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On the application of Discrete Time Optimal Control Concepts to Economic Problems

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Abstract

An extension of the use of the maximum principle to solve Discrete-time Optimal Control Problems (DTOCP), in which the state equations are in the form of general equations, rather than difference equations has been examined. Comparing the previous maximum principle to the proposed, revealed that the only difference, lies in the law of motion of the co- state variables and in particular, by applying the Bellman's optimality principle and backward recursion, showed that the previous maximum principle is a subclass of our maximum principle. An economic problem was considered using the Timber Supply Model (TSM) 2000 to exemplify the use of the maximum principle in solving DTOCP problems. Furthermore derivations of necessary conditions which will be used to identify the optimal time paths for the variables were derived.

Keywords: maximum principle, dynamic optimization, economic models

1.0 Introduction

Lots of extensive works exist in Literature and have been thoroughly used in areas like economics, physics, electronics etc, since the emergence of maximum principle by [8]. The key features of the maximum principle are the control variables which usually unambiguously determine the state variables; the co-state (ad joint) variables represent the shadow values of the state variables linked with them. In addition, the necessary conditions for the optimization substantially matched the sufficient conditions for the optimization [3]. This led to the consideration of optimal control theory as a more simple but powerful method for solving the problem of dynamic optimization.

Pontryagin et al and a host of other authors have proposed numerous maximum principles to be applied to various types of optimal control theory. Specifically these authors have provided for DTOCP, the maximum principle only for the situation where the state equation is in the form of the difference equation. But we know that in solving economic problem models, we do encounter cases in which the state equation is not in the form of the difference equation rather in the general equation. In view of the above observation, we propose or extend the maximum principle that can handle DTOCP in which the state equation is not only of the form of a difference equation but as well as a general equation. To accomplish this, we shall adopt the procedure of reviewing what the previous maximum principle states, thereafter present the new maximum principle for DTOCP as a general one. By identifying the law of motion of the co-state variables, we should be able to differentiate the two principles. Similarly we shall show that the previous principle is a subclass of the present one. We shall consider a numerical problem for the utilization of our maximum principle as a solution technique.

2.0 Pontryagin et all's and other maximum principles

It is our desire here to present a complete review of what the previous maximum principle entails. According to [2], the maximum principle for DTOCP for this case is

$$Max \sum_{t=0}^{T-1} f(u_t, x_t, t) + S(x_T) \qquad (2.1)$$

Subject to

$x_{t+1} - x_t = g(u_t, x_t)$, $x(0) = x_0$ specified.

where t = 0(1) T, is considered as the set of time periods and t = 0 and t = T represents the initial and terminal time periods respectively. u_t denote a control variable in time period t; x_t the state variable representing the system in time period t. $f(.)$ stands for the payoff function or the net economic return. $S(.)$ the scrap (or terminal) value function at the terminal time period. The law of motion stated in the state variable is $x_{t+1} - x_t = g(u_t, x_t)$ .From equation (2.1), we have the Lagrangian function

$$L = \sum_{t=0}^{T-1} f(u_t, x_t, t) + \lambda_{t+1}(x_t + g(u_t, x_t) - x_{t+1}) + S(x_T) \qquad (2.2)$$

where λ_{t+1} is the Lagrangian multiplier associated with x_{t+1}. The first order necessary condition for optimality can be obtained as follows:

$$\frac{\partial f(u_t, x_t, t)}{\partial u_t} + \lambda_{t+1} \frac{\partial g(u_t, x_t)}{\partial u_t} = 0, t = 0(1)T - 1 \qquad (2.3)$$

$$\lambda_{t+1} - \lambda_t = -\left(\frac{\partial f(u_t, x_t)}{\partial x_t} + \lambda_{t+1} \frac{\partial g(u_t, x_t)}{\partial x_t} \right), \text{ t = 0(1) T-1} \qquad (2.4)$$

$$x_{t+1} - x_t = g(u_t, x_t), t = 0(1)T - 1 \qquad (2.5)$$

$$\lambda_T = \frac{\partial S(x_T)}{\partial x_T} \qquad (2.6)$$

$$x(0) = x_0 \qquad (2.7)$$

The Hamiltonian at time period t can be defined as

$$\tilde{H}(u_t, x_t, t, \lambda_{t+1}) = f(u_t, x_t, t) + \lambda_{t+1} g(u_t, x_t) \qquad (2.8)$$

Similarly, the first order necessary conditions are interpreted in terms of the Hamiltonian such as;

$$\frac{\partial \tilde{H}(u_t, x_t, t, \lambda_{t+1})}{\partial u_t} = \frac{\partial f(u_t, x_t, t)}{\partial u_t} + \lambda_{t+1} \frac{\partial g(u_t, x_t)}{\partial u_t} = 0, \ t = 0(1)T - 1 \qquad (2.9)$$

$$\lambda_{t+1} - \lambda_t = -\frac{\partial \tilde{H}(u_t, x_t, t, \lambda_{t+1})}{\partial x_t} = -(\frac{\partial f(u_t, x_t, t)}{\partial x_t} + \lambda_{t+1} \frac{\partial g(u_t, x_t)}{\partial x_t}) = 0, t = 0(1)T - 1$$

(2.10)

$$x_{t+1} - x_t = \frac{\partial \tilde{H}(u_t, x_t, t, \lambda_{t+1})}{\partial \lambda_{t+1}} = \frac{\partial g(u_t, x_t)}{\partial \lambda_{t+1}} = 0, t = 0(1)T - 1 \qquad (2.11)$$

Equations (2.9), (2.10) and (2.11) plus (2.6) and (2.7) represents the maximum principle proposed by Pontryagin et al and others in literature for DTOCP in which the state equation is in the form of difference equation.

Generalization of the Maximum Principle for DTOCP.

We consider the DTOCP in which the state equation is not in the form of the difference equation but have the form of a general equation. That is

$$Max\sum_{t=0}^{T-1} f(u_t, x_t, t) + S(x_T) \quad (2.12a)$$

subject to

$$x_{t+1} = p(u_t, x_t) + q(u_t, x_t) \quad (2.12b)$$

where equation (2.12a) denotes the law of motion of the state variable and $q(u_t, x_t)$ varies between 0 and 2 inclusive.

The Lagrangian function for the equation (2.12) is

$$L = \sum_{t=0}^{T-1}\{ f(u_t, x_t, t) + \lambda_{t+1}[p(u_t, x_t) + q(u_t, x_t) - x_{t+1}] + S(x_T) = 0,\ t = 0(1)\ T\text{-}1 \quad (2.13)$$

Hence the first order necessary conditions for optimality are

$$\frac{\partial f(u_t, x_t, t)}{\partial u_t} + \lambda_{t+1}[\frac{\partial p(u_t, x_t)}{\partial u_t} + \frac{\partial q(u_t, x_t)}{\partial u_t}] = 0,\ t = 0(1)\ T\text{-}1, \quad (2.14)$$

$$\lambda_t = \frac{\partial f(u_t, x_t, t)}{\partial u_t} + \lambda_{t+1}[\frac{\partial p(u_t, x_t)}{\partial u_t} + \frac{\partial q(u_t, x_t)}{\partial u_t}] \quad (2.15)$$

$$x_{t+1} = p(u_t, x_t) + q(u_t, x_t) \qquad t = 0(1)\ T\text{-}1 \quad (2.16)$$

$$\lambda_T = \frac{dS(x_T)}{dx_T} \quad (2.17)$$

$$x(0) = x_0 \text{, specified} \quad (2.18)$$

The corresponding Hamiltonian at time period t is as follows:

$$\tilde{H}(u_t, x_t, t, \lambda_{t+1}) = f(u_t, x_t, t) + \lambda_{t+1}[p(u_t, x_t) + q(u_t, x_t)]$$

$$\frac{\partial f}{\partial u_t} + \lambda_{t+1}[\frac{\partial p(u_t, x_t)}{\partial u_t} + \frac{\partial q(u_t, x_t)}{\partial u_t}] \quad (2.19)$$

and the necessary conditions for optimality are

$$\frac{\partial \tilde{H}}{\partial u_t} = \frac{\partial f}{\partial u_t} + \lambda_{t+1}[\frac{\partial p(u_t, x_t)}{\partial u_t} + \frac{\partial q(u_t, x_t)}{\partial u_t}] = 0,\ t = 0(1)\ T\text{-}1 \quad (2.20)$$

$$\lambda_t = \frac{\partial \tilde{H}}{\partial x_t} = \frac{\partial f}{\partial u_t} + \lambda_{t+1}[\frac{\partial p(u_t, x_t)}{\partial u_t} + \frac{\partial q(u_t, x_t)}{\partial u_t}],\ t = 0(1)\ T\text{-}1 \quad (2.21)$$

$$\lambda_{t+1} = \frac{\partial \tilde{H}}{\partial \lambda_{t+1}} = [p(u_t, x_t) + q(u_t, x_t)], t = 0(1)T - 1\infty \quad (2.22)$$

Similarly our maximum principle can thus be obtained from these last three equations plus equations (2.17) and (2.18).This clearly differs from the maximum principle of equations (2.9, 2.10, 2.11) plus equations (2.6, 2.7) only in the law of motion of the state and co-state variables. We also viewed the function $S(x_T)$ as the solution variable for the same maximization problem over the time frame between T and ∞ and $S(x_{T-1})$. This is an application of the Bellman's optimality principle, which states that:

"an optimality policy has the property that regardless of what the previous decisions have been, the remaining decisions must be optimal with regard to the state resulting from those previous decisions" and backward recursion.

From this, we state that the state equation (2.17) holds for t = 0(1)T-1, such that when t = 0(1)T and specifically for time T-1, we have,

$$S^*(x_{T-1}) = f^*(u_{T-1}, x_{T-1}, T-1) + S^*[p(u_{T-1}, x_{T-1}) + q(u_{T-1}, x_{T-1})] \tag{2.23}$$

where the superscript(*)represents the optimal value. Using equation (17) for T-1, we have

$$\lambda_{T-1} = \frac{dS(x_{T-1})}{dx_{T-1}}$$

$$= \frac{\partial f^*(u_{T-1}, x_{T-1}, T-1)}{\partial x_{T-1}} + \left\langle S*, p(u_{T-1}, x_{T-1})\right\rangle \frac{\partial p(u_{T-1}, x_{T-1})}{\partial x_{T-1}}$$

$$+ \left\langle S^*, q(u_{T-1}, x_{T-1})\right\rangle \frac{\partial q(u_{T-1}, x_{T-1})}{\partial x_{T-1}} \tag{2.24}$$

Applying (2.17) to equation (2.24), we have,

$$\lambda_{T-1} = \frac{dS(x_{T-1})}{dx_{T-1}} = \frac{\partial f^*(u_{T-1}, x_{T-1}, T-1)}{\partial x_{T-1}} + \lambda_T\left[\frac{\partial p(u_{T-1}, x_{T-1})}{\partial x_{T-1}} + \frac{\partial q(u_{T-1}, x_{T-1})}{\partial x_{T-1}}\right] \tag{2.25}$$

Based on the above result, the law of motion for the co-state variables can be expressed as

$$\lambda_t = \frac{dS(x_t)}{dx_t}, t = 0(1)T$$

Or

$$\lambda_t = \frac{\partial f^*(u_t, x_t, t)}{\partial x_t} + \lambda_{t+1}\frac{\partial p(u_t, x_t, t)}{\partial x_t} + \lambda_{t+1}\frac{\partial q(u_t, x_t, t)}{\partial x_t}, \ t = 0(1)T \tag{2.26}$$

Next we apply our MP to situation where the state equation is in the form of the difference equation. To throw light into the MP using the Hamiltonian, we set up the following maximum problem by changing the difference equation of the state variable

$$x_{t+1} - x_t = g(u_t, x_t)$$

into

$$x_{t+1} = x_t + g(u_t, x_t)$$

Hence the maximum problem represented as in equation (2.12) is changed into

$$Max\sum_{t=0}^{T-1} f(u_t, x_t, t) + S(x_t + g(u_t, x_t)) \tag{2.27}$$

The necessary conditions arising from the above is

$$\frac{\partial f(u_t, x_t, t)}{\partial u_t} + \frac{dS(x_{t+1})}{dx_{t+1}}\frac{\partial g(u_t, x_t)}{\partial u_t} = 0, t = 0(1)T-1$$

Then the Hamiltonian function at time period t is

$$\tilde{H}(u_t, x_t, t, \lambda_{t+1}) = f(u_t, x_t, t) + \lambda_{t+1}(x_t + g(u_t, x_t))$$

Also we obtain the necessary condition as

$$\frac{\partial \tilde{H}(u_t, x_t, t, \lambda_{t+1})}{\partial u_t} = \frac{\partial f(u_t, x_t, t)}{\partial u_t} + \lambda_{t+1}\frac{\partial g(u_t, x_t)}{\partial u_t} = 0, t = 0(1)T - 1$$

(2.28)

From equations (2.27) and (2.28), we have that

$$\lambda_{t+1} = \frac{dS(x_{t+1})}{dx_{t+1}}, t = 0(1)T - 1$$

(2.29)

Arising from the above, let us derive the law of motion of co-state variable, but first bear in mind that at time t, optimum value of S become,

$$S^*(x_t) = f^*(u_t, x_t, t) + S^*(x_t + g(u_t, x_t))$$

(2.30)

Then

$$\frac{\partial S^*(x_t)}{\partial x_t} = \frac{\partial f^*(u_t, x_t, t)}{\partial x_t} + \frac{dS^*(x_{t+1})}{dx_{t+1}}[1 + \frac{\partial g(u_t, x_t)}{\partial x_t}], t = 0(1)T - 1$$

(2.31)

Combining (27) and (29), we have,

$$\lambda_t = \frac{\partial f^*(u_t, x_t, t)}{\partial x_t} + \lambda_{t+1}[1 + \frac{\partial g(u_t, x_t)}{\partial u_t}], t = 0(1)T - 1$$

(2.32)

$$\lambda_{t+1} - \lambda_t = -(\frac{\partial f^*(u_t, x_t, t)}{\partial x_t} + \lambda_{t+1}\frac{\partial g(u_t, x_t)}{\partial u_t}), t = 0(1)T - 1$$

(2.33)

With this, we have derived the first order necessary conditions and transversality conditions stated above using Hamiltonian. This thus has given a generalization of our maximum principle, since it embeds the previous maximum principle proposed by Pontryagin, et al and others in literature.

3.0 Example: An Economic Problem

We shall examine the Timber Supply Model 2000(TSM 2000) formulated by [7] in applying our maximum principle to the economic problem. The TSM is formulated to analyze the dynamic behavior of the global timber market by incorporating additional features of the global timber market that occurred in recent years. We first summarize the formulation of the TSM 2000then look at the procedure of deriving the equations that we have to solve to find the optimal time paths for economic attributes.

Derivation of the TSM 2000 Model

The major aim of this model TSM 2000 is to maximize the total benefit of the society as a whole not for an individual private profit of a landowner. Secondly the model factors the total industrial wood harvest into solidwood and pulpwood. As such the net surplus of year is defined as

$$T_{S\,k} = \int_0^{Q_j} D_k^p(n)dn + \int_0^{\tilde{Q}} D_k^q(n)dn - C_k$$

(3.1)

where

Q_k = the quantity of timber for solidwood harvested in year k;

$D_k^p(n) =$ the inverse demand function of industrial solidwood in year k;

$D_k^p(n) =$ the demand of industrial pulpwood in inverse form;

$C_k =$ the total cost in year k.

Total cost implies the summation of harvest, access, transportation cost (CT_k) and the regeneration cost (CR_k). Harvesting and transportation costs in the year k, is a function of the total volume harvested by land class and regeneration costs is a function of hectares harvested (regenerated) and the level of input used. The following definitions will be useful and so they are in order for the formulation of the model:

Xh_k - a state vector of hectares of trees in each age group for land class h in year k with element xh_{ik} .

Zh_k - the state vector for the regeneration input with element zh_{ik}, the level of regeneration input associated with age group i in year k for land class h.

Uh_k - the control vector of hectares harvested with element uh_{ik}, representing the land class h, the portion of the hectares of trees in age group i harvested in year k.

Ph_k - the price of regeneration input for land class h.

The possible volume of timber that can be placed on sale per hectare for land class h in year k for a stand regenerated i time periods is a function of i and on the magnitude of the regeneration input used on this stand (zh_{ik}) Let this possible volume on sale be

$$dh_{ik} = f_h(i., zh_{ik}) \tag{3.2}$$

This volume is splitted into solidwood and pulpwood using variable proportion which vary by land class with ϕh the portion going to solidwood and 1 - ϕh the portion for pulpwood. Based on this, the volume of the commercial timber harvested for solidwood and pulpwood from land class h in year k is given by

$$Qh_k = \langle \phi h_k uh_k, Xh_k dh_k \rangle \tag{3.3}$$

$$Qh_k = \sum_k^T Qh_k,\ \tilde{Q}h = \sum_k^T \tilde{Q}h_k \tag{3.4}$$

Where Xh_k is a diagonal matrix using the elements of xh_{ik} and the total volume harvested in the responsive sections is the summations of these over all the land classes. Costs including harvest, access and transportatioin cost for land class h is a function of the volume harvested in theat land class

$$CTh_k = ch(Qh_k + \tilde{Q}h_k) \tag{3.5}$$

and regeneration cost for land class in year k is

$$CRh_k = \langle uh_k, xh_k + vh_k \rangle pwh_k wh_k \tag{3.6}$$

where <. , > gives the hectares harvested in land class h, vh_k is the exogenously determined number of hectares of new forest land in land class h, the product of the last two terms give expenditure per hectare. This leads to the total cost of

$$C_k = \sum_h (CTh_k + CRh_k) \tag{3.7}$$

Ensuing from the above, the objective function of TSM will be the discounted present value of the net surplus stream as follows:

$$T_0(x_0, z_0, u, w) = \sum_k p^k s_k(x_k, z_k) + p^k T_k^*(x_k, z_k) \tag{3.8}$$

where p is the discount factor ; k is the last time period of the model time range ; u is any admissible set of control vectors, w is any set of admissible control scalars and $T_k^*(.)$ is the optimal terminal value function . Equations (3.7)

is to be maximized over the control variables subject to the state equations and the constraints. The constraints for control variables and the state equations for the given system are

$$\begin{aligned} &0 \le uh_k, \le 1, \forall h, i, k \\ &0 \le wh_k, \forall h, k \\ &Xh_{k+1} = (B + DUh_k)xh_k + vh_k b, \forall h, k. \\ &Zh_{k+1} = Bzh_k + wh_k b, \forall h, k. \end{aligned} \tag{3.9}$$

$$B = \begin{bmatrix} 0 & 0 & 0 & 0 & . & . & . & 0 & 0 \\ 1 & 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 1 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & 1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & 1 & 0 \end{bmatrix}, \tag{3.10}$$

,

$$b = [1,0,0,0,0\Lambda\ 0]^T$$

$$D = \begin{bmatrix} 1 & 1 & 1 & 1 & . & . & . & 1 & 1 \\ -1 & 0 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & -1 & 0 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & -1 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 0 & -1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & 0 & . & . & . & -1 & 0 \end{bmatrix}$$

B, D and U are N-square matrices; Uh_k is a diagonal matrix using the elements of uh_k (specified in)and b is an N –vector where N is equal to or greater than the index number of the oldest age group in the problem.

Application of the Maximum Principle on DTOCP

Maximizing the objective function (3.8) subject to the constraint (3.9) is the DTOCP, that has to be solved using the discrete time maximum principle. This principle states that the constrained maximization of equation (3.8) can be decomposed into a series of sub-problems and in each time period, the following Hamiltonian is maximized with respect to uh_k and wh_k subject to the constraints. The Hamiltonian is written for year k as,

$$\begin{aligned} H_k &= \int_0^Q D_k^p(n)dn + \int_0^{\tilde{Q}} D_k^q(n)dn - C_k \\ &+ \sum_h \left\langle \lambda_{h_{k+1}}, (B + DUh_k)Xh_k + Vh_k b \right\rangle + \sum_h \left\langle \psi_{h_{k+1}}, Bz_{hk} + wh_k b \right\rangle \end{aligned} \tag{3.11}$$

Where

$$\lambda h_k = \rho \left[\frac{dT_k^*(\hat{x}_k, z_k)}{dxh_k} \right]$$

$$\lambda h_k = \rho \left[\frac{dT_k^*(x_k, z_k)}{dxh_k} + \left\langle B + DUh_k, \lambda h_{k+1} \right\rangle \right] \tag{3.12}$$

and

$$\psi h_k = \rho [\frac{dT_k^*(x_k, z_k)}{dzh_k}] \tag{3.13}$$

$$\psi h_k = \rho \left[\frac{dT_k^*(x_k, z_k)}{dzh_k} + < B, \psi h_{k+1} > \right]$$

NB: The gradient vectors $g_{T_k} = \dfrac{dT_k^*(x_k, z_k)}{dzh_k}$, are the derivatives with vectors and $T_{k+1}^*(.)$ is the solution function in year k+1. Similarly λ_{hk} and ψ_{hk} , are the co state variables associated with λ_{hk} and z_{hk} respectively and identify the shadow values of the hectares of forest and regeneration input.

The Lagrangian function of the above problem is written as

$$\begin{aligned} L_k^H &= H_k + \sum_h \left\langle \xi_{hk}, (1 - u_{hk}) \right\rangle = \int_0^{Q_k} D_k^p(n)dn + \int_0^{\tilde{Q}_k} D_k^q(n)dn - C \\ &+ \sum_h \lambda_{hk} + [(B + DU_{hk})x_{hk} + v_{hk} b] + \sum_h \psi_{hk} + (Bz_{hk} + \psi_{hk} b) \\ &+ \sum_h \left\langle \xi_{hk}, (1 - u_{hk}) \right\rangle \end{aligned} \tag{3.14}$$

The Kuhn-Tucker necessary conditions are

$$\frac{\partial L_k^H}{\partial u_{hk}} = \phi_{hk} D_k^p(Q_k) + (1-\phi_{hk}) D_k^q(\tilde{Q}_k) - c_h(Q_{hk} + \tilde{Q}_{hk}) X_{hk} d_{hik}$$

$$- x_{hk} p_{wh} w_{hk} + X_{hk} D\lambda_{hk+1} - \xi_{hk} \leq 0, \forall h.$$

$$\left(\frac{\partial L_k^H}{\partial u_{hk}}\right)_{u_{hk}} = 0, h, i.$$

$$- \quad \frac{\partial L_k^H}{\partial w_{hk}} = -u_{hkp} x_{hk} p_{wh} + \psi_{hk+1} \leq 0, \forall h.$$

$$\left(\frac{\partial L_k^{?H}}{\partial w_{hk}}\right)_{w_{hk}} = 0, \forall h, i \tag{3.15}$$

$$\frac{\partial L_k^H}{\partial \xi_{hk}} = (1 - u_{hk}) \geq 0, \forall h$$

$$\left(\frac{\partial L_k^H}{\partial \xi_{hk}}\right) \xi_{hk} = 0, \forall h, i$$

Equations (3.8), (3.10), (3.13), and (3.15) known respectively as the state equations and the law of motion for the co state variables and the Kuhn-Tucker conditions identify a two-point boundary problem that can be used to solve both theoretical and numerical problems. These equations can be solved to determine the optimal time paths for economic variables.

Conclusion

We saw in this work that initial maximum principle was meant to deal with problems in which the state equations are in the form of the difference equation. In the light of this, we have successfully extended the idea to include the general equation, such that we have a maximum principle that can be used to solve DTOCP in which the state equations take form of the general equation. The only difference between the two, lies in the law of motion of the co-state variables; particularly, by applying the Bellman's Optimality Principle stated above and backward recursion, we observed that our maximum principle encloses the previous i.e. the previous maximum principle is a subclass of our maximum principle. In this sense, it can serve as a general solution method for DTOCP in economic problems. The TSM 2000 was taken as an illustrative problem to apply our (MP) as a solution technique; necessary equations that will be used to identify the optimal time path for the variables were derived.

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