{x_1 \to \infty}\lim_{x_2 \to \infty}...\lim_{x_n \to \infty}\psi(x_1, x_2, ...x_n) = \infty.$

The solution of equation (2.2) is $\psi(x_1, x_2, ..., x_n) = x_1^{\beta_1} + x_2^{\beta_2} + ... + x_n^{\beta_n}$. It is easy to show that the univariate marginal distributions of $X_i,\ i = 1, 2, ..., n$ are the univariate Pareto distributions of Pillai (1991), by taking $\psi(x_1) \equiv \psi(x_1, 0, ..., 0)$, $\psi(x_2) \equiv \psi(0, x_2, ..., 0)$,…, $\psi(x_n) \equiv \psi(0, 0, ..., x_n)$.
As an extension of Marshall and Olkin (1997), the multivariate extension of a family of distributions is given as follows:

Let $\underline{X} = (X_1, X_2 ..., X_n)$ be a random vector with joint survival function given by $\bar{F}_{\underline{X}}(\underline{X}) = \frac{1}{1+\left(x_1^{\beta_1} + x_2^{\beta_2} + ... + x_n^{\beta_n}\right)}$.

Then the modified multivariate survival function is in the form

$$\bar{G}_{\underline{X}}(\underline{X}) = \frac{\alpha \bar{F}_{\underline{X}}(\underline{X})}{1-(1-\alpha)\bar{F}_{\underline{X}}(\underline{X})} \quad \underline{X} = (x_1, x_2, ..., x_n) \geq 0,\ 0 < \alpha < 1, \qquad (2.3)$$

is a proper multivariate survival function. The family of distributions in this form is called Marshall and Olkin multivariate family of distributions.
From (2.3), we can see that the modified survival function is

$$\bar{G}(x_1, x_2, ..., x_n; \beta) = \frac{1}{1+\frac{1}{\alpha}\left(x_1^{\beta_1} + x_2^{\beta_2} + ... + x_n^{\beta_n}\right)} \quad \underline{X} = (x_1, x_2, ..., x_n) \geq 0,\ 0 < \alpha < 1.$$

(2.4)
The survival function in the form (2.4) is known as Marshall-Olkin multivariate Pareto distribution denoted by MO-MP. The density function is given by

$$g(x_1, x_2, ..., x_n) = 2\prod_{i=1}^{n}\beta_i \prod_{i=1}^{n} x_i^{\beta_i - 1}\frac{1}{\alpha^2}\left[1+\left(1/\alpha\right)\left(x_1^{\beta_1} + x_2^{\beta_2} + ... + x_n^{\beta_n}\right)\right]^{-3};\ x_i > 0, \forall i,\ \beta_i > 0, \forall i,\ 0 < \alpha < 1.$$

The marginal distributions of X_i are $g(x_i) = \frac{\beta_i}{\alpha}x_i^{\beta_i - 1}\left(1+\left(1/\alpha\right)x_i^{\beta_i}\right)^{-2}$, $x_i \geq 0,\ \beta_i > 0,\ 0 < \alpha < 1.$

Also,

$$E\left[X_i^r\right]=\left(\alpha^{1/\beta_i}\right)^r B\left(1+\tfrac{r}{\beta_i},1-\tfrac{r}{\beta_i}\right),\quad r<\beta_i.$$

Variance of X_i is given as:

$$Var(X_i)=\alpha^{2/\beta_i}\left\{\Gamma\left(1+\left(\tfrac{2}{\beta_i}\right)\right)\Gamma\left(1-\left(\tfrac{2}{\beta_i}\right)\right)-\left[\Gamma\left(1+\left(\tfrac{1}{\beta_i}\right)\right)\Gamma\left(1-\left(\tfrac{1}{\beta_i}\right)\right)\right]^2\right\};\ if\ \ \beta_i>2,\ i=1,2,...,n.$$

$$E\left[X_iX_j\right]=2\alpha^{\frac{1}{\beta_i}+\frac{1}{\beta_j}-2}B\left(\tfrac{1}{\beta_i}+1,2-\tfrac{1}{\beta_i}\right)B\left(\tfrac{1}{\beta_j}+1,1-\tfrac{1}{\beta_i}-\tfrac{1}{\beta_j}\right);\ i,j=1,2,...,n\ \ and\ \ i\neq j.$$

From the above information, we can get the correlation between X_i *and* X_j.

Similar multivariate distribution can be developed by considering multivariate Weibull and exponential survival functions. For instance, a multivariate Weibull distribution has a survival function of the form $\overline{F}_{\underline{X}}\left(\underline{X}\right)=EXP\left(-\sum_{i=1}^{n}x_i^{\beta_i}\right)$ where $\sum_{i=1}^{n}x_i^{\beta_i}$ satisfied the conditions specified above. Then the Marshall-

Olkin multivariate Weibull distribution has the survival function given by

$$\overline{G}\left(x_1,x_2,...,x_n;\beta\right)=\frac{\beta e^{-\sum_{i=1}^{n}x_i^{\beta_i}}}{1-(1-\beta)e^{-\sum_{i=1}^{n}x_i^{\beta_i}}}. \tag{2.5}$$

Theorem 2.1 : Let $\underline{X}=\left\{\left(X_i^{(1)},X_i^{(2)},...,X_i^{(n)}\right),i\geq 1\right\}$ be a multivariate sequence of non-negative random vectors independently and identically distributed as Marshall-Olkin multivariate Pareto, then

$$Z_n=\left[\left(\tfrac{n}{\alpha}\right)^{1/\beta_1}\min\left(X_1^{(1)},X_2^{(1)},...,X_n^{(1)}\right),\left(\tfrac{n}{\alpha}\right)^{1/\beta_2}\min\left(X_1^{(2)},X_2^{(2)},...,X_n^{(2)}\right),...,\left(\tfrac{n}{\alpha}\right)^{1/\beta_n}\min\left(X_1^{(n)},X_2^{(n)},...,X_n^{(n)}\right)\right]$$

; $\beta_i>0,\ \left(i=1,2,...,n\right),\ n>1,\ n>\alpha,$ is asymptotically distributed as multivariate Weibull as n goes to infinity.

Proof:- If $\underline{X}$ is distributed as MO-MP, then from equation (2.4), we have

$$\overline{G}\left(x_1,x_2,...,x_n;\alpha\right)=\frac{1}{1+\frac{1}{\alpha}\left(x_1^{\beta_1}+x_2^{\beta_2}+...+x_n^{\beta_n}\right)},$$

$$\overline{F}_{Z^{(n)}}\left(x_1,x_2,...,x_n\right)=P\left[\left(\tfrac{n}{\alpha}\right)^{1/\beta_1}\min\left(X_1^{(1)},X_2^{(1)},...,X_n^{(1)}\right)>x_1,...,\left(\tfrac{n}{\alpha}\right)^{1/\beta_n}\min\left(X_1^{(n)},X_2^{(n)},...,X_n^{(n)}\right)>x_n\right]$$

$$=\left[\overline{G}\left(\left(\tfrac{n}{\alpha}\right)^{-1/\beta_1}x_1,\left(\tfrac{n}{\alpha}\right)^{-1/\beta_2}x_2,...,\left(\tfrac{n}{\alpha}\right)^{-1/\beta_n}x_n\right)\right]^n$$

$$=\left(\frac{1}{1+\frac{\left(x_1^{\beta_1}+x_2^{\beta_2}+...+x_n^{\beta_n}\right)}{n}}\right)^n. \tag{2.6}$$

As n goes to infinity, equation (2.6) tends to $e^{-\left(x_1^{\beta_1}+x_2^{\beta_2}+\ldots+x_n^{\beta_n}\right)}$.
Hence the proof is complete.
From the [10] and discussed in detail by [6] with subsequent representation by [18], we can easily verify that Marshall-Olkin multivariate Pareto distribution is in the domain of attraction for minimum of the multivariate Weibull distribution.

3.0 Characterizations

The following theorems provide the characterizations properties of Marshall-Olkin multivariate Pareto distribution (MO-MP). Proofs of these theorems are also given in this section.

Theorem 3.1: Let N be a geometric random variable with parameter p such that $P\{N = n\} = pq^{n-1},\ n = 1,2,3,\ldots,\ 0 < p < 1,\ q = 1 - p$. Consider a sequence $\{X_i^{(1)}, X_i^{(2)}, X_i^{(3)}, \ldots, X_i^{(n)}, i \ge 1\}$ of independent and identically distributed random vectors with common survival function $\overline{F}(x_1, x_2, \ldots, x_n)$, where N and $\{X_i^{(1)}, X_i^{(2)}, X_i^{(3)}, \ldots, X_i^{(n)}, i \ge 1\}$ are independent for all $i \ge 1$. Let

$U_1 = \min_{1\le i\le N} X_i^{(1)}, U_2 = \min_{1\le i\le N} X_i^{(2)}, \ldots, U_n = \min_{1\le i\le N} X_i^{(n)}$. Then the random vectors $(U_1, U_2, \ldots, U_n)$ are distributed as Marshall-Olkin multivariate Pareto if and only if $\{X_i^{(1)}, X_i^{(2)}, X_i^{(3)}, \ldots, X_i^{(n)}, i \ge 1\}$ have the multivariate Pareto distribution.

Proof: Suppose

$$\overline{R}(x_1, x_2, \ldots, x_n) = P(U_1 > x_1, U_2 > x_2, \ldots, U_n > x_n)$$
$$= \sum_{n=1}^{\infty}\left[\overline{F}(x_1, x_2, \ldots, x_n)\right]^n pq^{n-1}$$
$$= p\overline{F}(x_1, x_2, \ldots, x_n)\sum_{n=1}^{\infty}\left[\overline{F}(x_1, x_2, \ldots, x_n)\right]^{n-1} q^{n-1}$$
$$= \frac{p\overline{F}(x_1, x_2, \ldots, x_n)}{1 - q\overline{F}(x_1, x_2, \ldots, x_n)}.$$

Let $\overline{F}(x_1, x_2, \ldots, x_n) = \dfrac{1}{1+\left(x_1^{\beta_1}+x_2^{\beta_2}+\ldots+x_n^{\beta_n}\right)}$, which is the survival function of multivariate Pareto distribution. Putting this into the equation above, we have

$$\overline{R}(x_1, x_2, \ldots, x_n) = \frac{1}{1+\frac{1}{p}\left(x_1^{\beta_1}+x_2^{\beta_2}+\ldots+x_n^{\beta_n}\right)} = \overline{G}(x_1, x_2, \ldots, x_n)$$

(3.1)
which is the survival function of Marshall-Olkin multivariate Pareto distribution.

Conversely, let

$$\overline{R}\left(x_1, x_2, \ldots, x_n\right) = \frac{1}{1+\frac{1}{p}\left(x_1^{\beta_1}+x_2^{\beta_2}+\ldots+x_n^{\beta_n}\right)}.$$

Then

$$\frac{p\overline{F}\left(x_1, x_2, \ldots, x_n\right)}{1-(1-p)\overline{F}\left(x_1, x_2, \ldots, x_n\right)} = \frac{1}{1+\frac{1}{p}\left(x_1^{\beta_1}+x_2^{\beta_2}+\ldots+x_n^{\beta_n}\right)}.$$

From this, we have

$\overline{F}\left(x_1, x_2, \ldots, x_n\right) = \dfrac{1}{1+\left(x_1^{\beta_1}+x_2^{\beta_2}+\ldots+x_n^{\beta_n}\right)}$, which is survival function of multivariate Pareto distribution. Hence the proof is complete.

Another characterization of the MO-MP distribution is given below.

Let $\{N_k : k \geq 1\}$ be a sequence of geometric random variables with parameters $p_k,\ 0 \leq p_k \leq 1.$

Define $\overline{F}_k\left(x_1, x_2, \ldots, x_n\right) = p\left(U_{N_{k-1}}^{(1)} > x_1, U_{N_{k-1}}^{(2)} > x_2, \ldots, U_{N_{k-1}}^{(n)} > x_n\right),\ k = 2, 3, \ldots$

$$= \frac{p_{k-1}\overline{F}_{k-1}\left(x_1, x_2, \ldots, x_n\right)}{1-\left(1-p_{k-1}\right)\overline{F}_{k-1}\left(x_1, x_2, \ldots, x_n\right)}. \qquad (3.2)$$

In this case, we refer $\overline{F}_k$ as the survival function of the geometric p_{k-1} minimum of independent and identically distributed random vectors with $\overline{F}_{k-1}$ as the common survival function.

Theorem 3.2: Let $\left\{X_i^{(1)}, X_i^{(2)}, X_i^{(3)}, \ldots, X_i^{(n)}, i \geq 1\right\}$ be a sequence of independent and identically distributed non-negative random vectors with common survival function $\overline{G}\left(x_1, x_2, \ldots, x_n\right)$. Define $\overline{G}_1 = \overline{G}$ and $\overline{F}_k$ as the survival function of the geometric p_{k-1} minimum of independent and identically distributed random vectors with $\overline{F}_{k-1},\ k = 2, 3, \ldots$ as the common survival function. Then

$$\overline{F}_k\left(x_1, x_2, \ldots, x_n\right) = \overline{G}\left(x_1, x_2, \ldots, x_n\right) \qquad (3.3)$$

if and only if $\left\{X_i^{(1)}, X_i^{(2)}, X_i^{(3)}, \ldots, X_i^{(n)}, i \geq 1\right\}$ has MO-MP distribution.

Proof: From the definition, the survival function $\overline{F}_k$ satisfies the equation (3.2). Therefore, we have

$$\overline{G}\left(x_1, x_2, \ldots, x_n\right) = \frac{1}{1+\frac{1}{p}\psi\left(x_1, x_2, \ldots, x_n\right)}$$

$= \dfrac{1}{1+\Phi\left(x_1, x_2, \ldots, x_n\right)}$, where $\Phi\left(x_1, x_2, \ldots, x_n\right)$ is a monotonically increasing function in $x_i\ (i = 1, 2, 3, \ldots n),\ x_i \geq 0, (i = 1, 2, 3, \ldots n)$ and also

$\lim_{x_1 \to 0}\lim_{x_2 \to 0}\ldots\lim_{x_n \to 0} \Phi\left(x_1, x_2, \ldots, x_n\right) = 0$ and $\lim_{x_1 \to \infty}\lim_{x_2 \to \infty}\ldots\lim_{x_n \to \infty} \Phi\left(x_1, x_2, \ldots, x_n\right) = \infty.$ Hence we can write $\overline{G}_k\left(x_1, x_2, \ldots, x_n\right) = \dfrac{1}{1+\Phi_k\left(x_1, x_2, \ldots, x_n\right)};\ k = 1, 2, 3, \ldots.$ Substituting this in (3.2), we get

$$\Phi_k\left(x_1, x_2, ..., x_n\right) = \frac{\Phi_{k-1}\left(x_1, x_2, ..., x_n\right)}{p_{k-1}},\ k = 2, 3, ...\ .$$

Using this relation recursively, we have $\Phi_k\left(x_1, x_2, ..., x_n\right) = \frac{\Phi_1\left(x_1, x_2, ..., x_n\right)}{p_1 p_2 \cdots p_{k-1}}$, as $\overline{G}_1 = \overline{G}$ we have

$\Phi_1 = \Phi$.

This implies that $$\Phi_k\left(x_1, x_2, ..., x_n\right) = \frac{\Phi_1\left(x_1, x_2, ..., x_n\right)}{p_1 p_2 \cdots p_{k-1}}. \qquad (3.4)$$

Hence, $\overline{F}_\kappa\left(x_1, x_2, ..., x_n\right) = \overline{G}\left(x_1, x_2, ..., x_n\right)$.

On the other hand, assume equation (3.3) holds. By the hypothesis of the theorem equation (3.4) follows. Therefore equations (3.3) and (3.4) together lead to the equation

$$\left\{1 + \frac{1}{p_1 p_2 \cdots p_{k-1}} \Phi_1\left(x_1, x_2, ..., x_n\right)\right\}^{-1} = \overline{G}\left(x_1, x_2, ..., x_n\right)$$

$$= \frac{1}{1 + \frac{1}{p}\psi\left(x_1, x_2, ..., x_n\right)}$$

$$= \frac{1}{1 + \Phi\left(x_1, x_2, ..., x_n\right)}.$$

This implies that $\Phi\left(x_1, x_2, ..., x_n\right) = \frac{\Phi_1\left(x_1, x_2, ..., x_n\right)}{p_1 p_2 \cdots p_{k-1}}$. Hence the theorem.

4. Marshall-Olkin multivariate Pareto AR (1) model

In this section, we consider the construction of first order autoregressive minification time series model, AR (1), with MO-MP distribution as stationary marginal distribution.

Theorem 4.1: Consider a multivariate autoregressive minification process $\left\{\left(X_n^{(1)}, X_n^{(2)}, ..., X_n^{(n)}\right)\right\}$ having the structure

$$X_n^{(1)} = \begin{cases} U_n^{(1)} & with\ \ probability\ \ \ p \\ \min\left(X_{n-1}^{(1)}, U_n^{(1)}\right) & with\ \ probability\ \ 1-p \end{cases}$$

$$X_n^{(2)} = \begin{cases} U_n^{(2)} & with\ \ probability\ \ \ p \\ \min\left(X_{n-1}^{(2)}, U_n^{(2)}\right) & with\ \ probability\ \ 1-p \end{cases} \qquad (4.1)$$

....

$$X_n^{(n)} = \begin{cases} U_n^{(n)} & with\ \ probability\ \ \ p \\ \min\left(X_{n-1}^{(n)}, U_n^{(n)}\right) & with\ \ probability\ \ 1-p \end{cases}$$

where $\left\{\left(U_n^{(1)},U_n^{(2)},\ldots,U_n^{(n)}\right)\right\}$ are the innovations which are independent of $\left\{\left(X_{n-k}^{(1)},X_{n-k}^{(2)},\ldots,X_{n-k}^{(n)}\right)\right\},\ k=1,2,\ldots,n.$ Then $\left\{\left(X_n^{(1)},X_n^{(2)},\ldots,X_n^{(n)}\right)\right\}$ has stationary marginal distribution as MO-MP if and only if $\left\{\left(U_n^{(1)},U_n^{(2)},\ldots,U_n^{(n)}\right)\right\}$ is jointly distributed as multivariate Pareto distribution.

Proof: From equation (4.1), we have

$$\overline{F}_{X_n^{(1)},\ldots,X_n^{(n)}}\left(x_1,x_2,\ldots,x_n\right) = p\overline{G}_{U_n^{(1)},\ldots,U_n^{(n)}}\left(x_1,x_2,\ldots,x_n\right)+\left(1-P\right)\overline{F}_{X_{n-1}^{(1)},\ldots,X_{n-1}^{(n)}}\left(x_1,x_2,\ldots,x_n\right)\overline{G}_{U_n^{(1)},\ldots,U_n^{(n)}}\left(x_1,x_2,\ldots,x_n\right) \tag{4.2}$$

Under stationary condition, we have

$$\overline{F}_{X_n^{(1)},\ldots,X_n^{(n)}}\left(x_1,x_2,\ldots,x_n\right)=\frac{p\overline{G}_{U^{(1)},\ldots,U^{(n)}}\left(x_1,x_2,\ldots,x_n\right)}{\left\{1-\left(1-P\right)\overline{G}_{U^{(1)},\ldots,U^{(n)}}\left(x_1,x_2,\ldots,x_n\right)\right\}}. \tag{4.3}$$

If we take

$\overline{G}_{U_n^{(1)},\ldots,U_n^{(n)}}\left(x_1,x_2,\ldots,x_n\right)=\dfrac{1}{1+\sum_{i=1}^{n}x_i^{\beta_i}}$, then

$\overline{F}_{X^{(1)},\ldots,X^{(n)}}\left(x_1,x_2,\ldots,x_n\right)=\dfrac{p}{p+\sum_{i=1}^{n}x_i^{\beta_i}}$, which is the survival function of MO-MP.

On the other hand, if we take $\overline{F}_{X^{(1)},\ldots,X^{(n)}}\left(x_1,x_2,\ldots,x_n\right)=\dfrac{p}{p+\sum_{i=1}^{n}x_i^{\beta_i}}$, which is the survival function of MO-MP and substituting it in (4.3), we have

$\overline{G}_{U_n^{(1)},\ldots,U_n^{(n)}}\left(x_1,x_2,\ldots,x_n\right)=\dfrac{1}{1+\sum_{i=1}^{n}x_i^{\beta_i}}$, which is the survival function of multivariate Pareto distribution and the process is stationary.

Stationarity can be established as follows.

Assume $\left\{\left(X_n^{(1)},X_n^{(2)},\ldots,X_n^{(n)}\right)\right\}\underline{\underline{d}}\ MO-MP$ and $\left\{\left(U_n^{(1)},U_n^{(2)},\ldots,U_n^{(n)}\right)\right\}\underline{\underline{d}}\ MP.$ Then from (4.2) $\overline{F}_{X_n^{(1)},\ldots,X_n^{(n)}}\left(x_1,x_2,\ldots,x_n\right)=\dfrac{p}{p+\sum_{i=1}^{n}x_i^{\beta_i}}$. This established that $\left\{\left(X_n^{(1)},X_n^{(2)},\ldots,X_n^{(n)}\right)\right\}$ is distributed as MO-MP.

It is also possible to show that $\left\{\left(X_n^{(1)},X_n^{(2)},\ldots,X_n^{(n)}\right)\right\}$ is stationary and is asymptotically marginally distributed as MO-MP.

5.0 Generalization to the k^{th} order autoregressive model

In this section, we develop the k^{th} order multivariate autoregressive model $\left\{\left(X_n^{(1)}, X_n^{(2)},..., X_n^{(k)}\right)\right\}$ having the structure given in the following theorem as an extension of first order multivariate autoregressive minification time series model presented earlier in section 4.0. The structure of the process is given in the following theorem.

Theorem 5.1: Consider a k^{th} order autoregressive time series, AR (k), model with structure:

$$X_n^{(1)} = \begin{cases} U_n^{(1)} & with\ probability\ \ p_0 \\ \min\left(X_{n-1}^{(1)}, U_n^{(1)}\right) & with\ probability\ \ p_1 \\ \min\left(X_{n-2}^{(1)}, U_n^{(1)}\right) & with\ probability\ \ p_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \min\left(X_{n-k}^{(1)}, U_n^{(1)}\right) & with\ probability\ \ p_k \end{cases}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \qquad (5.1)$$

$$X_n^{(k)} = \begin{cases} U_n^{(k)} & with\ probability\ \ p_0 \\ \min\left(X_{n-1}^{(k)}, U_n^{(k)}\right) & with\ probability\ \ p_1 \\ \min\left(X_{n-2}^{(k)}, U_n^{(k)}\right) & with\ probability\ \ p_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots & \\ \min\left(X_{n-k}^{(k)}, U_n^{(k)}\right) & with\ probability\ \ p_k \end{cases}$$

where $0 < p_i < 1,\quad \sum_{i=0}^{k} p_i = 1.$ Then $\left\{\left(X_n^{(1)}, X_n^{(2)},..., X_n^{(k)}\right)\right\}$ has stationary marginal distribution as MO-MP if and only if $\left\{\left(U_n^{(1)}, U_n^{(2)},..., U_n^{(k)}\right)\right\}$ has a multivariate Pareto distribution.

The proof of this theorem is similar to that of theorem (4.1) above with first order autoregressive minification process model replaced by k^{th} order autoregressive minification process model.

Acknowledgements: The Author thanks all those that contributed with suggestions which led to the improved version of this paper.

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