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Rigid Body Motion Calculated From Spatial Co-ordinates of Markers

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Abstract

In this paper, we present a unified method for calculating spatial co-ordinates of markers for a rigid body motion such as in bones. Kinematical analysis of bone movement in cadaveric specimens or living objects had been developed. Here, we show how spatial co-ordinates of markers in or on bone can be calculated from the co-ordinates of projections of these markers in two different directions on one or two planes. This rigid body motion can be described by a rotation matrix and a translation vector or by the position of screw axis, the angle of rotation about this axis and the translation along the axis. Our method shows that our solution process is different and our results show that three or more non-collinear points are used and no initial approximation is needed.

1.0 Introduction

Several methods have been developed for the kinematical analysis of bone movements in cadaveric specimens or living subjects ([1], [2], [3], and [4]). Methods based on X-ray or light photogrammetry of markers connected to bone are usually relatively accurate as compared to electro-goniometry ([5] and [6]).
Spatial coordinates of markers in or on bone can be calculated from the coordinates of projections of these markers in two different directions on one or two planes. These spatial co-ordinates are used to determine kinematical parameters. The object in study is considered rigid and its movement between two subsequent positions is taken to be a screw motion. Such a motion can be described by a rotation matrix and translation vector or by the position of the screw axis, the angle of rotation about this axis and the translation along this axis.
Rodriguez [1] needed the spatial co-ordinates of three non-collinear points before and after the movements in order to calculate the direction vector **n** of the helical axis and the rotation angle ϕ. If $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ and $\mathbf{p_1}$, $\mathbf{p_2}$, $\mathbf{p_3}$ are the radius vectors of these points before and after movement, the equations for **n** and ϕ are (see [3]).

$$\Omega = n \tan\frac{\phi}{2} \tag{A}$$

$$p_1 + p_2 - a_1 + a_2 = \Omega^* \ (p_1 - p_2 + a_1 - a_2) \tag{1.1}$$

$$p_1 - p_3 - a_1 + a_3 = \Omega^* \ (p_1 - p_3 + a_1 - a_3) \tag{1.2}$$

where * denotes cross product of vectors . The vector $\mathbf{\Omega}$ can be solved from the latter two equations if these equations are not inconsistent. If they are, a least squares method is needed and the results may be slightly different from the results according to [3] or our method.

Kinzel *et al* [2] used the co-ordinates of four non-planar points in order to calculate the 4*4 matrices that described both rotation and translation.
Chao [4] calculated the rotation matrix **R** from two vectors pointing from one of three markers to the other two.

The extension of his method in case of more than three markers was not shown.
Selvik [3] used a least squares method and minimized

$$\sum_{i=1}^{n}(p_i - Ra_i - v)^2 \tag{1.3}$$

where n (n ≥3) is the number of markers and **v** is the translation vector. Variables were the three components of **v** and the three Eulerian angles in which **R** was expressed. [3] needed an initial approximation for these variables.
Our method comes close to [3] method. The expression to be minimized is the same but the solution process is different. Three or more non-collinear points are used and no initial approximation is needed.

2.0 Determination Of The Rotation Matrix R And The Translational Vector.

The movement of a rigid body from a position 1 into another position 2 can be characterized by a transition vector **v** and a rotation matrix R [7]. This matrix is orthogonal and therefore satisfies

$$R^T R = I \tag{2.1}$$

where **I** is the 3*3 unit matrix and the super script T indicates transposition.
Let $\mathbf{a_1, a_2, ...a_n}$ denote the radius vector of **n (n ≥3)** non-collinear points $\mathbf{P_1, P_2,........,P_n}$ of the body in position 1**,** then the radius vectors $\mathbf{q_1, q_2,..........q_n}$ of these in position are given by

$$\mathbf{q_i} = Ra_i + \mathbf{v} \quad \text{for } i = 1,2,.............\mathbf{n}. \tag{2.2}$$

R and **v** are unknown and must be determined from the measured radius vectors $\mathbf{p_1, p_2,..........p_n}$ of $P_1, P_2,.............P_n$ in position 2. In general, these vectors will differ from the exact vectors $\mathbf{q_1, q_2,..........q_n}$**.** An overall measure for this difference is given by the function f of **v** and R, defined by:

$$f(v,R) = \frac{i}{n}\sum_{i=1}^{n} (R a_i + v - p_i)^T (R a_i + v - p_i) \tag{2.3}$$

Introducing average vectors **a** and **p ,** a matrix M and a scalar quantity f_{0}:

$$a = \frac{1}{n}\sum_{i=1}^{n} a_i \quad , \quad p = \frac{1}{n}\sum_{i=1}^{n} p_i \tag{2.4}$$

$$M = \frac{1}{n}\sum_{i=1}(p_i a_i^T) - p a^T \tag{2.5}$$

$$f_0 = \frac{1}{n}\sum_{i=1}(a_i^T a_i + p_i^T p_i) - (a^T a + p^T p) \tag{2.6}$$

The expression for f(**v**,R) can be written as :

$$f(v,R) = f_0 + (Ra+v-p)^T (Ra+v-p) - 2\,trace(M^T R) \tag{2.7}$$

Here trace (M R) is equal to the sum of the components on the main diagonal of the 3 *3 matrix M R.
The Lagrangian multiplier theorem is used to determine the matrix R and the vector **v** that minimize f under the constraint condition (2.1). To use this theorem, a 3*3 matrix S of Lagrangian multipliers and a function F of **v ,**R and S are introduced.

$$F(v,R,S) = f(v,R) + trace(S(R^T R - I)) \tag{2.8}$$

The above mentioned theorem now states that if F = F(v,R,S) is stationary for some, then f=f(**v**, R) is stationary and equation (2.1) is satisfied for that **v** and R. Stationary points of F(**v**,R,S)are found by requiring the first variation δF of F(**v**,R,S) to be zero for each variation δ**v,** δR and δS of **v**, R and S.

$$\delta F = 2(Ra + v - p)^T\,(\delta Ra + \delta v) - 2trace(M^T\delta R)\,trace(\delta S((R^T R - I)) + trace(S(\delta R^T R + R^T \delta R)) \quad (2.9)$$

Requiring δF = 0 for each δS results in the constraint condition (2.1). Next δF = 0 for each δ**v** gives an equation for v;

$$v = p - Ra \quad (2.10)$$

Finally, using equation (2.10) and trace (S δRT R) = trace (ST RT δR), the requirement δF = 0 for each δR leads to the matrix equation

$$M = \tfrac{1}{2} R(S + S^T) \quad (2.11)$$

It is easy to solve equation (2.11) for the symmetric matrix $\tfrac{1}{2}(S + S^T)$. With

$$\bar{S} + \frac{1}{2}(S - S^T) \quad (2.12)$$

and with M = RS and equation (2.12), it follows that

$$\bar{S}^2 = \bar{S}^T R^T R\bar{S} = M^T M \quad (2.13)$$

$M^T M$ is a symmetric matrix with eigenvalues $D_{11}^2 \geq D_{22}^2 \geq D_{33}^2 \geq 0$ and a corresponding set of three orthonormal eigenvectors. The eigenvalues are arranged on the principal diagonal of a diagonal matrix D^2 while the eigenvectors are considered as the columns of a 3*3 matrix V. From the definition of eigenvalues and eigenvectors, it is seen that

$$M^T M = S^2 = VD^2V^T; \quad VV^T = \mathbf{I} \quad (2.14)$$

A solution for the symmetric matrix $\bar{S}$ is therefore given by

$$\bar{S} = VDV^T \quad (2.15)$$

where the signs of the principal diagonal components D_{11}, D_{22}, D_{33} of D are up to now indeterminate. Insertion of this solution into $M = R\bar{S}$ gives

$$M = RVDV^T \quad (2.16)$$

The signs of D_{11}, D_{22}, and D_{33} follow the condition that f(**v**,**R**)must be minimal. With (2.7), (2.10), (2.16), it follows

$$f = f_0 - 2\,trace(VDV^T) = f_0 - 2\left(D_{11} + D_{22} + D_{33}\right) \quad (2.17)$$

and in order to make f minimal, D_{11}, D_{22}, D_{33} must be chosen non-negative.
To elucidate the geometrical significance of D_{11}, D_{22}, and D_{33}, the measured vector $\mathbf{p}_i$ (i =1,2,.....,n) is related to the exact vector $\mathbf{q_i} = R\,\mathbf{a_i} + \mathbf{v}$ by writing

$$p_i = Ra_i + v + \delta_i, \quad for\; i = 1, 2, ..., n \quad (2.18)$$

The error vector $\boldsymbol{\delta}_\mathbf{i}$ is due to measuring errors and to the fact the body is not perfectly rigid. Insertions of equation (2.18) in (2.4) and (2.5) leads to

$$M = R(M_0 + E) \qquad (2.19)$$

where M_0 depends only on the radius vectors $\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots\ldots\ldots\boldsymbol{a}_n$, before the movement while E depends on **R**, $\boldsymbol{a_1}, \boldsymbol{a_2}, \ldots\ldots\ldots\boldsymbol{a}_n$ *and* $\boldsymbol{\delta_1}, \boldsymbol{\delta_2} \ldots\ldots\ldots\boldsymbol{\delta_n}$;

$$M_0 = \frac{1}{n}\sum_{i=1}^{n}\{(a_i - a)^T\}; \qquad E = \frac{1}{n}\sum_{i=1}^{n}\{R^T\delta(a_i - a)^T\} \qquad (2.20)$$

M_0 is a symmetric (semi) positive definite matrix. Therefore M_0 can be written as

$$M_0 = V_0 D_0 V_0^T; \qquad V_0 V_0^T = I \qquad (2.21)$$

where V_0 is the matrix of eigenvectors of M_0 and D_0 is a diagonal matrix with the non-negative eigenvalues $D_{01} \geq D_{02} \geq D_{03} \geq 0$ of M_0. It can easily be proved that all eigenvalues are unequal zero if and only if more than three non-coplannar markers are used. If all markers are lying in one plane, then $D_{03} = 0$, whereas $D_{02} = D_{03}$ if all markers are collinear.

With equations (2.19) and (2.21), the eigenvalue problem (2.14) can be written in a form that is more suitable for further analysis :

$$V D^2 V^T = V_0 D_0^2 V_0^T + M_0^T E + E^T M_0 + E^T E \qquad (2.22)$$

From this it is clear that $V = V_0$, $D = D_0$ give a solution of equation (17) if $E = 0$. IN practice, E will be unequal zero but very small compared with M_0. Then V_0 and D_0 will be realistic approximation for V and D. Using equations (2.22) and (2.20), it is possible to establish bounds for the differences between V and V_0 and between D and D_0 if bounds for the error vectors $\boldsymbol{\delta_1}, \boldsymbol{\delta_2} \ldots\ldots\ldots\boldsymbol{\delta_n}$ are given. However, this is not the subject for this paper and will not be analyzed here.

The rotation matrix R can be determined reliably from equation (2.16) if and only if at least two of the eigenvalues $D_{11}^2 \geq D_{22}^2 \geq D_{33}^2$ differ significantly from zero. From the foregoing, it is seen that this will be the case if three or more non-collinear points are used. From equation (2.16) follows

$$RVD = MV = [\, \mathbf{m_1} \quad \mathbf{m_2} \quad \mathbf{m_3} \,] \qquad (2.23)$$

where $\mathbf{m_i}$ (i = 1, 2, 3) represents columns i of RV, multiplied by D_{ii}. So, if D_{11} and D_{12} differ significantly from zero,(and this will always be true in realistic situations), columns 1 and 2 of RV is equal to the cross product ($1/D_{11} \cdot D_{22}$) $\mathbf{m_1} \bullet \mathbf{m_2}$ of first two columns. So, the final result for R is given by:

$$R = \left(\frac{1}{D_{11}} m_1 \;\; \frac{1}{D_{22}} m_2 \;\; \frac{1}{D_{11} \bullet D_{22}} m_1 * m_2 \right) \bullet V^T \qquad (2.24)$$

For the calculations of R and **v** are sufficient : Equations (2.1) and (2.2) can be used to determine M, a subroutine for the eigenvalues and eigenvectors of $M^T M$, the calculation of the columns $\mathbf{m_1}$, $\mathbf{m_2}$, and $\mathbf{m_3}$ of MV and finally equation (2.24) can be used to determine R and equation (2.10) for **v**.

3. 0 Determination of Helical Axis

In the foregoing, the movement of the body was characterized by the rotation matrix R and translation vector **v**. Such a movement can be considered the result of a rotation through an angle φ about the helical axis and a translation t along this axis. Let **n** denote a unit vector along the helical axis and let **s** be the radius vector of a point on this axis, such that **n** and **s** are orthogonal:

$$\mathbf{n}^T\mathbf{n} = \mathbf{1}; \qquad \mathbf{n}^T\mathbf{s} = \mathbf{0}. \qquad (3.1)$$

The sense of rotation and the direction of **n** will correspond with the right-hand screw rule and φ will always be non-negative and less than or equal π radian.

The connection between both description of the movement of the body is given by the requirement that

$$\mathrm{R}\mathbf{w} + \mathrm{v} = \mathbf{w} + t\,\mathbf{n} + (\mathbf{1} - \cos\varphi)\,\mathbf{n}\bullet(\mathbf{n} \bullet (\mathbf{w} - \mathbf{s})) + \sin\varphi\,\mathbf{n} \bullet (\mathbf{w} - \mathbf{s}) \tag{3.2}$$

must hold for every vector w. Consequently :

$$\mathbf{v} = t\,\mathbf{n} + (1 - \cos\varphi)\,\mathbf{s} - \sin\varphi\mathbf{n}\bullet\mathbf{s} \tag{3.3}$$

$$\mathrm{R}\mathbf{w} = \mathbf{cos}\,\varphi\mathbf{w} + (\mathbf{1} - \cos\varphi)\,\mathbf{nn}^{\mathrm{T}}\,\mathbf{w} + \sin\varphi\,\mathbf{n}\bullet\mathbf{w}\text{, for every } \mathbf{w}, \tag{3.4}$$

where the last equation is seen to be equivalent with

$$½\,(\mathrm{R} - \mathrm{R}^{\mathrm{T}})\mathbf{w} = \sin\varphi\,\mathbf{n}\bullet\mathbf{w} \text{ for every } \mathbf{w} \tag{3.5}$$

$$½\,(\mathrm{R} - \mathrm{R}^{\mathrm{T}}) - \cos\varphi\,I + (1 - \cos\varphi)\mathbf{nn}^{\mathrm{T}} \tag{3.6}$$

The matrix ½ (R - R^T) is skew-eymmetric and it can easily be shown that sin φ**n** is given by :

$$\sin\phi\, n = \begin{bmatrix} R_{32} & - & R_{23} \\ R_{13} & - & R_{31} \\ R_{21} & - & R_{12} \end{bmatrix} \tag{3.7}$$

With $\mathbf{n}^{\mathbf{T}}\mathbf{n} = I$ and sin φ ≥ 0, this equation can be solved for sin φ, which results in :

$$\sin\phi = \frac{1}{2}\sqrt{(R_{12} - R_{23})^2 + (R_{13} - R_{31})^2 + (R_{21} - R_{12})^2} \tag{3.8}$$

Apart from this, cosφ can be calculated from equation (3.6) by adding up the components on the principal diagonal of the matrices. Then

$$3\cos\varphi + (1 - \cos\varphi)\mathrm{trace}\,(\mathbf{nn}^{\mathrm{T}}) = \mathrm{trace}\,(½\,(\mathrm{R} + \mathrm{R}^{\mathrm{T}})) \tag{3.9}$$

and because of trace ($\mathbf{nn}^T$) = $\mathbf{n}^T\mathbf{n}$ = 1, it follows that

$$\cos\phi = \frac{1}{2}(R_{11} + R_{22} + R_{33} - 1) \tag{3.10}$$

Both equations (3.8) and (3.10) can be used to calculate φ. For numerical reasons, it is preferred to use equation (3.8) if sin φ ≤ ½$\sqrt{2}$ and equation (3.10) if sin φ > ½$\sqrt{2}$.
As soon as sin φ is known, **n** can be determined from equation (3.7) if sin φ ≠0. From a numerical point of view, this is not recommendable if φ approaches π. With known cos φ, it is preferred to use (3.6) if φ > ¾ π . While

$$(1 - \cos\varphi)\,\mathbf{nn}^{\mathrm{T}} = ½\,(\mathrm{R} + \mathrm{R}^{\mathrm{T}}) - \cos\varphi\,I = [\,\mathbf{b}_1\ \mathbf{b}_2\ \mathbf{b}_3\,], \tag{3.11}$$

it is seen that each of the columns $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$ of the matrix ½ (R + R^T) - cos φ I is a vector in the same direction as **n**. So, apart from a factor, **n** is equal to $\mathbf{b}_1$, $\mathbf{b}_2$ and $\mathbf{b}_3$. Let $\mathbf{b}_i$ (i = 1,2,or 3)be the column with the greatest length $\sqrt{b_i^T\, b_i}$. Then **n** is determined by;

$$b_i^T\, b_i \;=\max(b_1^T\, b_1,\; b_2^T\, b_2,\; b_3^T\, b_3); \qquad n=\pm\frac{b_i}{\sqrt{b_i^T\, b_i}} \quad . \tag{3.12}$$

The sign must be chosen such that sin φ in equation (3.7) is positive. The translation t along the helical axis and the radius vector **s** of a point on the axis follow from equations (3.1) and (3.3):

$$t \;=\; \mathbf{n}^T\mathbf{v} \tag{3.13}$$

$$s=-\frac{1}{2}n*(n*v)+\frac{\sin\phi}{2(1-\cos\phi)}n*v \tag{3.14}$$

The relations (3.12), (3.13) and (3.14) hold if $\varphi \neq 0$ If $\varphi = 0$, there is no rotation at all. In this case the helical axis is not defined and therefore **n, s** and t are not unique. If $\varphi = 0$ and $\mathbf{v} \neq 0$, then one can put;

$$t=\sqrt{v^T\, v} \quad ; \qquad n=\frac{1}{t}v; \qquad s=0 \tag{3.15}$$

If $\varphi = 0$ and $\mathbf{v} = 0$, there is no movement at all. This case is of no interest.
Sufficient for the computation of **n**, φ, **s and** t are equations (3.8) and (3.10), for φ, equations (3.7) and (3.11) until (3.14) for **n**, t and **s** if $\varphi \neq 0$ and (3.14) if $\varphi = 0$. The calculation of these parameters will be the focus of our next paper.

4.0 Discussion

Accurate results are quickly achieved by the described method. Although alternative methods exist (see [3,4]), the described procedure here seems more elegant and needs no initial approximation. The analysis is rather long and the details are not given here as it will be a subject of another paper. But only few of its equations are required to achieve numerical results.

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