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On the Payoff Valuations of Investment Strategies: A Case of Multiple Investment Companies

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Abstract

We consider the payoff valuation of an investor for investing in N investment companies. The investor paid some percentage as cost of running the investment to the investment companies. Since stock price is naturally random over time, we assume that it follows a standard geometric Brownian motion with a consumption process $\Lambda_t, 0 \le t \le T$ which is a nonnegative and $F-$ adapted process such that $\int_1^T \Lambda_t dt < \infty$. This consumption process is the sum total amount consumed by the investment companies from time period t=0 to time period T on behalf of the investor. We are to maximize the investor's investment in N companies. We also determine among the companies, which of them yield the highest returns at time t. We find that investors may not invest in some of the companies as a result of poor performance that arises from the high risk involve in the investments.

Keywords: Payoff valuation; Investment strategy; Stock price; Stochastic.

1.0 Introduction

An investor invested S_0^i in i investment company at time $t = 0$. At period $t > 0$, the investor is expected to have S_t^i amount from investment company i. The investor incurred a cost of $\Lambda_t = \Lambda_t^i$ for the investment to investment company i at time t. Let
$G(S_t^i)$ be the payoff function of investment company i at period t, so that
$G(S_t^i) = S_t^i - \Lambda_0 - \int_1^T \Lambda_t dt$, where T is the terminal period of the investment and

$\int_1^t \Lambda_s ds,$ represents the total amount spent (costs) in the investment period up to time t by the investment companies on behalf of the investor. The cost is charge on only the profit from the investment. In this paper, $\Lambda_0 = S_0$ represents the initial costs (i.e, initial investment). The investor aim is to maximize her investment. In this paper, we adopt the approach of Black-Scholes model for option pricing process. We did not look at our work as option but as investment settings involving transaction costs and competitive markets.
This paper is based on previous research work. [3] constructed a portfolio process that involved a riskless asset and a risky asset. The risk associated with such portfolio process cannot be hedged out completely, therefore the need for

diversification. [10] constructed a discounted portfolio process involving a riskless asset and a risky asset. [6] showed that a hedge portfolio could be constructed for an option to buy at the historical maximum, and that closed-form valuation formulas exist in the European case. [6] developed another valuation model, which separates the lookback option into two underlying options and further give the ability to price a European option on dividend paying assets. [9] used Morte Carlo simulation with a specific variance reduction method to compute the price of fixed-strike average-rate options. [5] derived explicit valuation formulas of most European barrier options, as well as some upper bounds in the American case. [2] analyzed the problem of pricing path-dependent contingent claims. [12] considered the valuation of put spread option (i.e. an option to sell) of futures contract under Black-Scholes setting. [1] obtained analytical solution for the optimal terminal portfolio values and derived solutions for the portfolio strategies and numerical simulations under a multivariant Black-Scholes framework. In this paper, we construct and diversify the investor's investment into N number of stocks and a riskless bond with all the companies having a unique constant interest rate. We also determine the optimal investment strategy of the investor at time t, adopting Black-Scholes derivative pricing process. We assume in this paper that the market is complete and frictionless i.e the market is characterized with transaction costs.

2.0 Basic Notations and Definitions:

S_t – *the stock price at time t*;

Λ_t – *the amount spent for managing the investment by the companies at time* $t > 0$;

Λ_0 – *the initial costs (i.e., initial investment)*;

r – *the percentage amount of return from bond market*;

S_0 – *the initial price of the stock*;

σ - the volatity of the stock;

μ - the instantaneous expected return from the stock.

r_{c_i} – *the percentage amount charge as costs of managing the investment by investmet company i*;

T – *the terminal period*;

N – *the number of risky assets*;

R^i – *the net payoff from investment company i*;

Definition1: The total return , R, per stock is the sum of payoff and initial price of the stock.
In other words, $R = f(S_t) + S_0$.

Definition 2: Let Ω be a non-empty set. Let T be a fixed positive number, and assume that for each $t \in [0, T]$ there is a $\sigma -$ algebra $F(t)$. Assume further that if $s \le t$, then every set in $F(s)$ is also in $F(t)$. Then we call the collection of $\sigma -$ algebra $F(t)$, $0 \le t \le T$, a filtration. In other words, let (Ω, F) denote a measurable space. A family of a $\sigma -$ algebra a $\{F(t)\}_{t \ge 0}$, where a $F(s) \subseteq F(t) \subseteq F,\ for\ 0 \le s \le t$ is called a filtration on (Ω, F).
A filtration tells us the information we will have at future times. More precisely, when we get to time t, we will know for each set in $F(t)$ whether the true w lies in that set.

Definition 3: Let Ω be a non-empty sample space equipped with a filtration $F(t)$, $0 \le t \le T$. Let G(t) be a collection of random variables indexed by $t \in [0, T]$. We say that this collection of random variables is an adapted stochastic process if, for each t, the random variable G(t) is $F(t)$-measurable.
In this work, asset prices and wealth process are all adapted to a filtration that we regarded as a model of the flows of public information.

Definition 4: An adapted stochastic process is a collection of random variables $\{G(t),\ 0 \le t \le T\}$ also indexed by time such that for every t, G(t) is $F(t)-measurable;$ the information at time t is sufficient to evaluate the random variable X(t).

In this work, we think of G(t) as the wealth generated from the price of assets at time t and $F(t),\ 0\le t\le T,$ as the information obtained by watching all the prices in the market up to time t.

Definition 5: Let (Ω, F, P) be a probability space. For each $w\in\Omega,$ suppose there is a continuous function $W(t)\ of\ \ t\ge 0$ that satisfies $W(0)=0$ and that depends on w. Then $W(t),\, t\ge 0$ is a Brownian motion if for all $0=t_0<t_0<...<t_m$ the increments

$$W(t_1)=W(t_1)-W(t_0)\,,W(t_2)-W(t_1),...,W(t_m)-W(t_{m-1})$$

are independent and each of these increments is normally distributed with

$$E[W(t_{i+1})-W(t_i)]=0$$

$$Var[W(t_{i+1})-W(t_i)]=t_{i+1}-t_i.$$

Definition 6: Let (Ω, F, P) be a probability space on which is defined a Brownian motion $W(t),\, t\ge 0$. A filtration for the Brownian motion is a collection of $\sigma-$algebra of $F(t),\, t\ge 0,$ satisfying

- (Information accumulates) For $0\le s\le T,$ every set in $F(s)$ is also in $F(t).$ In other words, there is at least as much information available at the later time $F(t)$ as there is at the earlier time $F(s)$.
- (Adaptivity) For each $t\ge 0,$ the Brownian motion $W(t)$ at time t is $F(t)-measurable.$ In other words, the information available at time t is sufficient to evaluate the Brownian motion $W(t)$ at that time.
- (Independence of future increments). For $0\le s\le u,$ the increment $W(u)-W(t)$ is independent of $F(t)$. In other words, any increment of the Brownian motion after time t is independent of the information available at time t.

Definition 7: Let $W(t),\, t\ge 0$ be a Brownian motion and let $F(t),\, t\ge 0$ be an associated filtration. An Ito process is a stochastic process of the form

$$G(t)=G(0)+\int_0^t \Delta(u)dW(u)+\int_0^t \Theta(u)du,$$

where G(0) is nonrandom and $\Delta(u)\ and\ \ \Theta(u)$ are adapted stochastic processes.

Definition 8: An n-dimensional Brownian motion is a process $W(t)=(W_1(t),...,W_n(t))$ with the following properties:

- Each $W_i(t),\ i=1,2,...,n$ is a one dimensional Brownian motion.
- if $i\ne j$, then the process $W_i(t)$ and $W_j(t)$ are independent.

Associated with an n-dimensional Brownian motion we have a filtration $F(t),\, t\ge 0$ such that the following holds.

- (Information accumulates) For $0\le s\le t,$ every set in $F(s)$ is also in $F(t)$.
- (Adaptivity) For each $t\ge 0$, the random vector $W(t)$ is $F(t)$-measurable.
- (Independence of future increment) For $0\le s\le u,$ the vector of increments $W(u)-W(t)$ is independent of $F(t)$.

Definition 9: A market model is complete if every derivative security associated with the model can be hedged.

Definition 10: (Stochastic process): A family $G=\{G(t)\,|\,0\le t<\infty\}$ of random variables

$$G(t):(\Omega,F)\to(\Re,B(\Re))$$

is called continuous time stochastic process. If

$G(t):(\Omega, F)\rightarrow(\Re^n, B(\Re^n))$, we say that G is an n-dimensional stochastic process. The family X may also be interpreted as a $X:[0,\infty)\times\Omega,\rightarrow S$:

$$G(t,w):=G(t) \ \ for\ all\ (t,w)\in[0,\infty)\times\Omega.$$

Definition 11: A cumulative consumption process is a non-negative progressively measurable process $\{\Lambda(t), 0\le t\le T\}$ with increasing, right continuous with left limits paths on [0, T], and with $\{\Lambda(0)=0\ and\ \Lambda(T)<\infty\ almost\ surely\}$.

Definition 12: The arbitrage-free condition means that there does not exist a zero investment portfolio that yields only positive earnings and strictly positive earnings with strictly positive probability.

Definition 13: (Risk Premium) Investors assume risk so that they are rewarded in the form of higher return. Hence, risk premium may be defined as the additional return investors expect to get, or investors earned in the past, for assuming additional risk. Risk premium may be calculated between two classes of securities that differ in their risk level.

3.0 Dynamics of the Stock Pricing Process

Our stock pricing process satisfies the stochastic differential equation,

$$\frac{dS_t^i}{S_t^i}=\mu_{it}dt+\sum_{j=1}^{N}\sigma_{ijt}dW_j(t) \tag{3.1}$$

This says that the infinitesimal change dS_t^i in the stock price at time t under i investment company, as a percentage of the value S_t^i, is given by a drift term $\mu_{it}dt$ and a 'fluctuation' or small movement upwards and downwards given by $\sum_{j=1}^{N}\sigma_{ijt}dW_j(t)$ at time t of i investment company. The randomness $W(t)=\{W_1(t),...,W_N(t)\}'; t\in[0,T]$ is an N-dimensional Brownian motion defined on a complete probability space (Ω,F,P), where P is the real world probability measure and $\sigma_{i,j}$ is the volatility of asset i with respect to changes in $W_j(t)$. $\mu:=\{\mu_i,...,\mu_N\}$ is the appreciation rate vector. Moreover, $\sigma=\{\sigma_{i,j}\}_{i,j}^N$ is the volatility matrix referred to as the coefficients of the market. The volatility matrix $\{\sigma=\{\sigma_{i,j}\}_{1\le i,j\le N}\}$ is progressively measurable with respect to the filtration F and satisfies the condition

$$\int_0^T\left(|r(t)|+\|\mu_i(t)\|+\sum_{i=1}^{N}\|\sigma_i(t)\|^2\right)dt<\infty\ almost\ surely, \tag{3.2}$$

where $\|.\|$ denotes the Euclidean norm in $\Re^N$ and where σ_i denotes the $i-th$ row of σ. The filtration $F=(F(t))_{t\ge0}$ represents the information structure generated by the Brownian motion and is assumed to satisfy equation (3.2).

It can be shown that equation (1) is solved by

$$S_t^i=S_0^i\exp\left[\mu_it+\sum_{j=1}^{N}\left(\sigma_{ij}W_j(t)-\frac{1}{2}\sigma_{ij}^2t\right)\right], i=1,...,N. \tag{3.3}$$

4.0 Dyanamics of Our Derivative Price

We can now define our derivative price dynamics. Let df_t be our derivative price, then for $f=\left(f_t^1,\ f_t^2,\Lambda\ ,f_t^N\right)^T=\left(f_t^i\right)^T_{t\ge0,\ i=1,\Lambda\ ,n}$, we have

$$df_t^i=\mu_{it}^f\ f_t^idt+\sum_{j=1}^{N}\sigma_{ijt}^{f^i}\ f_t^idW_j(t),$$

where $\mu_{it}^{f^i}$ *and* $\sigma_{ijt}^{f^i}$ are the drift and volatility of the derivative price respectively.

Lemma 1: Given that $df_t^i = \mu_{it}^{f^i} f_t^i dt + \sum_{j=1}^{N} \sigma_{ijt}^{f^i} f_t^i dW_j(t), i = 1,\ldots, N$

and

$$dS_t^i = \mu_{it} S_t^i dt + \sum_{j=1}^{N} \sigma_{ijt} S_t^i dW_j(t), i = 1,\ldots, N \text{, then for}$$

$$dv_t = r_r v_t dt \,,$$

we have
$$\frac{\mu_{it}^{f^i} - r_t}{\sigma_{ijt}^{f^i}} = \frac{\mu_{it} - r_t}{\sigma_{ijt}}, i, j = 1,\ldots, N .$$

Theorem 1: Let $f(S_t, t) = f^i(S_t^i, t), i = 1,\ldots, N$ and

$$df_t^i = \mu_{it} f_t^i dt + \sum_{j=1}^{N} \sigma_{ij}^{f^i} f_t^i dW_j(t), i = 1,\ldots, N .$$

Suppose that Lemma 1 holds, then

$$\frac{\partial f_t^i}{\partial t} + \sum_{j=1}^{N} \frac{1}{2} S_t^{i^2} \left(\sigma_{ijt}^{f^i}\right)^2 \frac{\partial^2 f_t^i}{\partial S_t^{i^2}} + r_t \frac{\partial f_t^i}{\partial S_t^i} S_t^i - r_t f_t^i = 0, i = 1,\ldots, N \tag{4.1}$$

This is the Black-Schole partial differential equation for derivative pricing process.

Theorem 2: Let $f(S_t^i, t) = \dfrac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G\left(S_t^i e^{r(T-t) + \sum_{j=1}^{N}\left(\sigma_{ij}\sqrt{T-t}\theta - \frac{1}{2}\sigma_{ij}^2(T-t)\right)} \right) e^{-\frac{1}{2}\sigma^2} d\theta, i = 1,\ldots, N$,

then $f(S_t^i, t), i = 1,\ldots, N$ solves equation (4.1).

5.0 Valuation of the Payoff Function

Given that the payoff function $G(S_T^i) = S_t^i - \left(\Lambda_0 + \int_1^T \Lambda_t dt \right)$, then

$$f_t^i = \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} G\left(S_0^i \exp\left[\mu_i (T-t) + \sum_{j=1}^{N} \sigma_{ij} \sqrt{T-t}\, \theta - \frac{1}{2} \sum_{j=1}^{N} \sigma_{ij}^2 (T-t) \right] e^{-\frac{1}{2}\theta^2} d\theta, i = 1,\ldots, N\right.$$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left(S_0^i \exp\left[\mu_i (T-t) + \sum_{j=1}^{N} \sigma_{ij} \sqrt{T-t}\, \theta - \frac{1}{2} \sum_{j=1}^{N} \sigma_{ij}^2 (T-t) \right] - \int_0^T \Lambda_t dt \right) e^{-\frac{1}{2}\theta^2} d\theta, i = 1,\ldots, N$$

A consumption process Λ_t, $0 \le t \le T$, is a nonnegative and $F-$adapted process such that

$$\Lambda_0 + \int_1^T \Lambda_t dt = \int_0^T \Lambda_t dt < \infty.$$

This is the sum total amount consumed by the investment companies from time period t=0 to time period T on behalf of the investor. The adapted condition means that the investor cannot anticipate the future.

Every investor wants to maximize her investment. Hence, it is expected that $S_t^i \ge \int_0^T \Lambda_t dt, i = 1,\ldots, N$. If $S_t^i = \int_0^T \Lambda_t dt$, then the investor neither experiences gain nor loss. Hence, the investment is at the money (i.e. the return from the investment is equal to consumption process). The investor will have maximum investment if

$$S_t^i > \int_0^T \Lambda_t dt, i = 1,\ldots, N .$$

Suppose that $S_t^i > \int_0^T \Lambda_t dt$, then

$$S_0^i \exp\left[\mu_i(T-t) + \sum_{j=1}^N \sigma_{ij}\sqrt{T-t}\,\theta - \frac{1}{2}\sum_{j=1}^N \sigma_{ij}^2(T-t)\right] > \int_0^T \Lambda_t dt$$

$$\Rightarrow \exp\left[\mu_i(T-t) + \sum_{j=1}^N \sigma_{ij}\sqrt{T-t}\,\theta - \frac{1}{2}\sum_{j=1}^N \sigma_{ij}^2(T-t)\right] > \int_0^T \frac{\Lambda_t dt}{S_0^i}$$

$$\Rightarrow \mu_i(T-t) + \sum_{j=1}^N \sigma_{ij}\sqrt{T-t}\;\theta - \frac{1}{2}\sum_{j=1}^N \sigma_{ij}^2(T-t) > \log\left[\int_0^T \frac{\Lambda_t dt}{S_0^i}\right], i = 1,\ldots, N$$

In order to obtain the integration region $(ie.\,[\theta_{\min},\infty])$ where function is nonzero, we isolate the integration variable θ.

$$\theta > \frac{\log\left[\int_0^T \frac{\Lambda_t dt}{S_0^i}\right] - \mu_i(T-t) + \frac{1}{2}\sum_{j=1}^N \sigma_{ij}^2(T-t)}{\sum_{j=1}^N \sigma_{ij}\sqrt{T-t}}, i = 1,\ldots, N$$

The critical value $\theta_{\min}$ may be defined as

$$\theta_{\min} = \frac{\log\left[\frac{1}{S_0^i}\int_0^T \Lambda_t dt\right] - \mu_i(T-t) + \frac{1}{2}\sum_{j=1}^N \sigma_{ij}^2(T-t)}{\sum_{j=1}^N \sigma_{ij}\sqrt{T-t}}, i = 1,\ldots, N. \qquad (5.1)$$

So, that, the function

$$G(S_t^i) = S_0^i \exp\left[\mu_i(T-t) + \sum_{j=1}^N \sigma_{ij}\sqrt{T-t}\,\theta - \frac{1}{2}\sum_{j=1}^N \sigma_{ij}^2(T-t)\right] - \int_0^T \Lambda_t dt, i = 1,\ldots, N$$

for $\theta > \theta_{\min}$ and zero otherwise.

The function $G(S_t^i) > 0, i = 1,\ldots, N$, hence the derivative pricing becomes

$$f_t^i = \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\left(\int_{\theta_{\min}}^{\infty} S_0^i \exp\left[\mu_i(T-t) + \sum_{j=1}^N \sigma_{ij}\sqrt{T-t}\,\theta - \frac{1}{2}\sum_{j=1}^N \sigma_{ij}^2(T-t)\right] - \int_0^T \Lambda_t dt\right) e^{-\frac{1}{2}\theta^2} d\theta, i = 1,\ldots, N$$

$$= \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{\theta_{\min}}^{\infty} S_0^i \exp\left[\mu_i(T-t) + \sum_{j=1}^N \sigma_{ij}\sqrt{T-t}\,\theta - \frac{1}{2}\sum_{j=1}^N \sigma_{ij}^2(T-t)\right] e^{-\frac{1}{2}\theta^2} d\theta - \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{\theta_{\min}}^{\infty}\int_0^T e^{-\frac{1}{2}\theta^2} \Lambda_t dt d\theta$$

Let $\Phi = \frac{e^{-r(T-t)}}{\sqrt{2\pi}}\int_{\theta_{\min}}^{\infty} S_0^i \exp\left[\mu_i(T-t) + \sum_{j=1}^N \sigma_{ij}\sqrt{T-t}\,\theta - \frac{1}{2}\sum_{j=1}^N \sigma_{ij}^2(T-t)\right] e^{-\frac{1}{2}\theta^2} d\theta, i = 1,\ldots, N$

and

$$\Psi = -\int_{\theta_{\min}}^{\infty} r\int_0^T \Lambda_t \frac{e^{-r(T-t)}}{\sqrt{2\pi}} e^{-\frac{1}{2}\theta} dt d\theta .$$

Now,

$$\Phi = \frac{e^{(\mu_i - r)(T-t)}}{\sqrt{2\pi}} \int_{\theta_{\min}}^{\infty} S_0^i \exp\left[-\frac{1}{2}\,\theta^2 + \sum_{j=1}^{N} \sigma_{ij}\sqrt{T-t}\,\theta - \frac{1}{2}\sum_{j=1}^{N}\sigma_{ij}^2 (T-t) \right] d\theta$$

$$= \frac{S_0^i e^{(\mu_i - r)(T-t)}}{\sqrt{2\pi}} \int_{\theta_{\min}}^{\infty} \exp\left[-\frac{1}{2}\,\theta^2 - 2\sum_{j=1}^{N} \sigma_{ij}\sqrt{T-t}\;\theta + \sum_{j=1}^{N}\sigma_{ij}^2 (T-t) \right] d\theta$$

$$= \frac{S_0^i e^{(\mu_i - r)(T-t)}}{\sqrt{2\pi}} \int_{\theta_{\min}}^{\infty} \exp\left[-\frac{1}{2}\left(\theta - \sum_{j=1}^{N} \sigma_{ij}\sqrt{T-t} \right)^2 \right] d\theta, i = 1,\ldots,N. \tag{5.2}$$

Next,

$$\Psi = -\frac{1}{\sqrt{2\pi}} \int_{\theta\min}^{\infty} r \int_0^T \Lambda_t e^{-r(T-t)} e^{-\frac{1}{2}\theta^2} dt\, d\theta = -r\int_0^T \frac{e^{-r}(T-t)}{\sqrt{2\pi}} \Lambda_t dt \int_{\theta_{\min}}^{\infty} e^{-\frac{1}{2}\theta^2} d\theta$$

$$= -rN(-\theta_{\min}) \int_0^T e^{-r(T-t)} \Lambda_t dt, \tag{5.3}$$

where $N(\theta_{\min})$ is the standard normal cumulative probability density function.

We now express (5.1) in terms of h^+ *and* h^- rather than $\theta_{\min}$. By definition, the constants become

$$h^{\pm} = \frac{\log\left[\int_0^T \frac{\tilde{S}_t}{\Lambda_t dt} \right] \pm \frac{1}{2}\sum_{j=1}^{N}\sigma_{ij}^2 (T-t)}{\sum_{j=1}^{N} \sigma_{ij}\sqrt{T-t}}, i = 1,\ldots,N., \tag{5.4}$$

where $\tilde{S}_t^i$ is the forward amount $S_0^i e^{r(T-t)}, i = 1,\ldots,N$.

We now evaluate $-\theta_{\min}$ as appeared in Eq.(5.3)

$$-\theta_{\min} = -\left\{ \frac{\log\left[\int_0^T \frac{\Lambda_t}{S_0^i} dt \right] - r(T-t) + \frac{1}{2}\sum_{j=1}^{N}\sigma_{ij}^2 (T-t)}{\sum_{j=1}^{N} \sigma_{ij}\sqrt{T-t}} \right\}$$

$$= \frac{\log\left[\int_0^T \frac{\tilde{S}_t^i}{\Lambda_t} dt \right] - \frac{1}{2}\sum_{j=1}^{N}\sigma_{ij}^2 (T-t)}{\sum_{j=1}^{N} \sigma_{ij}\sqrt{T-t}}, i = 1,\ldots,N. \tag{5.5}$$

Obviously, $-\theta_{\min} = h^-$ *and* $\theta_{\min} + \sum_{j=1}^{N} \sigma_{ij}\sqrt{T-t} = h^+, i = 1,\ldots,N$. Hence, we write Ψ as follows:

$$\Psi = -rN(h^-) \int_0^T e^{-r(T-t)} \Lambda_t dt \tag{5.6}$$

From equation (5.2), we have that

$$\Phi = S_0^i \frac{e^{(\mu_i - r)(T-t)}}{\sqrt{2\pi}} \int_{\theta_{\min}}^{\infty} \exp\left[-\frac{1}{2}\left(\theta - \sum_{j=1}^{N} \sigma_{ij}\sqrt{T-t} \right)^2 \right] d\theta, i = 1,\ldots,N.$$

Let $\xi = \theta - \sum_{j=1}^{N} \sigma_{ij}\sqrt{T-t}, i = 1,\ldots,N.$, then $d\theta = d\xi$ and the lower limit of the integration variable $\theta = \theta_{\min}$, Let $\xi = \theta - \sum_{j=1}^{N}\sigma_{ij}\sqrt{T-t}, i=1,...,N.$, then $d\theta = d\xi$ and the lower limit of the integration variable $\theta = \theta_{\min}$, becomes the new lower limit $\xi = \theta_{\min} - \sum_{j=1}^{N}\sigma_{ij}\sqrt{T-t} = -h^{+}$.

Hence, the integral, Φ becomes

$$\Phi = \frac{S_0^i e^{(\mu_i - r)(T-t)}}{\sqrt{2\pi}} \int_{\xi=-h^+}^{\xi=\infty} e^{-\frac{1}{2}\xi^2} d\xi, i=1,...,N.$$

We now write the integral in terms of $N(x)$.

$$\Phi = S_0^i e^{(\mu_i - r)(T-t)} N(h^+), i=1,...,N.$$

But $f_t^i = \Phi + \Psi$

$$= S_0^i e^{(\mu_i - r)(T-t)} N(h^+) - rN(h^-)\int_0^T e^{-r(T-t)}\Lambda_t dt$$

$$= S_0^i e^{(\mu_i - r)(T-t)} e^{-r(T-t)} N(h^+) - rN(h^-)\int_0^T e^{-r(T-t)}\Lambda_t dt, i=1,...,N. \quad (5.7)$$

We now determine the integral

$$\int_0^T e^{-r(T-t)}\Lambda_t dt.$$

Which gives the following

$$\int_0^T e^{-r(T-t)}\Lambda_t dt = \frac{e^{-rT}}{r}\left[\Lambda_T e^{rT} - \Lambda_0 - \int_1^T e^{rt} d\Lambda_t\right]. \quad (5.8)$$

Substituting (5.8) into (5.7), we have

$$f_t^i = S_0^i e^{\mu_i(T-t)} e^{-rT} \cdot e^{rt} N(h^+) - rN(h^-)\left[\frac{e^{-rT}}{r}\left\{\Lambda_T e^{rT} - \Lambda_0 - \int_1^T e^{rt} d\Lambda_t\right\}\right], i=1,...,N.$$

$$= e^{-rT}\left\{S_0^i e^{\mu_i(T-t)} e^{rt} N(h^+)\right\} - N(h^-)\left[\exp[-rT]\left(\Lambda_T e^{rT} - \Lambda_0 - \int_1^T e^{rt} d\Lambda_t\right)\right], i=1,...,N.$$

$$= e^{-rT}\left[S_0^i e^{\mu_i(T-t)} e^{rt} N(h^+) - N(h^-)\left(\Lambda_T e^{rT} - \Lambda_0 - \int_1^T e^{rt} d\Lambda_t\right)\right], i=1,...,N.$$

$$= e^{-rT}\left[E(S_{T-t}^i) e^{rt} N(h^+) - N(h^-)\left(\Lambda_T e^{rT} - \Lambda_0 - \int_1^T e^{rt} d\Lambda_t\right)\right], i=1,...,N., \quad (5.9)$$

where $E[S_{T-t}^i] = S_0^i e^{\mu_i(T-t)}, i=1,...,N$ and $\Lambda_t = \exp[r_{c_i} t], i=1,...,N, t>0$, where r_{c_i} is the perentage costs for running the investment by investment companies i.

6.0 Numerical Results

Table 1 shows the parameters and the payoff from eight investment companies after one year. Table 2 shows the payoff after two years and Table 3 shows the payoff after nine years.

Table 1: Payoff and Percentage Payoff Per Stock After a Year

i	r_{c_i} %	σ_{ij} %	S_0^i $(in\ Naira$	Λ^i $(in\ Naira$	r %	μ_i %	h^+	h^-	$N(h^+)$	$N(h^-)$	f^i $(in\ Naira)$	$R^i = f^i + S_0^i$ $(in\ Naira)$	% Increase
1	0.10	0.35	2.0	1.11	0.05	0.25	2.16	1.81	0.9846	0.9648	2.7439	4.7439	137.2
2	0.09	0.33	1.6	1.09	0.05	0.20	1.59	1.26	0.9441	0.8962	1.8125	3.4125	113.2
3	0.15	0.49	2.6	1.16	0.05	0.30	2.20	1.71	0.9861	0.9564	3.9378	6.5378	151.5
4	0.18	0.52	3.0	1.20	0.05	0.40	2.37	1.85	0.9911	0.9678	5.0577	8.0577	168.6
5	0.08	0.28	1.1	1.08	0.05	0.15	0.48	0.20	0.6844	0.5793	0.6496	1.7496	059.1
6	0.12	0.36	2.4	1.13	0.05	0.24	2.61	2.25	0.9955	0.9878	3.5036	5.9036	146.0
7	0.14	0.39	2.3	1.15	0.05	0.28	2.33	1.94	0.9901	0.9738	3.3315	5.6315	144.8
8	0.20	0.49	3.2	1.22	0.05	0.45	2.62	2.13	0.9956	0.9834	5.6809	8.8809	177.5

Table 2: Payoff and Percentage Payoff Per Stock After Two Years

i	r_{c_i} %	σ_{ij} %	S_0^i (*in Naira*)	Λ^i (*in Naira*)	r %	μ_i %	h^+	h^-	$N(h^+)$	$N(h^-)$	f^i (*in Naira*)	$R^i = f^i + S_0^i$ (*in Naira*)	% Increase
1	0.10	0.35	2.0	1.35	0.05	0.25	2.97	2.48	0.9985	0.9934	3.0648	05.0648	153.2
2	0.09	0.33	1.6	1.31	0.05	0.20	2.66	2.20	0.9961	0.9861	2.0129	03.6129	125.8
3	0.15	0.49	2.6	1.57	0.05	0.30	2.62	1.93	0.9956	0.9732	4.5133	07.1133	173.6
4	0.18	0.52	3.0	1.72	0.05	0.40	2.68	1.94	0.9964	0.9738	6.2901	09.2901	209.7
5	0.08	0.28	1.1	1.27	0.05	0.15	2.13	1.74	0.9834	0.9591	0.8843	01.9843	080.4
6	0.12	0.36	2.4	1.43	0.05	0.24	3.24	2.74	1.0000	0.9969	3.8059	06.2059	158.6
7	0.14	0.39	2.3	1.52	0.05	0.28	2.92	2.37	0.9982	0.9911	3.7443	06.0443	162.8
8	0.20	0.49	3.2	1.82	0.05	0.45	2.87	2.18	0.9979	0.9854	7.3490	10.5490	229.4

Table 3: Payoff and Percentage Payoff Per Stock After Nine Years

i	r_{c_i} %	σ_{ij} %	S_0^i (*in Nai*	Λ^i (*in Naira*	r %	μ_i %	h^+	h^-	$N(h^+)$	$N(h^-)$	f^i (*in Naira*)	$R^i = f^i + S_0^i$ (*in Naira*)	% Increase
1	0.10	0.35	2.0	2.46	0.05	0.25	3.07	2.02	0.9999	0.9783	009.4742	011.4742	0473.7
2	0.09	0.33	1.6	2.25	0.05	0.20	3.02	2.03	0.9993	0.9788	004.1231	005.7231	0257.7
3	0.15	0.49	2.6	3.86	0.05	0.30	2.60	1.13	0.9953	0.8708	019.6863	022.2863	0757.2
4	0.18	0.52	3.0	5.05	0.05	0.40	2.56	1.00	0.9948	0.8413	059.3310	062.3310	1977.7
5	0.08	0.28	1.1	2.05	0.05	0.15	2.99	2.15	0.9986	0.9842	000.9109	002.0109	0082.8

6	0.12	0.36	2.4	2.94	0.05	0.24	3.11	2.03	1.0000	0.9788	010.1818	012.5800	0424.2
7	0.14	0.39	2.3	3.53	0.05	0.28	2.86	1.69	0.9979	0.9545	013.9188	016.2188	0605.2
8	0.20	0.49	3.2	6.05	0.05	0.45	2.62	1.15	0.9956	0.8749	100.7370	103.9370	3148.0

Discussion:

From table 1, we have that after a year, company 8 has the net payoff of 5.68 naira with percentage increase of 177.5 and company 5 has the lowest net payoff of 0.65 naira with percentage increase of 59.1. From table 2, we also have that after two years, company 8 also has the highest percentage increase (229.4%) and company 5 has the lowest percentage increase (80.4%). Furthermore, from table 3, we have that after nine years of operation, company 8 continue to dominate in terms of returns from the investment while company 5 continue to yield the lowest return. Therefore, investors are adviced to invest more of their resources into company 4 and 8, since they yield the highest percentage returns. Again, investors are adviced not to invest their resource into company 5, since the percentage increase is not encouraging.

We found that investors should invest more of their short position into company 4 and 8 as they continue to yield the highest returns. We also found that there is the need for the investors to invest into other companies in order to hedge out the risks associated with their investment. Again, we found that the returns from company 5 is not encouraging when compared with other stocks. We therefore conclude that investors should not invest their resources into company 5.

Conclusion

Since company 4 and 8 yield the highest percentage increase of wealth, we conclude that investors should invest more of their short positions (i.e. resources) into these companies so as to maximize returns. There is also the need for the investors to invest into other companies in order to reduce risks associated with their investment. Again, investors are adviced not to invest their resource into company 5, since the return from such stock is not encouraging when compared with other stocks as we can see in Table 1 to 3.

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