Journal of the Nigerian Association of Mathematical Physics
***Volume* 16 (May, 2010), pp 503 - 514**
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On The Existence and Uniqueness of Wealth Dynamics and Derivation of Optimal Portfolio-Consumption Strategies For an Investor

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Abstract

We consider the existence and uniqueness of investor's wealth dynamics and optimization of investment portfolio and consumption processes. We described the existence and uniqueness of our dynamics using already existing approaches. Using the method of successive approximation, we found that

$$P\left\{\int_0^t \mu\left(s, X^{(k)}(s)\right)ds \to \int_0^t \mu\left(s, \tilde{X}(s)\right)ds\right\}=1$$

and

$$P\left\{\int_0^t \sigma\left(s, X^{(k)}(s)\right)dW(s) \to \int_0^t \mu\left(s, \tilde{X}(s)\right)dW(s)\right\}=1,$$

in probability as $k \to \infty$ for each $t \in [0,T]$. This shows that the limit process X(t) satisfies our Stochastic wealth equation (1.9). We assume that the investor invested his short positions into a riskless asset and N risky assets. We also assume that the market is complete, arbitrage-free and continuously open. We derived the optimal portfolio as well as the optimal consumption strategies for an investor.

Keywords: Optimal Portfolio, Consumption, Wealth Dynamics, Existence and Uniqueness.

1.0 Introduction

We consider the intertemporal consumption optimization problem. We assume that an infinitely-lived small individual with initial capital, works only before age, T and consume continuously throughout his/her lifetime. The consumption terminates when the individual died. At that point, the wealth becomes zero. This paper is based on the already existing works. [7] and [4], who provide an approximate solution and analytical results to the intertemporal consumption problem. [4] assumed that the asset return is non-stochastic. [8] considered a tractable model of precautionary savings in continuous time and assume that the uncertainty is about the timing of the income loss in addition to the assumption of non-stochastic asset return. [2] considered labour supply flexibility and portfolio choice of individual life cycle. They determine the objective of maximizing the expected discounted lifetime utility and assume that the utility function has two argument (consumption and labour/leiture). [1] used the quadratic utility function that has the characterization of linear marginal utility. This utility function is not attractive in describing the behaviour of individual towards risk as it implies increasing absolute risk aversion. [9] invesigated the continuous-time consumption model with stochastic asset returns and stochastic labour income using Martingale approach. He derived analytically a closed-form solution for the consunption and labour supply process. [2] concluded that labour income induces the individual to invest an additional amount of wealth to the risky asset. They show that labour income and investment choices are related, while they failed to analyze the optimal consumption process. [9] analyzed the optimal consumption process and treated consumption and leisure as a ‘’composite’’ good. He assumes

that people work for their whole lifetime which is unrealistic. In this paper, we assume that people work up to a certain age, T and continue to enjoy their investment returns throughout their lifetime. The above authors considered investment of the individual income stream into a riskless and a risky assets using Martingale method. [6] established the global existence and uniqueness of Stochastc differential equation. We consider in this paper, the existence and uniqueness locally as related to our work. Again, we consider the investment of individual short position into a riskless asset and N risky assets. This to a great extent minimize the risk associated with the investment. We also consider the the derivation of optimal portfolio strategies for the investor. Our aim is to find the optimal value functions of investment portfolio and consumption strategies using Hamilton-Jacobi-Bellman (HJB) equation.

1.2 Problem Formulation

Let the short position of the investor be invested into a riskless asset with a nominal return r and N risky assets with instantaneous expected gross return $\mu_i, i = 1,..., N.$ When the investor retires, his/her post-retirement consumption is financed by his/her investment and savings at time $t \le T$. The aim of the investor is to maximize his/her expected lifetime utility by choosing the portfolio and optimal consumption at time $t \le T$.

Let $X(t) \equiv X^{\Delta,C}(t)$ be the wealth process, where $\Delta(t)$ is the portfolio process at time t. Let $\Delta_i(t)$ be the proportion of wealth invested in the risky asset i at time t, then $1 - \sum_{i=1}^{N} \Delta_i(t)$ is the proportion wealth invested in the riskless asset.

1.3 Continuous-time model of financial stock markets

1.3.1 Riskless bond:

The rickless bond with price process, $B(t)$ is given by the dynamics

$$\frac{dB(t)}{B(t)} = r(t)dt \qquad (1.1)$$
$$B(0) = 1,$$

where $r(t)$ represents the short term interest rate at time t . It is also known as norminal return.

1.3.2 Financial stock price dynamics:

The n assets are the risky financial assets, whose prices are denoted by $S_i(t), i = 1,2,..., N.$ The dynamics of $S_i(t)$ given by

$$\frac{dS_i(t)}{S_i(t)} = \mu_i(t)dt + \sum_{j=1}^{N} \sigma_{i,j}(t) dW_j^S(t), \; i = 1,2,..., N, \qquad (1.2)$$
$$S_i(0) = s_i \in (0, +\infty), \; i = 1,2,..., N$$

where the randomness $W(t) = \{W_1(t),..., W_N(t)\}^T; t \in [0,T]$ is N-dimensional Brownian motion defined on a complete probability space (Ω, F, P) , where P is the real world probability measure and

$\sigma_{i,j}(t)$ is the volatility of asset i at time t with respect to changes in $W_j(t)$. $\mu(t) := \{\mu_1(t),...,\mu_N(t)\}$ is the appreciation rate vector.

Moreover, $\sigma(t) = \{\sigma_{i,j}(t)\}_{i,j}^{N}$ is the volatility matrix referred to as the coefficients of the market. The volatility matrix $\{\sigma(t) = \{\sigma_{i,j}(t)\}_{1 \le i,j \le N}, 0 \le t \le T\}$ are progressively measurable with respect to the filtration F and satisfy the condition

$$\int_0^T \left(|r(t)| + \|\mu_i(t)\| + \sum_{i=1}^{N} \|\sigma_i(t)\|^2 \right) dt < \infty \text{ almost surely.} \tag{1.3}$$

where $\|.\|$ denotes the Euclidean norm in $\Re^N$ and where $\sigma_i(t)$ denotes the $i-th$ row of $\sigma(t)$. The filtration $F = (F(t))_{t \ge 0}$, represents the information structure generated by the Brownian motion and is assume to satisfy equation (1.3).
We assume that the financial market is arbitrage-free, complete and continuously open between time 0 and T, where T is a strictly positive real number i.e., there is only one process $\theta(t)$ satisfying

$$\theta(t) = (\sigma_i)^{-1}(t)(\mu_i(t) - r(t)), 0 \le t \le T, 1 \le i \le N,$$

with *where* $\sigma(t,w)$ is non-singular, for $(\lambda \otimes P)$ almost everywhere and $(t,w) \in [0,T] \times \Omega$. The exponential process

$$Z(t) = \exp\left[-\int_0^t (\theta)^T(u)dW(u) - \frac{1}{2}\int_0^t \|\theta(u)\|^2 du \right], 0 \le t \le T. \tag{1.4}$$

is assumed to be a martingale, and the risk-neutral equivalent martingale measure, denoted by

$$\widetilde{P}(A) = E[Z(T)1_A], A \in F(T). \tag{1.5}$$

Using Ito lemma on Eq.(1.1) and (1.2), we respectively obtain the following solutions

$$B(t) = B(0)\exp\left[\int_0^t r(u)du \right] \tag{1.6}$$

$$= \exp\left[\int_0^t r(u)du \right]$$

$$B(0) = 1;$$

$$S_i(t) = S_i(0)\exp\left[\int_0^t \{\mu_i(u) - \frac{1}{2}\sum_{j=1}^{N} \sigma_{i,j}(u)^2\} + \int_0^t \sum_{j=1}^{N} \sigma_{i,j}(u)dW_j(u) \right]; \tag{1.7}$$

Definition 1: Let $\Delta(t)$ portfolio process and $C(t)$ the consumption process, then the pair (Δ, C) is said to be self-financing if the corresponding wealth process $X^{\Delta,C}(t), t \in [0,T]$, satisfies

$$dX^{\Delta,C}(t) = \sum_{i=1}^{N} \Delta_i(t) X^{\Delta,C}(t) \frac{dS_i(t)}{S_i(t)} + \left(1 - \sum_{i=1}^{N} \Delta_i(t) \right) X^{\Delta,C}(t) \frac{dB(t)}{B(t)} - C(t)dt. \tag{1.8}$$

The requirement of being self-financing states that the change in wealth must equal the different of the capital gains and infinitesimal consumption. Substituting the assets returns in Eq.(1.1) and (1.2) into Eq.(1.8), we obtain the following

$$dX^{\Delta,C}(t) =$$
$$\sum_{i=1}^{N}\Delta_i(t)X^{\Delta,C}(t)\left\{\mu_i(t)dt+\sum_{j=1}^{N}\sigma_{i.j}(t)dW_j\ (t)\right\}+\left(1-\sum_{i=1}^{N}\Delta_i(t)\right)X^{\Delta,C}(t)r(t)dt-C(t)dt$$

$$= X^{\Delta,C}(t)r(t)dt+\sum_{i=1}^{N}\Delta_i(t)X^{\Delta,C}(t)(\mu_i(t)-r)dt+\sum_{i,j=1}^{N}\sigma_{i.j}(t)\Delta_i(t)X^{\Delta,C}(t)dW_j\ (t)-C(t)dt$$
$$=\left(X^{\Delta,C}(t)r(t)+\sum_{i=1}^{N}\Delta_i(t)X^{\Delta,C}(t)(\mu_i(t)-r)-C(t)\right)dt+\sum_{i,j=1}^{N}\sigma_{i.j}(t)\Delta_i^S(t)X^{\Delta,C}(t)dW_j\ (t) \quad (1.9)$$

This is our stochastic differential equation which represents the wealth process of the investor.

1.4 Existence and Uniqueness of our Stochastic Wealth Dynamics

From Eq.(1.9), setting

$$\mu(t,X(t)) = X^{\Delta,C}(t)r(t)+\sum_{i=1}^{N}\Delta_i(t)X^{\Delta,C}(t)(\mu_i(t)-r)-C(t) \text{ and}$$

$$\sigma(t,X(t)) = \sum_{i,j=1}^{N}\sigma_{i.j}(t)\Delta_i^S(t)X^{\Delta,C}(t)dW_j\ (t)$$

We have the following stochastic differential equation:

$$dX(t) = \mu(t,X(t))dt+\sigma(t,X(t))dW(t) \quad (1.10)$$

Our aim is to show that if Eq.(1.10) exists and is unique, then Eq.(1.9) is exists and is unique.
When we integrate Eq.(1.10), we obtain the following stochastic integral equation:

$$X(t) = X(0)+\int_0^t\mu(s,X(s))ds+\int_0^t\sigma(t,X(t))dW(t) \quad (1.11)$$

Where the first integral is a Lebesgue (or Riemann) integral for each sample path and the second integral is an Ito integral.
Let $\mu,\sigma:[0,T]\times\Re^N\to\Re.$ Then, we have the following assumptions for μ and σ.

Assumptions:

(i) $\mu=\mu(t,x)$ and $\sigma=\sigma(t,x)$ are jointly L^2-measurable in $(t,x)\in[0,T]\times\Re^N$;

(ii) There exists a constant K>0 such that

$$|\mu(t,x)-\mu(t,y)|\le K|x-y| \text{ and}$$
$$|\sigma(t,x)-\sigma(t,y)|\le K|x-y| \text{ for all } t\in[0,T] \text{ and } x,y\in\Re.$$

(iii) There exists a constant K>0 such that

$$|\mu(t,x)|^2\le K^2(1+|x|)^2 \text{ and}$$
$$|\sigma(t,x)|^2\le K^2(1+|x|)^2 \text{ for all } t\in[0,T] \text{ and } x\in\Re.$$

(iv) $X(0)$ is A_0-measurable with $E\left(|X(0)|^2\right)<\infty,$ where A_0 is a $null-\sigma-$algebra.

(v) There exists a constant K>0 such that

$$|C(t,x)-C(t,y)|\le K|x-y| \text{ for all } t\in[0,T] \text{ and } x,y\in\Re.$$

The assumption (ii) provides the key estimates in both the proofs of existence and uniqueness by the method of successive approximations. Hence, we required the following Gronwall inquality.

Lemma 1: Let $f, g : [0,T] \to \Re$ be integrable with $0 \le f(t) \le g(t) + L\int_0^t f(s)ds$ for $t \in [0,T]$, where $L > 0$. Then

$$f(t) \le g(t) + L\int_0^t \exp[L(t-s)]g(s)ds. \tag{1.12}$$

Lemma 2: Suppose that assumption (i) and (ii) hold, then the solution of Eq.(1.10) with the same initial value and
Brownian motion are pathwise unique.

Let $X(t)$ and $\tilde{X}(t)$ be two solution to Eq.(10) on the time interval $[0,T]$. Let the solutions be almost surely locally lipschitz continuous sample paths on $\mu(t,x)$ and $\sigma(t,x)$ in the spatial variable x. For $\tilde{N} > 0$ and $t \in [0,T]$, we define the following truncation process:

$$H_t^{(\tilde{N})} = \begin{cases} 1 : \left|\tilde{X}_s(w)\right|, \left|X_s(w)\right| \le N \;\; for\ 0 \le s \le t \\ 0 : elsewhere. \end{cases}$$

We have that $H_t^{(\tilde{N})}$ is $\mathrm{A}_t -$measurable and $H_t^{(\tilde{N})} = H_t^{(\tilde{N})} H_s^{(\tilde{N})}$ $for\ 0 \le s \le t.$

Hence, we write

$$G_t^{(\tilde{N})} = H_t^{(\tilde{N})}\int_0^t H_t^{(\tilde{N})}\left(\mu(s, X(s)) - \mu(s, \tilde{X}(s)\right)ds + H_t^{(\tilde{N})}\int_0^t H_s^{(N)}\left(\sigma(s, X(s)) - \sigma(s, \tilde{X}(s)\right)dW(s)$$

$$= H_t^{(\tilde{N})}\left(X(t) - \tilde{X}(t)\right).\ for\ 0 \le s \le t.$$

(1.13)

Applying assumption (ii), we have the following:

$$\max\left\{\left|H_s^{(\tilde{N})}\left(\mu(s, X(s)) - \mu(s, \tilde{X}(s)\right)\right|, \left|H_s^{(\tilde{N})}\left(\sigma(s, X(s)) - \sigma(s, \tilde{X}(s)\right)\right|\right\}$$

$$\le KH_s^{(\tilde{N})}\left|X(t) - \tilde{X}(t)\right| \le KH_s^{(\tilde{N})}\left(\left|X(t)\right| + \left|\tilde{X}(t)\right|\right),\ for\ 0 \le s \le t.$$

(1.14)

We now define the following sequences:

$$\Lambda(s) = \sup\left\{t \ge 0 : \left|X(t)\right| \le M\right\} \text{ and } \tilde{\Lambda}(s) = \sup\left\{t \ge 0 : \left|\tilde{X}(t)\right| \le M\right\}.$$

Therefore,

$$\tilde{N} = \max\left\{\Lambda(s), \tilde{\Lambda}(s)\right\}.$$

(1.15)

Using Eq.(1.15), Eq.(1.14) becomes

$$KH_s^{(\tilde{N})}\left|X(t) - \tilde{X}(t)\right| \le KH_s^{(\tilde{N})}\left(\left|X(t)\right| + \left|\tilde{X}(t)\right|\right) \le 2K\tilde{N},\ for\ 0 \le s \le t.$$

(1.16)

This shows that the second moments exist for $G_t^{(\tilde{N})}$. Using the inequality $(a+b)^2 \le 2\left(a^2 + b^2\right)$, we now evaluate

$$E\left[\left|G_t^{(\tilde{N})}\right|^2\right] = E\left|H_t^{(\tilde{N})}\left(X(t) - \tilde{X}(t)\right)\right|^2$$

$$\leq 2E\left(\left|\int_0^t H_s^{(\tilde{N})}\left(\mu(s,X(s))-\mu(s,\tilde{X}(s))\right)ds\right|^2\right)+2E\left(\left|\int_0^t H_s^{(\tilde{N})}\left(\sigma(s,X(s))-\sigma(s,\tilde{X}(s))\right)dW(s)\right|^2\right)$$

(1.17)

Considering the second term, we use the Ito isometry, we have

$$E\left(\int_0^t H_s^{(\tilde{N})}\left(\sigma(s,X(s))-\sigma(s,\tilde{X}(s))\right)dW(s)\right)^2=E\left(\int_0^t \left(H_s^{(\tilde{N})}\right)^2\left|\sigma(s,X(s))-\sigma(s,\tilde{X}(s))\right|^2 ds\right)$$

(1.18)

Applying the Classical Holder's inquality (or Cauchy Schwartz inequality) for the Lebsegue integrals which states that for $p,q\in(1,\infty)$, $with\ \frac{1}{p}+\frac{1}{q}=1$, the following inquality is satisfied.

$$\int\left|f(x)g(x)\right|d\hat{\mu}(x)\leq\left(\int\left|f(x)\right|^p d\hat{\mu}(x)\right)^{\frac{1}{p}}\left(\int\left|f(x)\right|^q d\hat{\mu}(x)\right)^{\frac{1}{q}}.$$

Now, taking p=q=2, we have the following

$$\begin{aligned}E\left(\int_0^t H_s^{(\tilde{N})}\left(\mu(s,X(s))-\mu(s,\tilde{X}(s))\right)ds\right)^2 &\leq E\left(\int_0^t\left|H_s^{(\tilde{N})}\left(\mu(s,X(s))-\mu(s,\tilde{X}(s))\right)ds\right|^2\right)\\ &\leq E\left[\int_0^t 1^2\,ds\int_0^t\left(H_s^{(\tilde{N})}\left(\mu(s,X(s))-\mu(s,\tilde{X}(s))\right)\right)^2 ds\right]\\ &\leq E\left[t\int_0^t\left(H_s^{(\tilde{N})}\left(\mu(s,X(s))-\mu(s,\tilde{X}(s))\right)\right)^2 ds\right]\end{aligned}$$

(1.19)

Thus, substituting Eq.(18) and Eq.(19) into Eq.(17), we obtain

$$E\left[\left|G_t^{(\tilde{N})}\right|^2\right]\leq 2tE\left[\int_0^t\left\{H_t^{(\tilde{N})}\left(\mu(s,X(s))-\mu(s,\tilde{X}(s))\right)\right\}^2 ds\right]+2E\left[\int_0^t\left|H_t^{(\tilde{N})}\left(\sigma(s,X(s))-\sigma(s,\tilde{X}(s))\right)\right|^2 ds\right]$$

$$\leq 2(T+1)K^2E\left[\int_0^t\left(H_t^{(\tilde{N})}\left(X(s)-\tilde{X}(s)\right)\right)^2 ds\right]$$

$$\Rightarrow E\left(\left|G_t^{(\tilde{N})}\right|^2\right)\leq 2(T+1)K^2E\left[\int_0^t E\left(\left|G_s^{(\tilde{N})}\right|^2\right)ds\right]$$

By Gronwall's Lemma, we have that

$$E\left(\left|G_t^{(\tilde{N})}\right|^2\right)\leq LE\left[\int_0^t E\left(\left|G_s^{(\tilde{N})}\right|^2\right)ds\right],\ for\ t\in[0,T]$$

(1.20)

where, $L=2(T+1)K^2$.

We now apply the Gronwall inquality on Eq.(1.20), with $f(t)=E\left(\left|G_t^{(\tilde{N})}\right|^2\right)$ and $g(t)\equiv 0$.

Hence,

$$E\left[\left|G_t^{(\tilde{N})}\right|^2\right]=E\left|H_t^{(\tilde{N})}\left(X(t)-\tilde{X}(t)\right)\right|^2=0$$

We conclude that $H_t^{(\tilde{N})}X(t)=H_t^{(\tilde{N})}\tilde{X}(t)\ for\ t\in[0,T]$.

Hence, $P\left(H_t^{(\tilde{N})}X(t) = H_t^{(\tilde{N})}\tilde{X}(t)\right) = 1 \ for \ t \in [0,T].$

Theorem 1: Using assumption (i)-(iv), the stochastic differential equation (1.10) has a pathwise unique strong solution X(t) on $[0,T]$ with

$$\sup_{0\le t\le T} E\left(|X(t)|^2\right) < \infty.$$

We are to show the existence of a continuous solution of the SDE (10) on $[0,T]$ associated with the Brownian motion W(t) and initial condition ξ. We use the method of successive approximations. We define $X^{(k)}(t)$ recursively with $X^{(0)}(t) = \xi$, and

$$X^{(k+1)}(t) = X(0) + \int_0^t \mu\left(s, X^{(k)}(s)\right)ds + \int_0^t \sigma\left(s, X^{(k)}(t)\right)dW(s) \tag{1.21}$$

For k=0,1,2,...

If for a fixed $k \ge 0$, the approximation $X^{(k)}(t)$ is $\mathrm{A}_t -$measurable and continuous on $[0,T]$, then by assumption (i), (ii) and (iii), then Eq.(1.21) is well define and $X^{(k+1)}(t)$ is $\mathrm{A}_t -$measurable and continuous on $[0,T]$, so also $X^{(0)}(t).$ By assumption (iv) and the definition of $X^{(0)}(t)$, we have that

$$\sup_{0\le t\le T} E\left(\left|X^{(0)}(t)\right|^2\right) < \infty.$$

Using the following inquality $(a+b+c)^2 \le 3\left(a^2+b^2+c^2\right)$, the Classical Holder's inquality and the linear growth bound to Eq.(1.21), we obtain

$$\begin{aligned}
E\left(\left|X^{(k+1)}(t)\right|^2\right) &\le 3E\left(|X(0)|^2\right) + 3E\left(\left|\int_0^t \mu\left(s, X^{(k)}(t)\right)ds\right|^2\right) + 3E\left(\left|\int_0^t \sigma\left(s, X^{(k)}(t)\right)dW(s)\right|^2\right) \\
&\le 3E\left(|X(0)|^2\right) + 3TE\left(\int_0^t \left|\mu\left(s, X^{(k)}(t)\right)\right|^2 ds\right) + 3E\left(\int_0^t \left|\sigma\left(s, X^{(k)}(t)\right)\right|^2 ds\right) \\
&\le 3E\left(|X(0)|^2\right) + 3(T+1)K^2 E\left(\int_0^t \left(1 + \left|X^{(k)}(t)\right|^2\right)ds\right)
\end{aligned}$$

For k=0,1,2,...

By induction, we have that

$$\sup_{0\le t\le T} E\left(\left|X^{(k)}(t)\right|^2\right) \le M_0 < \infty \quad \text{for k=1,2,3,...} \tag{1.22}$$

Now,

$$\begin{aligned}
E\left(\left|X^{(k+1)}(t) - X^{(k)}(t)\right|^2\right) &\le 2tE\left(\int_0^t \left(\mu(s, X(s)) - \mu\left(s, \tilde{X}(s)\right)\right)^2 ds\right) + 2E\left(\int_0^t \left(\sigma(s, X(s)) - \sigma\left(s, \tilde{X}(s)\right)\right)^2 ds\right) \\
&\le 2(T+1)K^2 E\left(\int_0^t \left|X^{(k+1)}(s) - X^{(k)}(s)\right|^2 ds\right) \\
&\le 2(T+1)K^2 \int_0^t E\left(\left|X^{(k+1)}(s) - X^{(k)}(s)\right|^2\right)ds \\
&\le L\int_0^t E\left(\left|X^{(k+1)}(s) - X^{(k)}(s)\right|^2\right)ds
\end{aligned} \tag{1.23}$$

For $t \in [0,T]$ and k=1,2,3,..., where, $L = 2(T+1)K^2$. Then using the Cauchy formula:

$$\int_0^t \int_0^{t_{k-1}} \ldots \int_0^{t_1} h(s) ds dt_1 \ldots dt_{k-1} = \frac{1}{(k-1)!} \int_0^t (t-s)^{k-1} h(s) ds .$$

Repeating the iteration of Eq.(1.23), we obtain

$$E\left(\left| X^{(k+1)}(t) - X^{(k)}(t) \right|^2 \right) \le \frac{4(T+1)^2 K^4}{(k-1)!} \int_0^t \left((t-s)^{k-1} E\left| X^{(1)}(s) - X^{(0)}(s) \right|^2 \right) ds$$

$$\le \frac{L^2}{(k-1)!} \int_0^t \left((t-s)^{k-1} E\left| X^{(1)}(s) - X^{(0)}(s) \right|^2 \right) ds \qquad (1.24)$$

For $t \in [0, T]$ and k=1,2,3,...,.

By assumption (iii), we find that, for k=0,

$$E\left(\left| X^{(k+1)}(t) - X^{(k)}(t) \right|^2 \right) \le 2(T+1) K^2 \int_0^t \left(1 + E\left| X^{(0)}(s) \right|^2 \right) ds$$

$$\le L \int_0^t \left(1 + E\left| X^{(0)}(s) \right|^2 \right) ds$$

$$\le LT \left(1 + E\left| X^{(0)}(s) \right|^2 \right) = M_1 \qquad (1.25)$$

Substituting Eq.(1.25) into Eq.(1.24), we obtain

$$E\left(\left| X^{(k+1)}(t) - X^{(k)}(t) \right|^2 \right) \le \frac{M_1 L^k t^k}{k!},$$

For $t \in [0, T]$ and k=0,1,2,...

Therefore,

$$\sup_{0 \le t \le T} E\left(\left| X^{(k+1)}(t) - X^{(k)}(t) \right|^2 \right) \le \frac{M_1 L^k T^k}{k!}, \text{ k=0,1,2,3,...} \qquad (1.26)$$

This implies that the mean-square integrable martingale converges uniformly on $[0, T]$.

Theorem 2: Suppose that Lemma 1 and Theorem 1 above hold, then

$$P\left\{ \left| \int_0^t \mu\left(s, X^{(k)}(s)\right) ds - \int_0^t \mu\left(s, \tilde{X}(s)\right) ds \right| \le K \int_0^t \left| X^{(k)}(s) - \tilde{X}(s) \right| ds \to 0 \right\} = 1 \text{ and}$$

$$P\left\{ \int_0^t \left| \sigma\left(s, X^{(k)}(s)\right) - \int_0^t \sigma\left(s, \tilde{X}(s)\right) \right|^2 ds \le K^2 \int_0^t \left| X^{(k)}(s) - \tilde{X}(s) \right|^2 ds \to 0 \right\} = 1$$

Implies that

$$P\left\{ \int_0^t \mu\left(s, X^{(k)}(s)\right) ds \to \int_0^t \mu\left(s, \tilde{X}(s)\right) ds \right\} = 1 \text{ and}$$

$$P\left\{ \int_0^t \sigma\left(s, X^{(k)}(s)\right) dW(s) \to \int_0^t \mu\left(s, \tilde{X}(s)\right) dW(s) \right\} = 1$$

In probability as $k \to \infty$ for each $t \in [0, T]$.

We are to show that it is almost surely convergence of their sample paths uniformly on $[0, T]$. We define the following

$$Y_k = \sup_{0 \le t \le T} \left| X^{(k+1)}(t) - X^{(k)}(t) \right|, k = 0,1,2,...$$

$$Y_k \le \int_0^t \left| \mu\left(s, X^{(k)}(s) - \mu\left(s, X^{(k-1)}(s)\right)\right) \right| ds + \sup_{0 \le t \le T} \left| \int_0^t \left(\sigma\left(s, X^{(k)}(s)\right) - \sigma\left(s, X^{(k-1)}(s)\right) \right) dW(s) \right|, k = 0,1,2,...$$

Using the Doob inquality, the Cauchy-Schartz inequality and assumption (ii), we determine

$$E\left[\left|Y_k\right|^2\right] \le 2TK^2 \int_0^T E\left(\left|X^{(k)}(s) - X^{(k-1)}(s)\right|^2\right) ds + 8K^2 \int_0^T E\left(\left|X^{(k)}(s) - X^{(k-1)}(s)\right|^2\right) ds$$

$$\le 2(T+4)K^2 \int_0^T E\left(\left|X^{(k)}(s) - X^{(k-1)}(s)\right|^2\right) ds \qquad (1.27)$$

Combinning Eq.(1.26) and (1.27), we conclude that

$$E\left[\left|Y_k\right|^2\right] \le \frac{M_2 L^{k-1} T^{k-1}}{(k-1)!},\ k = 1,2,3,\ldots$$

where $M_2 = 2TM_1K^2(T+4)$. Then using the Markov inequality, we have that

$$\sum_{k=1}^{\infty} P\left(Y_k > \frac{1}{k^2}\right) \le M_2 \sum_{k=1}^{\infty} \frac{k^4}{(k-1)!} L^{k-1} T^{k-1},$$

where the series on the right hand side converges by the ratio test. Hence, the series on the left hand side also converges, so by the Borel-Cantelli lemma, we conclude that Y_k converges to 0, almost surely. This implies that the successive approximation $X^{(k)}(t)$ converge almost surely, uniformly on $[0,T]$ to a limit $\tilde{X}(t)$ defined by

$$\tilde{X}(t) = X(0) + \sum_{k=0}^{\infty} \left\{X^{(k+1)}(t) - X^{(k)}(t)\right\}.$$

It follows from Eq.(1.22) that $\tilde{X}(t)$ is mean square bounded on $[0,T]$. Since the limit of A^*-adapted process is adapted and the uniform limit of continuous process is continuous, then $\tilde{X}(t)$ is A^*-adapted process and continuous. By assumption (iii), the right hand side of the integral equation (11) is well defined for this process $\tilde{X}(t)$. Now, taking the limit as $k \to \infty$ in Eq.(1.21), we see that $\tilde{X}(t)$ is the solution of Eq.(1.11). The left hand side of Eq.(1.21) converges to $\tilde{X}(t)$ uniformly on $[0,T]$. Compareing the right hand side, the result follows.

Hence, the limit process $\tilde{X}(t)$ satisfies the stochastic integral equation (1.22).

We shall use the following results in solving our problem.

2. Optimization Program

We now determine the optimization process of our problem.

Theorem 3: Suppose the value funtion is defined and $X \in C^{1,2}\left([0.T] \times \Re^N\right)$. Then V is a solution of the following second order partial differential equation:

$$\begin{cases} -v_t + \sup_{\Delta \in \Pi} G(t, X, \Delta, -v_X, -v_{XX}) = 0 \\ v\big|_{t=T} = h(X) \end{cases} \qquad (2.1)$$

where $G(t, X, \Delta, p, P) = -\frac{1}{2} tr\left(P\sigma(t, X, \Delta)\sigma(t, X, \Delta)^T\right) + \langle p, b(t, X, \Delta)\rangle - f(t, X, \Delta),$ (2.2)

for any $(t, X, \Delta, p, P) \in [0,T] \times \Re^N \times \Pi \times \Re^N \times \Re^N$.

Eq.(2.1) is called the Hamilton-Jacobi-Bellman (HJB) equation. The function $G(t, X, \Delta, p, P)$ is called the generalized Hamiltonian.

Definition 2: A function $v \in C([0.T] \times \Re^N)$ is called a viscosity subsolution of Eq.(28) if

$$v(T, X) \le h(X) \ \text{for any}\ X \in \Re^N; \qquad (2.3)$$

for any $\phi \in C^{1,2}([0.T] \times \Re^N)$, where $v - \phi$ attains a local maximum at $(u, X) \in [0, T] \times \Re^N$, we have

$$-\phi_t(t, X) + \sup_{\Delta \in \Pi} G(t, X, \Delta, -\phi_X(t, X), -\phi_{XX}(t, X)) \le 0. \qquad (2.4)$$

A function $v \in C([0.T] \times \Re^N)$ is called a viscosity supersolution of Eq.(2.1)

Theorem 4: The Hamilton-Jacobi-Bellman equation associated to this problem is given by

$$\sup_{\substack{\Delta \in \Pi \\ c \in C}} \left\{ L^{\Delta, C} U(t, V(t)) + \exp(-\rho t) U(U(C)) \right\} = 0 \qquad (2.5)$$

with boundary conditions

$$U(T, V(T)) = \exp(-\rho T) U(U(V)) \qquad (2.6)$$

If there is an optimal portfolio process $\{\Delta^*, C^*\}$, then it is given by the solution in Eq.(2.5).
Proof: (see [5]).

Proposition 1: Let U be a function that belongs to the class of functions with continuous derivatives of first order in t on [τ ,T] and first and second order in $X \in \Re$ almost everywhere. The infinitesimal generator of $\{V(t, X(t)), \tau \le t \le T\}$, with $U(t, X(t))$ satisfies Eq.(2.8), Δ satisfies assumption (ii)-(iii) and C satisfies assumption (v), is given by

$$LU(V(t, X(t))) = \frac{\partial U(t, X(t))}{\partial t} + rX \frac{\partial U(t, X(t))}{\partial X} - \frac{1}{2} \frac{(\mu_i - r)^2}{\sum_{j=1}^{N} (\sigma_{i,j})^2} \frac{(\partial U(t, X(t))/\partial X)^2}{\partial^2 U(t, X(t))/\partial X^2} - C \frac{\partial U(t, X(t))}{\partial X}.$$

Proof: (see [3]).

2.1 Optimization of Portfolio and Consumption Process

In this section, we derived the optimal portfolio and consumption process using HJB equation. The theorem below give us the optimal portfolio and consumption process at time t.

Theorem 1.6 : Let

$$dX^{\Delta,C}(t) = \left(X^{\Delta,C}(t) r(t) + \Delta_i(t) X^{\Delta,C}(t)(\mu_i(t) - r) - C(t) \right) dt + \sum_{j=1}^{N} \sigma_{i.j}(t) \Delta_i^S(t) X^{\Delta,C}(t) dW_j(t), i = 1, \dots, N. \qquad (2.7)$$

be the change in wealth process at time t.

Let $U(X,t)=\sup\limits_{\{\Delta\}\in\Pi_X} E[U(X(T))\mid X(t)=X]$

(2.8)

be the value function, where Π_X is the set of admissible policy that are F_X – progressively measureable, that satisfy the integrability condition

$$E\left[\int_0^T \Delta(s)^2\,ds\right]<\infty,$$

Then,

$$\Delta_i^*(t)=-\frac{U_X(X,t)(\mu_i-r)}{X\sum_{j=1}^{N}\sigma_{i,j}^2 U_{XX}(X,t)},\ i=1,\dots,N \quad \text{and}$$

$$C^*=I(U_X(X,t)\exp[\rho t]),$$

where,

$$I=\left(\frac{dU(C)}{dC}\right)^{-1}.$$

Proof: Given $U(X,t)=\sup\limits_{\{\Delta\}\in\Pi_X^s} E[U(X(T))|X(t)=X]$ and by theorem 1, 2, and 5, Eq.(2.7) becomes:

$$U(X,t)\ge E\left[U\left(X^{\Delta,C}(t+h),t+h\right)\middle|X^{\Delta,C}(t)=X\right]$$

$$\ge U(X,t)+$$

$$E\left[\int_t^{t+h}U_t\left(X^{\Delta,C}(u),u\right)+U_X\left(X^{\Delta,C}(u),u\right)\left(rX^{\Delta,C}(u)+\Delta_i(u)X^{\Delta,C}(u)(\mu_i-r)-C(u)\right)\right]du\,\Big|X^{\Delta}(u)=X$$

$$+E\left[\int_t^{t+h}\frac{1}{2}U_{XX}\left(X^{\Delta}(u),u\right)\left[\sum_{j=1}^{N}(\sigma_{i,j})^2\Delta_i(u)^2X^{\Delta,C}(u)^2\right]du\,\Big|X^{\Delta,C}(u)=X\right]$$

(2.9)

Since we are only interested in the final utility, we set $f(t,X,\Delta)$=0 in theorem 3. By subtracting $U(X,t)$ from bothsides of Eq.(2.9) and then divide bothsides by h and allow h approaches zero, we obtain:

$$0\ge U_t(X,t)+U_X(X,t)\left(rX+\Delta_i(t)X(\mu_i-r)\right)+\frac{1}{2}U_{XX}(X,t)\Delta_i(t)^2X^2\sum_{j=1}^{N}(\sigma_{i,j})^2-C(t)U_X(X,t)$$

Using theorem 4 on Eq.(2.9), we obtain the following:

$$0\ge U_t(X,t)+U_X(X,t)\left(rX+\Delta_i(t)X(\mu_i-r)\right)+\frac{1}{2}U_{XX}(X,t)\Delta_i(t)^2X^2\sum_{j=1}^{N}(\sigma_{i,j})^2-C(t)U_X(X,t)+U(C(t))\exp(-\rho t)$$

This yields the HJB equation for the value function

$$U_t(X,t)+\max_{\substack{\Delta\in\Pi_X\\ C\in\Pi_C}}\left\{\begin{matrix}U_X(X,t)\left(rX+\Delta_i(t)X(\mu_i-r)\right)+\\ \frac{1}{2}U_{XX}(X,t)\Delta_i(t)^2X^2\sum_{j=1}^{N}(\sigma_{i,j})^2-C(t)U_X(X,t)+U(C(t))\exp(-\rho t)\end{matrix}\right\}=0$$

$$U_t(X,t)+\max_{\substack{\Delta\in\Pi_X\\ C\in\Pi_C}}\left\{\begin{array}{l}U_X(X,t)\left(\Delta_i(t)X(\mu_i-r)\right)+\\ \frac{1}{2}U_{XX}(X,t)\left[\Delta_i(t)^2X^2\sum_{j=1}^{N}\left(\sigma_{i,j}\right)^2\right]-C(t)U_X(X,t)+U(C(t))\exp(-\rho t)\end{array}\right\}+$$

$$rXU_X(X,t)=0$$

(2.10)

where $\Delta=\{\Delta_i(t)\}_{i=1}^{N}$ and $C\in\Pi_C$ are sets of admissible strategies.

By theorem 1 and 2, we have that $U((X,t)\in C^{1,2}(\Re\times[0,T])$, then Eq.(2.10) has a unique smooth solution and the maximum in Eq.(2.10) is well-defined. Hence, we have the following:

$$\max_{\substack{\Delta\in\Pi_X\\ C\in\Pi_C}}\left\{U_X(X,t)\left(\Delta_i(t)X(\mu_i-r)\right)+\frac{1}{2}U_{XX}(X,t)\left[\Delta_i(t)^2X^2\sum_{j=1}^{N}\left(\sigma_{i,j}\right)^2\right]-C(t)U_X(X,t)+U(C(t))\exp(-\rho t)\right\}=0$$

$$U_X(X,t)\left(\Delta_i^*(t)X(\mu_i-r)\right)+\frac{1}{2}U_{XX}(X,t)\left[\Delta_i^*(t)^2X^2\sum_{j=1}^{N}\left(\sigma_{i,j}\right)^2\right]-C^*(t)U_X(X,t)+U(C^*(t))\exp(-\rho t)=0$$

$$\Rightarrow U_X(X,t)\left(\Delta_i^*(t)X(\mu_i-r)\right)+\frac{1}{2}U_{XX}(X,t)\left[\Delta_i^*(t)^2X^2\sum_{j=1}^{N}\left(\sigma_{i,j}\right)^2\right]=0$$

(2.11)

$$\Rightarrow -C^*(t)U_X(X,t)+U(C^*(t))\exp(-\rho t)=0$$

(2.12)

From Eq.(2.11), we obtain:

$$\Delta_i^*(t)=-\frac{U_X(X,t)(\mu_i-r)}{X\sum_{j=1}^{N}\left(\sigma_{i,j}\right)^2U_{XX}(X,t)},\ i=1,\ldots,N$$

(2.13)

From Eq.(2.12), we obtain:

$$C^*(t)=I\left(U_X(X,t)\exp[\rho t]\right),$$

(2.14)

where,

$$I=\left(\frac{dU(C)}{dC}\right)^{-1}.$$

Conclusion

We derived the existence and uniqueness of our investor's wealth dynamics following the existing approaches. We show that the limit process X(t) satisfies our Stochastic wealth equation (1.9). We derived the optimal portfolio as well as the optimal consumption strategies for an investor, which are presented in Eq.(2.13) and (2.14) respectively.

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