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Multivariate Exponential Autoregressive and Autoregressive Moving Average Models

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Abstract

Autoregressive (AR) and autoregressive moving average (ARMA) processes with multivariate exponential (ME) distribution are presented and discussed. The theory of positive dependence is used to show that in many cases, multivariate exponential autoregressive (MEAR) and multivariate autoregressive moving average (MEARMA) models consist of associated random variables. Also, we present special cases of the multivariate exponential autoregressive process in which the multivariate process is stationary and has well-known multivariate exponential distribution.

Keywords: Multivariate exponential distribution; Multivariate autoregressive and autoregressive moving average models in exponential random vectors; Association; Joint stationarity.

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1.0 Introduction

One of the major stationary model in time series analysis is the $p \times 1$ linear process given by

$$X(n) = \sum_{j=-\infty}^{\infty} A(j) \in (n-j), \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{1.1}$$

where $A(j), j = 0, \pm 1, \pm 2, \ldots,$ is a sequence of $p \times p$ parameter matrices such that $\sum_{j=-\infty}^{\infty} \|A(j)\| < \infty,$ and $\in (n), n = 0, \pm 1, \pm 2, \ldots,$ is a sequence of independent and identically distributed $p \times 1$ random vectors with mean 0 and common covariance matrix. It is well known that equation (1.1) includes the stationary vector autoregressive (AR) process and the stationary and invertible vector autoregressive moving average (ARMA) process. According to [6], in some physical situations where the random vectors $X(n)$ are either positive or discrete, the preceding assumptions on the $\in (n)$ sequence are inappropriate.

Several researchers, addressing themselves to this problem, have constructed univariate stationary AR type models and stationary ARMA type models where the random variables $X(n)$ have exponential or gamma distributions, and discrete models where $X(n)$ assumes values in a common set. It was stated in [13] and [14] that stationary autoregressive and autoregressive moving average type models where the random variables $X(n)$ have exponential distributions; [6] consider stationary autoregressive moving average-type models where the random variables $X(n)$ have gamma distributions. [15] proposed a modification to the stationary autoregressive and autoregressive moving average type models where the random variables $X(n)$ have exponential distributions. Similarly, [16] considered modification to the stationary autoregressive moving average-type models where the random variables $X(n)$ have gamma distributions. In [9] and [10] as well as [11] an attempt has been made to construct autoregressive moving average type models where the random variables $X(n)$ are discrete and assume values in a common finite set. [4], presented the bivariate exponential and geometric autoregressive and autoregressive moving average models that are extension of all the above mentioned models. According to [4], all the above models have been used in various fields of applied probability and time series analysis.

According to [9] and [10], these models can be used to model and analyse univariate point processes with correlated service and correlated inter arrival times. More detail information about bivariate exponential moving average type processes and the corresponding point processes can be seen in [12] as well as [4].

In this paper, we present a class of autoregressive and autoregressive moving average type sequence of multivariate random vector. This class has exponential marginal distributions. We denote this class of autoregressive and autoregressive moving average type models by $MEAR(m)$ and $MEARMA\ (m_1, m_2, \ldots m_k)$ for multivariate exponential autoregressive, order m and autoregressive moving average , order $(m_1, m_2, \ldots m_k)$, respectively, where m and $(m_1, m_2, \ldots m_k)$ parametrize the order of the dependence on the past. We use the theory of positive dependence to show that in many cases the classes of sequences are associated to each other.

In section 2, we define the multivariate exponential distribution which in this case, is the underlying distribution. We also present a variety of examples of this distribution. Furthermore, in this section we define the concept of association and present a variety of multivariate exponential distributions that are associated. Still, In section 2, we describe the multivariate dependence mechanisms which are used in generating the various models. In section 3, we construct the general $MEAR\ (m)$ model showing that the sequences have multivariate exponential distribution. We also, discuss the autocorrelation structure of the variety of sequences of the models. In section 4, we present special cases of the multivariate exponential autoregressive of order one sequences. In this section, we show that defined appropriately, the multivariate processes are stationary, and obtain well-known multivariate exponential distribution. In section 5, we introduce and present the multivariate exponential autoregressive moving average, $MEARMA\ (m_1, m_2, \ldots m_k)$ model. In this section, we conclude by describing the association properties of the sequences and discussing how to utilise association to obtain some probability bounds and moment inequalities for the multivariate processes and the corresponding point processes.

2.0 Preliminaries

In this section, we present some definitions and prove some basic results that are going to be used later. First, we provide the definition of multivariate exponential distribution and provide some examples to illustrate the basic concept of this distribution. The concept of association and some of its examples are also presented in this section. Finally, we discuss some multivariate dependence mechanisms

Definition 2.1: Let $E_1, E_2, \ldots, E_k$ be random variables taking values in $(0, \infty)$. We say that $(E_1, E_2, \ldots, E_k)$ has a multivariate exponential distribution if $E_i,\ i = 1,2, \ldots, k,$ have exponential distributions.

Example 2.1: (a) Let E be exponential. Then $(E_1, E_2, \ldots, E_k)$ is multivariate exponential. (b) Let $E_1, E_2, \ldots, E_k$ be independent exponentials. Then $(E_1, E_2, \ldots, E_k)$ has a multivariate exponential distribution. (c) Let $X_1, X_2, \ldots, X_k$ be independent exponentials and let $E_i = \{min(X_i, X_j)\}, i \neq j , i = 1,2, \ldots, k \text{ and } j = 1,2, \ldots, k.$ Then, each (E_i, E_j) , $i \neq j,$ has the Marshall-Olkin bivariate exponential distribution. (d) Let $(M_1, M_2, \ldots, M_k)$ have a multivariate geometric distribution and let $[E_1(j), E_2(j), \ldots, E_k(j)]$, $j = 1,2, \ldots,$ be an independent and identically distributed sequence of random vectors with multivariate exponential distributions, independent of all $M_i,\ i = 1,2, \ldots, k.$ Then $\left[\sum_{j=1}^{M_1} E_1(j), \sum_{j=1}^{M_2} E_2(j), \ldots, \sum_{j=1}^{M_k} E_k(j),\right]$ has a multivariate exponential distribution. (e) Let $|\alpha| \leq 1$. Then $(E_1, E_2, \ldots, E_k)$ determined by $P\{E_1 \leq x_1, E_2 \leq x_2, \ldots, E_k \leq x_k\} = (1 - e^{-x_1})(1 - e^{-x_2}) \times \ldots \times (1 - e^{-x_k}) \times (1 + \alpha e^{-x_1 - x_2 - \cdots - x_k}),$ $x_1, x_2, \ldots, x_k > 0,$
has a [7] multivariate exponential distribution. (f) Let $\alpha \geq 1$. Then $(E_1, E_2, \ldots, E_k)$ determined by

$P\{E_1 > x_1, E_2 > x_2, \dots, E_k > x_k\} = exp\left[-(x_1^{\alpha} + x_2^{\alpha} + \cdots + x_k^{\alpha})^{\frac{1}{\alpha}}\right], \; x_1, x_2, \dots, x_k > 0,$ is multivariate exponential distribution. (g) Let $(X_1, X_2, \dots, X_k)$ be a random vector with continuous marginal distributions $F_1, F_2, \dots, F_k$, respectively. Then the random vector $\{-ln[1 - F_1(X_1)], -ln[1 - F_2(X_2)], \dots, -ln[1 - F_k(X_k)]\}$ is a multivariate exponentially distributed.
The bivariate version of example 2.1 (d), has been used by the following researchers to generate bivariate distributions. These researchers are [1], [5], [8] and [18]. Similarly, example 2.1 (d) above can be used to generate multivariate distributions.
The following remarks illustrate how multivariate exponential distribution is obtained from Example 2.1 (d).

Remarks 2.1: Let $M_1 = M_2 = \cdots = M_k$ and let $E_1(j), E_2(j), \dots, E_k(j)$ be independent exponentials, $j = 1,2, \dots$. Then we obtain the multivariate version of the distribution introduced by [5].
Now, let us consider a concept of positive dependence.

Definition 2.2: Let $\mathbf{T} = (T_1, T_2, \dots, T_k),\; k = 1,2, \dots$ be a multivariate random vector. We say that the random variables $T_1, T_2, \dots, T_k$ are associated if $cov[f(t), g(t)] \geq 0$ for all $f(t)$ *and* $g(t)$ monotonically non-decreasing in each argument, such that the expectations exist.

Remarks 2.2: Independent random variables are associated and that non-decreasing functions of associated random variables are also associated. This remarks can also be seen in [2]. Therefore, the components of the vector given in example 2.1 (b) and (c) above, are associated. Similarly, the components of the multivariate exponential distribution given in example 2.1 (d) are associated provided that $M_1, M_2, \dots$ *and* M_k, and $E_1(1), E_2(1), \dots, E_k(1)$ are associated by the same reasoning given in remarks 2.1 above.
We are now ready to discuss the various dependence mechanisms used in obtaining multivariate exponential distribution. It turns out that many of these mechanisms are related and an attempt has been made to describe the relationships that exist among them.

Lemma 2.1: Let $(X_1, X_2, \dots, X_k)$ and $(Z_1, Z_2, \dots, Z_k)$ be independent random vectors with exponential marginals where $X_i \triangleq Z_i,\; i = 1,2, \dots, k$ and $(X_i, X_j), (Z_i, Z_j)\; i \neq j,\;\; i,j = 1,2, \dots, k$ have mean vector $(\beta_1^{-1}, \beta_2^{-1}, \dots, \beta_k^{-1})$. Let $(I_1, I_2, \dots, I_k)$ be a multivariate Bernoulli random vector independent of $(X_1, X_2, \dots, X_k)$, $(Z_1, Z_2, \dots, Z_k)$. Also, assume

$$P(I_i = m, I_j = n) = P_{mn}, \quad i \neq j, \quad m, n = 0,1. \tag{2.1}$$

Such that $\sum_{m,n} P_{mn} = 1;\; 1 - \pi_1 = P_{10} + P_{11} < 1, \quad and \quad 1 - \pi_2 = P_{01} + P_{11} < 1.$ Then a random vector given by

$$(Y_1, Y_2, \dots, Y_k) \triangleq (I_1 Z_1, I_2 Z_2, \dots, I_k Z_k) + (\pi_1 X_i, \pi_2 X_j), \quad i \neq j,\; i,j = 1,2, \dots, k. \tag{2.2}$$

has a multivariate exponential distribution with the same marginals as $(X_1, X_2, \dots, X_k)$ *and* $(Z_1, Z_2, \dots, Z_k)$.

Proof: The prove of his lemma follows easily by computing the marginal characteristic functions.

Lemma 2.2: Let $(X_{1j}, X_{2j}, \dots, X_{kj}),\; j = 1,2,3, \dots,$ be independent and identically distributed multivariate exponential random vectors with mean vector
$\left({}^{\pi_i}/_{\beta_i}\right), i = 1,2, \dots, k,\; 0 < \pi_i < 1,\; for\; all\; i = 1,2, \dots, k.$ Let $(N_1, N_2, \dots, N_k)$ have a multivariate geometric distribution with mean vector $(\pi_1^{-1}, \pi_2^{-1}, \dots, \pi_k^{-1})$ and be independent of $(X_{1j}, X_{2j}, \dots, X_{kj})$ and $(M_{1j}, M_{2j}, \dots, M_{kj})$. Then random vectors given by

$$(Y_1, Y_2, \dots, Y_k) \triangleq \left(\sum_{j=1}^{N_1} X_{1j}, \sum_{j=1}^{N_2} X_{2j}, \dots, \sum_{j=1}^{N_k} X_{kj}\right) \tag{2.3}$$

has a multivariate exponential distribution with mean vector $(\beta_1^{-1}, \beta_2^{-1}, \dots, \beta_k^{-1})$.

Proof: The prove of this lemma can easily be seen by computing the marginal characteristic functions. The following lemma describes how lemmas (2.1) and (2.2) above are related. The connection between the concepts stated in the above two lemmas is a key element in the development of the multivariate exponential autoregressive (AR) and autoregressive moving average (ARMA) models that are going to be discussed in the following sections.

Lemma 2.3: Let $(X_{1j}, X_{2j}, \dots, X_{kj})$ and $(M_{1j}, M_{2j}, \dots, M_{kj})$, $j = 1,2,\dots$, be as defined in lemma 2.2. Let $(N_1, N_2, \dots, N_k)$ have the multivariate geometric distribution with $1 - \pi_1 = P_{10} + P_{11} < 1;\ 1 - \pi_2 = P_{01} + P_{11} < 1,$ and be independent of the $(X_{1j}, X_{2j}, \dots, X_{kj})$ and $(M_{1j}, M_{2j}, \dots, M_{kj})$. Furthermore, let $(X_1, X_2, \dots, X_k)$, $(I_1, I_2, \dots, I_k)$, $(M_1, M_2, \dots, M_k)$ and $(K_1, K_2, \dots, K_k)$ be as defined in lemma 2.1. Then $(Y_1, Y_2, \dots, Y_k) = \left(\sum_{j=1}^{N_1} X_{1j}, \sum_{j=1}^{N_2} X_{2j}, \dots, \sum_{j=1}^{N_k} X_{kj}\right)$ has the representation $(I_1Z_1, I_2Z_2, \dots, I_kZ_k) + (\pi_1 X_i, \pi_2 X_j)$, $i \neq j$, $i, j = 1,2,\dots,k$.

That is

$$E[exp\{it_1Y_1 + it_2Y_2 + \cdots + it_kY_k\}] = E[exp\{it_1(I_1Z_1 + \pi_1X_1) + \cdots + it_k(I_kZ_k + \pi_kX_k)\}] \qquad (2.4)$$

Where $i = \sqrt{-1}$ and $-\infty < t_1, t_2, \dots, t_k < \infty$.

Proof: Suppose that $\left(\sum_{j=1}^{N_1} X_{1j}, \sum_{j=1}^{N_2} X_{2j}, \dots, \sum_{j=1}^{N_k} X_{kj}\right) = \left(\aleph^{(N_1>1)} \sum_{j=1}^{N_1} X_{1j}, \dots, \aleph^{(N_k>1)} \sum_{j=1}^{N_k} X_{kj}\right)$, where $\aleph$ denotes the indicator function. Now, we show that $\left(\aleph^{(N_1>1)} \sum_{j=1}^{N_1} X_{1j}, \dots, \aleph^{(N_k>1)} \sum_{j=1}^{N_k} X_{kj}\right)$ has the same distribution as $(I_1Z_1, I_2Z_2, \dots, I_kZ_k)$. Note that

$$P\left\{\aleph^{(N_1>1)} \sum_{j=1}^{N_1} X_{1j} \le x_1, \dots, \aleph^{(N_k>1)} \sum_{j=1}^{N_k} X_{kj} \le x_k\right\}$$

$$= \sum_{i_1}^{1} \dots \sum_{i_k}^{1} P\left\{\aleph^{(N_1>1)} \sum_{j=1}^{N_1} X_{1j} \le x_1, \dots, \aleph^{(N_k>1)} \sum_{j=1}^{N_k} X_{kj} \le x_k\right\} \qquad (2.5)$$

$where \quad \aleph^{(N_1>1)} = i_1, \aleph^{(N_2>1)} = i_2, \dots, \aleph^{(N_k>1)} = i_k.$

Now, for $(i_1, i_2, \dots, i_k) = (1,1,\dots,1)$ in equation (2.5) above, we have

$$P\left\{\sum_{j=1}^{N_1} X_{1j} \le x_1, \dots, \sum_{j=1}^{N_k} X_{kj} \le x_k,\ N_1 > 1, \dots, N_k > 1\right\} = \sum_{n_1}^{\infty} \dots \sum_{n_k}^{\infty} P\left\{\sum_{j=1}^{n_1} X_{1j} \le x_1, \dots, \sum_{j=1}^{n_k} X_{kj} \le x_k\right\} P\{N_1 = n_1, \dots, N_k = n_k\}. \qquad (2.6)$$

But $P\{N_1 = n_1, \dots, N_k = n_k\} = P\{N_1 = n_1 - 1, \dots, N_k = n_k - 1\} P\{N_1 > 1, \dots, N_k > 1\}$, so that equation (2.6) is equal to

$$P\left\{\sum_{j=1}^{N_1} X_{1j} \le x_1, \dots, \sum_{j=1}^{N_k} X_{kj} \le x_k\right\} P\{I_1 = 1, \dots, I_k = 1\} = P\{Z_1 \le x_1, \dots, Z_k \le x_k\} P\{I_1 = 1, \dots, I_k = 1\}.$$

3.0 Generalized multivariate exponential autoregressive model (MEAR)

In this section, we construct a class of autoregressive sequences of multivariate random vectors. In this class, the sequences are labeled by the parameter m. We denote the sequences of this class by $\{X(m,n) = [X_1(m,n), X_2(m,n), \dots, X_k(m,n)]^I,\quad m = 1,2,\dots \text{ and } n = 0,1,2,\dots\}$. We show that the random vector $X(m,n)$ has a multivariate exponential distribution with mean vectors that do not depend on m or n. We also discuss the association property for any finite number of random variables belonging multivariate exponential autoregressive model. We conclude this section by discussing the autocorrelation structure of the sequences of this model.

Now we construct the class of multivariate exponential autoregressive sequences, MEAR(m).
Let $\beta_1, \beta_2, \dots, \beta_k \in (0,\infty)$, let $\pi_1(n), \pi_2(n), \dots, \pi_k(n) \in (0,1)$, and let $B(n)$ be a $k \times k$ diagonal matrix with $B(n) = diag\{\pi_1(n), \pi_2(n), \dots, \pi_k(n)\}$. Furthermore, let $E^I(n) = \{E_1(n), E_2(n), \dots, E_k(n)\}$ be a sequence of independent multivariate exponential random vectors with mean vector $(\beta_1^{-1}, \beta_2^{-1}, \dots, \beta_k^{-1})^I$ let e_j be an $m-$dimensional vector with component j equal to one and the other components equal to zero, $j = 1,2, \dots, m$, and let 0 denote the $m-$dimensional zero vector. Finally, let $I^I(n) = \{I_1(n,1), \dots, I_1(n,m), \dots, I_k(n,1), \dots, I_k(n,m)\}$ be a sequence of $2m-$dimensional independent random vectors with components assuming values one or zero independent of all $E^I(n)$, and let $A(n,q)$ be a $k \times k$ random diagonal matrix with $A(n,q) = diag\{I_1(n,q), \dots, I_k(n,q)\}$, $q = 1,2, \dots, m$.
We assume that for $l = 1,2, \dots, k$,

$$\sum_{j=1}^{m} P\{(I_l(n,1), \dots, I_l(n,m)) = e_j^I\} = 1 - \pi_l(n), \qquad (3.1)$$

and that

$$P\{(I_l(n,1), \dots, I_l(n,m)) = 0^I\} = \pi_l(n). \qquad (3.2)$$

We define the multivariate exponential autoregressive, MEAR(m) sequences as follows:

$$X(n) = \begin{cases} E(n), & n = 0,1,2, \dots, m-1 \\ \sum_{q=1}^{m} A(n,q)X(n-q) + B(n)E(n), & n = m, m+1, \dots \end{cases} \qquad (3.3)$$

[19] proposed a multivariate exponential autoregressive model which is to the one developed here; however, neither model is a generalization of the other. The major difference between the [19] model and the current model is the way that the components of the vector series depend on each other. In the [19] model, the components are functionally related to each other with positive probability, so that, for example, the first component can get a
contribution from the second component at previous time points. However, in the current multivariate exponential autoregressive model, the dependence between components come from the dependence structure inherent in the exponential noise vector $E(n)$.
Next, we show that $X(n)$ has a multivariate exponential distribution.

Lemma 3.1: For $n = 0,1,2, \dots,$ $X(n)$ has a multivariate exponential distribution with mean vector $(\beta_1^{-1}, \beta_2^{-1}, \dots, \beta_k^{-1})$.

Proof: Mathematical induction on n is used to prove this lemma. For $n = 0,1, \dots, m-1$, the results of the lemma follow by the definition of $X(n)$. Let us assume that the results of the lemma hold for all non-negative integers that are less than or equal to r, $r \geq m-1$, and prove that the results of the lemma hold for $r+1$.
Let $E^I = (E_1, E_2, \dots, E_k)$ be a multivariate exponential random vector with mean vector $(\beta_1^{-1}, \beta_2^{-1}, \dots, \beta_k^{-1})$ independent of all $E(n)$. Then, by the induction assumption, we have for $l = 1,2, \dots, k$ that

$$X_l(r+1) \triangleq \begin{cases} E_l + \pi_l(r+1)E_l(r+1), & with\ probability\ \ 1 - \pi_l(r+1) \\ \pi_l(r+1)E_l(r+1), & with\ probability\ \ \pi_l(r+1). \end{cases}$$

It is easy to check that $X_l(r+1)$ has an exponential distribution with mean β_l^{-1}. Hence, the results of the lemma follow.
Now we consider the association of any finite collection of the $X_l(n)$.

Lemma 3.2: Let us assume that for $j = 0,1,2, \dots, m-1$, the random variables $X_1(j), X_2(j), \dots, X_k(j)$ in equation (3.3) are associated; let $n_1 < n_2 < \cdots < n_r$, be non-negative integers and let $l_1, l_2, \dots, l_r \in \{1,2, \dots, k\}$, $r = 1,2, \dots$. Then the random variables $X_{l_q}(n_q)$, $q = 1,2, \dots, r$, are associated.

Proof: Let $T_{2j} = X_2(j-1)$ and let $T_{2j-1} = X_1(j-1),\ j = 1,2,\dots$. To prove the result of the lemma it suffices, according to [2], to show

that the random variables $T_1, T_2, \dots, T_r$ are associated for all $r = 1,2,\dots$. (3.4)

We prove (3.4) by an induction argument on r. For $r \le 2m$, equation (3.4) follows by the lemma assumption and by [2]. Let us assume that equation (3.4) holds for $r,\ r \ge 2m$ and prove that equation (3.4) holds for $r+1$.

From equation (3.4), the conditional random variable $T_{r+1}/T_1,\dots T_r$, is stochastically non-decreasing in $T_1, T_2, \dots, T_r$. Therefore, by Barlow and Proschan (1981), there is an $r+1$ argument function h, non-decreasing in each argument, and a random variable U independent of $T_1, T_2, \dots, T_r$, such that $(T_1, T_2, \dots, T_{r+1}) \triangleq \left(T_1, T_2, \dots, T_r, h(U, T_1, T_2, \dots, T_r)\right)$. According to [2], U is associated and hence the random variables $U, T_1, T_2, \dots, T_r$ are associated. Consequently, by [2], $T_1, T_2, \dots, T_{r+1}$ are associated.

Finally, for a class of sequences, we compute the autocorrelation functions in the case when the marginal processes are stationary.

For the exponential models, put $\pi_l(n) = \pi_l, \mathit{for\ all}\ n;\ l = 1,2,\dots,r,$ and let

$P\{I_l(n,q) = 1\} = \varphi_l(q), \quad l = 1,2,\dots,k;\ q = 1,2,\dots,m$

Such that

$(i)\ \varphi_l(q) \ge 0,\ \ and\ \ (ii)\ \sum_{q=1}^{m} \varphi_l(q) = 1 - \pi_l,\ \ l = 1,2,\dots,k,$ as specified in equation (3.1). Define $\rho_{X_l}(k) = Corr\{X_l(n), X_l(n+k)\},\ l = 1,2,\dots,k;\ n = m, m+1,\dots; k = 1,2,\dots$.

Then

$$\rho_{X_l}(k) = \varphi_l(1)\rho_{X_l}(k-1) + \varphi_l(2)\rho_{X_l}(k-2) + \cdots + \varphi_l(m)\rho_{X_l}(k-m) \qquad (3.5)$$

with $Variance\{X_l(n)\} = \beta_l^{-2},\ l = 1,2,\dots,k.$

The marginal correlation structure of the multivariate exponential sequences, as given in (3.5) is similar to that of the Gaussian autoregressive process. We note that, in general, even when the marginal processes are stationary, the joint process is not stationary. This is easily seen, for example, by letting m=1 in (3.3) with $\pi_l(n) = \pi_l,\ \ for\ all\ n;\ l = 1,2,\dots,k;$ choosing $E(n)$ to be an independent and identically distributed sequence of random vectors where $E_1(n), E_2(n), \dots, E_k(n)$ are independent and identically distributed exponential random variables for all n, and letting $I(n)$ be an independent and identically distributed sequence of random vectors for which $P\{I_1(n) = 1,\dots, I_k(n) = 1\} \neq (1-\pi_1) \times \dots \times (1-\pi_k)$. A simple computation shows that $Cov\{X_1(1),\dots,X_k(1)\} \neq Cov\{X_1(2),\dots,X_k(2)\}$ in this example.

In the following section, we develop models in which joint processes are also stationary.

4.0 Stationary multivariate exponential autoregressive model (MEAR(1))

In this section, we consider special case of the MEAR(m) model given in section 3, in which the joint processes are stationary. Throughout this section we put $m = 1$, assume that $\pi_l(n)$ does not vary with n, and put more structure on the $E(n)$ sequences. We show that for this model, the multivariate distribution of $X(n)$ has a form of the type studied by [1]. By selecting the $E(n)$ sequences, as defined in section 3, appropriately, we can obtain well-known multivariate distribution. For a stationary multivariate exponential first autoregressive model, MEAR(1), we obtain the following: [1], [5], [8] and [18] multivariate exponential distributions.

We conclude this section by computing the auto covariance matrices for this model.

To present stationary multivariate exponential first autoregressive model, the following notations and assumptions are required. Let $m = 1$, and let us assume that $[I_1(N,1), I_2(N,1), \dots, I_k(N,1)]$, given in equation (3.1), is an independent and identically distributed sequence of multivariate random vectors. For simplicity of notation, we denote $\pi_i(n)$ by π_i, for $i = 1,2,\dots,k$ and let $P_{ij} = P\{(I_1(N,1), I_2(N,1), \dots, I_k(N,1)) = (i,j)\}$, $i,j = 0,1$. Note that by equations (3.2) and (3.3), we have

$$P_{10} + P_{11} = 1 - \pi_1, \qquad P_{01} + P_{11} = 1 - \pi_2 \tag{4.1}$$

Furthermore, let $(N_1, N_2, \dots, N_k)$ be a multivariate geometric random vector with parameters P_{ij}, $i,j = 0,1$ given in equation (2.1), and let $E(r)$, $r = \pm 1, \pm 2, \dots$ be an independent and identically distributed sequence of multivariate random vectors with mean vectors $(\beta_1^{-1}, \beta_2^{-1}, \dots, \beta_k^{-1})$ independent of $(N_1, N_2, \dots, N_k)$ and all $(I_1(N,1), I_2(N,1), \dots, I_k(N,1))$. Note that by lemma (2.1) above, $\left(\sum_{j=1}^{N_1} \pi_1 E_1(-j), \sum_{j=1}^{N_2} \pi_2 E_2(-j), \dots, \sum_{j=1}^{N_k} \pi_k E_k(-j)\right)$ is a multivariate exponential random vector with mean vector $(\beta_1^{-1}, \beta_2^{-1}, \dots, \beta_k^{-1})$. We assume that

$$E(0) = \left(\sum_{j=1}^{N_1} \pi_1 E_1(-j), \sum_{j=1}^{N_2} \pi_2 E_2(-j), \dots, \sum_{j=1}^{N_k} \pi_k E_k(-j)\right)^I. \tag{4.2}$$

Define $A(n)$ to be a $k \times k$ diagonal random matrix $A(n) = diag\,\{I_1(N,1), I_2(N,1), \dots, I_k(N,1)\}$ and B to be a $k \times k$ diagonal random matrix $B = diag\,\{\pi_1, \pi_2\}$. The stationary MEAR(1) model is defined as follows:

$$X(n) = \begin{cases} E(0), & n = 0 \\ A(n)X(n-1) + BE(n), & n = 1,2,\dots \end{cases} \tag{4.3}$$

We now state and prove a characterization of $X(n)$.

Lemma 4.1: Let $X(n)$ be defined by equation (4.3). Then for $n = 0,1,2,\dots$, $X(n) \triangleq E(0)$, where $E(0)$ is as given in equation (4.2).

Proof: We prove the result of the lemma by an induction argument on n. By definition the result of the lemma holds for $n = 0$. Let us assume that the result of the lemma holds for $n, n \geq 0$. Note that

$$E^I(0) = \left(\sum_{j=1}^{N_1} \pi_1 E_1(-j), \sum_{j=1}^{N_2} \pi_2 E_2(-j), \dots, \sum_{j=1}^{N_k} \pi_k E_k(-j)\right)$$

$$= \left(\aleph^{(N_1>1)} \pi_1 \sum_{j=1}^{N_1} E_1(-j), \dots, \aleph^{(N_k>1)} \pi_k \sum_{j=1}^{N_k} E_k(-j)\right) + \left(\pi_1 E_1(-1), \dots, \pi_k E_k(-1)\right),$$

where $\aleph(\cdot)$ denotes the indicator function, and that the k summands are independent random vectors. Now, by lemma 2.3 and the induction assumption,

$$\left(\aleph^{(N_1>1)} \pi_1 \sum_{j=1}^{N_1} E_1(-j), \dots, \aleph^{(N_k>1)} \pi_k \sum_{j=1}^{N_k} E_k(-j)\right)$$

$$\triangleq \left(I_1(n,1)\pi_1 \sum_{j=1}^{N_1} E_1(-j), \dots, I_k(n,1)\pi_k \sum_{j=1}^{N_k} E_k(-j)\right)$$

$$\triangleq \left(I_1(n,1)X_1(n), I_2(n,1)X_2(n), \dots, I_k(n,1)X_k(n)\right).$$

Furthermore, by the definition of $E(r)$,

$$\left(\pi_1 E_1(-j), \pi_2 E_2(-j), \dots, \pi_k E_k(-j)\right) \triangleq \left(\pi_1 E_1(n), \pi_2 E_2(n), \dots, \pi_k E_k(n)\right).$$

Since the random vectors $\left(\pi_1 E_1(n), \pi_2 E_2(n), \dots, \pi_k E_k(n)\right)$ and $\left(I_1(n,1)X_1(n), I_2(n,1)X_2(n), \dots, I_k(n,1)X_k(n)\right)$ are independent, we have by equation (3.3) that

$$\left(\pi_1 \sum_{j=1}^{N_1} E_1(-j), \dots, \pi_k \sum_{j=1}^{N_k} E_k(-j)\right) \triangleq$$

$$\left(\pi_1 E_1(n), \pi_2 E_2(n), \dots, \pi_k E_k(n)\right) + \left(I_1(n,1)X_1(n), I_2(n,1)X_2(n), \dots, I_k(n,1)X_k(n)\right) = X^I(n+1).$$

Hence the prove is completed.

According to following remarks, many interesting multivariate distributions are possible in lemma 4.1. For details, see [3].

Remarks: Let E be an exponential random variable with mean θ, $0 < \theta < (\beta_1, \beta_2, \ldots, \beta_k)^{-1}$, let $\pi_1 = \beta_1\theta, \pi_2 = \beta_2\theta, \ldots, \pi_k = \beta_k\theta$ and let $E(1) = (\pi_1^{-1}E, \pi_2^{-1}E, \ldots, \pi_k^{-1}E)$. Then, if $P_{00} = 0$, $P_{01} = \pi_1$, $P_{10} = \pi_2$ and $P_{11} = 1 - (\beta_1 + \beta_2)\theta$, then the resulting $X(n)$ has independent components.

Now, we give the auto covariance matrices for the stationary multivariate exponential autoregressive of order one, MEAR(1) model. Let $\sum_X = Var\{X(n)\}$ be the variance covariance matrix of $X(n)$. Define $\sqrt{}_X(k) = Cov\{X(n+k), X(n)\}$, $k = 0,1,2, \ldots$, and note that $\sqrt{}_X(0) = \sum_X$. In view of equation (4.3), it is easy to see that $\sqrt{}_X(k) = A\sqrt{}_X(k-1)$, $k = 1,2, \ldots$ where A is the $k \times k$ diagonal matrices defined by $A = diag\,\{1-\pi_1, 1-\pi_2, \ldots, 1-\pi_k\}$. Therefore, for stationary multivariate exponential autoregressive, MEAR(1), model we have

$$\sqrt{}_X(k) = A^k \textstyle\sum_X; \quad \sqrt{}_X(-k) = \sqrt{}_X^{\,l}(k), \quad k = 1,2, \ldots . \qquad (4.4)$$

5.0 Multivariate exponential autoregressive moving average model (MEARMA(m))

Considering the results of section 3 and the results of [12] for moving average sequences, we construct two classes of autoregressive moving average sequences of multivariate random vectors. In each class, the sequences are labelled by the parameters m_1 and m_2. We denote these two classes of sequences by $\{Z^I(j, m_1, m_2, n) = (Z_1(j, m_1, m_2, n), Z_2(j, m_1, m_2, n)), \quad n = 0,1,2, \ldots\}$, $j = 1,2$. We show that the random vector $Z^I(j, m_1, m_2, n)$ has a multivariate exponential distribution with a mean vector that does not depend on j, m_1, m_2 or n. Then we discuss the association property of any finite number of random variables belonging to one of the two autoregressive moving average (ARMA) classes.

Let $X(n)$ be a multivariate exponential autoregressive (MEAR(m_1)) sequence given by equation (3.3), and let Y_n be an m_2 −dependent multivariate exponential moving average sequence as given by Langberg-Stoffer (1987), independent of the $X(n)$ sequence. Further let $V^I(n) = \{V_1(n), V_2(n), \ldots, V_k(n)\}$ be a sequence of independent multivariate random vectors with components assuming the values 0,1,...k, independent of the $X(n)$ and Y_n sequences and let $P\{V_i(n) = 1\} = \pi_i(n)$, $0 < \pi_i(n) < 1$, $l = 1,2, \ldots, k$.

We define the two multivariate exponential autoregressive moving average, MEARMA(m_1, m_2), sequences as follows:

$$\{Z_1(1,n), Z_2(1,n), \ldots, Z_k(1,n)\} = \Big((1-\pi_1(n))Y_1(n), \ldots, (1-\pi_k(n))Y_k(n)\Big) + (V_1(n)X_1(n), \ldots, V_k(n)X_k(n)) \qquad (5.1)$$

$$\{Z_1(2,n), Z_2(2,n), \ldots, Z_k(2,n)\} = \Big((1-\pi_2(n))X_2(n), \ldots, (1-\pi_k(n))X_k(n)\Big) + (V_1(n)Y_1(n), \ldots, V_k(n)Y_k(n)). \qquad (5.2)$$

Next we show that $Z(j, n)$ has multivariate exponential distribution.

Lemma 5.1: For $j = 1,2, \ldots, k$ and $= 0,1,2, \ldots$, $Z(j,n)$ has a multivariate exponential distribution with mean vector $(\beta_1^{-1}, \beta_2^{-1}, \ldots, \beta_k^{-1})$.

Proof: Considering lemma (3.1), $X(n)$ has a multivariate exponential distribution. By [12], $Y(n)$ also has a multivariate exponential distribution. Consequently, from the two definitions and lemma 2.1 above, the results of this lemma follow.

Now we consider the association property of any finite number of random variables belonging to one of the two autoregressive moving average classes. We assume that the assumptions of [12], lemmas 3.1 and 3.2 are satisfied.

Lemma 5.2: Let $S_1, S_2, \dots, S_r$, $T_1, T_2, \dots, T_r$, be non-negative random variables. Let us assume that $S_1, S_2, \dots, S_r$ and $T_1, T_2, \dots, T_r$ are associated, and that the random vectors $(S_1, S_2, \dots, S_r)$ and $(T_1, T_2, \dots, T_r)$ are independent. Then the random variables $S_1T_1, S_2T_2, \dots, S_rT_r$ are associated.

Proof: Let $\boldsymbol{T} = (T_1, T_2, \dots, T_r)$. Let $\boldsymbol{W} = (S_1T_1, S_2T_2, \dots, S_rT_r)$, and let f, g be two non-negative functions each with r-arguments, non-decreasing in each argument.
The components of the conditional random vector $W/_T$ are non-decreasing functions of the associated random variables $S_1, S_2, \dots, S_r$. Therefore, by [2], the components of $W/_T$ are associated. Hence,

$\boldsymbol{E}\left[cov(f(W), g(W))/_T\right] \geq 0.$ $\boldsymbol{E}\left[f(W)/_T\right]$ and $\boldsymbol{E}\left[g(W)/_T\right]$ are two non-decreasing functions of the associated random variables $T_1, T_2, \dots, T_r$. Thus, by Barlow and Proschan (1981), the two random variables $\boldsymbol{E}\left[f(W)/_T\right]$ and $\boldsymbol{E}\left[g(W)/_T\right]$ are associated. Hence $cov\left[\boldsymbol{E}\left\{f(W)/_T\right\}, \boldsymbol{E}\left\{g(W)/_T\right\}\right] \geq 0.$ Note that

$$Cov[f(W), g(W)] = \boldsymbol{E}\left[cov(f(W), g(W))/_T\right] + cov\left[\boldsymbol{E}\left\{f(W)/_T\right\}, \boldsymbol{E}\left\{g(W)/_T\right\}\right].$$

Consequently, the lemma follows and the prove is completed.

6.0 Conclusion

Langberg and Stoffer, present inequalities and probability bounds for the bivariate point processes related to the bivariate exponential moving average sequence. We note that all the the results given by [12], hold for the multivariate point processes related to the multivariate exponential autoregressive sequences given in sections 3 and 4 and to the autoregressive moving average sequences stated in section 5 provided that they are associated.

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References

[1] Arnold, B.C (1975): A characterization of the exponential distribution by multivariate geometric compounding, Sankhya, A 37, 164-173.

[2] Barlow, R.E and Proschan, F (1981): Statistical Theory of reliability and life-testing: Probability Models. To begin with, Silver Spring, MD.

[3] Block, H.W (1977): A family of bivariate life distributions, In the theory and applications of life Reliability, Vol. 1. C.P. Tsokos and I. Simi. Eds. Academic Press. New York.

[4] Block, H.W and Paulson, A.S. (1988): A note on finite divisibility of some bivariate exponential and geometric distributions arising from a compounding process. Sankhya 46, A, 102-109.

[5] Downton, F (1970): Bivariate exponential distributions in reliability theory. J.R. Statis. Soc. B 32, 408-417.

[6] Gaver, D.P and Lewis, P.A.W (1980): First-order autoregressive gamma sequences and point processes. Adv. Appl. Prob. 12, 727-745.

[7] Gumbel, E.J. (1960): Bivariate exponential distributions. J. Amer. Statist. Assoc. 55, 698-707.

[8] Hawkes, A.G (1972): A bivariate exponential distribution with applications to reliability. J.R. Statist. Soc. B 34: 129-131.

[9] Jacobs, P.A and Lewis, P.A.W (1978a): Discrete time series generated by mixtures I: Correlational and runs properties. J.R. Statist. Soc. B 40, 94-105.

[10] Jacobs, P.A and Lewis, P.A.W (1978b): Discrete time series generated by mixtures II: Asymptotic properties. J.R. Statist. Soc. B 40, 222-228.

[11] Jacobs, P.A and Lewis, P.A.W (1983): Stationary discrete autoregressive moving average time series generated by mixtures. J. Times Series Anal. 4, 18-36.

[12] Langberg, N.A. and Stoffer, D.S (1987): Moving average models with bivariate exponential and geometric distributions. J. Appl. Prob. 24, 48-61.

[13] Lawrence, A.J. and Lewis, P.A.W (1977): A moving average exponential exponential point process (EMAI) J. Appl. Prob. 14, 98-113.

[14] Lawrence, A.J. and Lewis, P.A.W (1980): The exponential autoregressive moving average process EARMA. J.R. Statist. Soc, B 42, 150-161.

[15] Lawrence, A.J. and Lewis, P.A.W (1981): A new autoregressive times series model in exponential variables (NEAR(1)). Adv. Appl. Prob. 13, 826-845.

[16] Lawrence, A.J. and Lewis, P.A.W (1985): Modeling and residual analysis of nonlinear autoregressive time series in exponential variables J. R. Statist. Soc. B 47, 165-202.

[17] Marshall, A.W. and Olkin, I. (1967): A multivariate exponential distribution. J. Amer. Statist. Assoc. 62, 30-44.

[18] Paulson, A.S. (1973): A characterization of the exponential distribution and a bivariate exponential distribution. Sankhya A 35, 69-78.

[19] Raftery, A.E. (1984): A continuous multivariate exponential distribution. Commun. Statist. Theory Meth. A 13, 947-965.