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Effects of Perturbation on The Stability of The Autonomized Triangular Points in The Restricted Three- Body Problem With Variable Masses Under a Luminous Primary

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Abstract

We present a survey of the stability of triangular equilibrium points of the autonomized system in the restricted three-body problem under the influence of small perturbation in the Coriolis force, together with the effects of radiation pressure of the less massive primary, in which the masses of the main bodies vary isotropically in accordance with the unified Meshcherskii's law, and their motion take place within the framework of the Gylden-Meshcherskii problem. It is seen that the constant of a particular integral κ of the Gylden-Meshcherskii problem does not enter into the positions of the triangular points; a reason due to keeping the centrifugal force constant. It is observed that the triangular points are stable for all mass ratios υ when $0 < \upsilon < \upsilon_{c_\kappa}$ and all values of κ ($0.714531 < \kappa \leq 1$), where υ_{c_κ} are the critical mass ratios. Further it is observed that the radiation pressure of the less massive primary always has a destabilizing tendency in the presence or absence of κ, while the Coriolis force always have a stabilizing tendency. The overall effect is that the region of stability increases.

Keywords: Perturbations, Gylden-Meshcherskii problem, Photogravitational.

1.0 Introduction

The restricted three-body problem describes the motion of an infinitesimal mass moving under the gravitational effects of the two finite masses, called primaries, which move in circular orbits around their center of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. The approximate circular motion of the planets around the sun and the small masses of asteroids and the satellites of planets compared to the planet's masses, originally suggested the formulation of the restricted problem. The restricted three-body model, as constructed by Lagrange, is the framework within which many modern studies have produced new results. Not surprisingly, it serves as the backdrop for research efforts with a man-made satellite assuming the role of the infinitesimal mass in the Sun - Earth/Moon system; the Earth/Moon is then assumed to be a single entity located at the Earth- Moon barycenter.

It is well known that the infinitesimal mass can be at rest in a rotating coordinate frame, at five libration points (three collinear $L_{1,2,3}$ and two triangular $L_{4,5}$), where the gravitational and centrifugal forces just balance each other. The collinear points are unstable where as triangular points are linearly stable, when the mass ratio of the primaries

is less than the Routhian value [24]. Their stability occurs in spite of the fact that the potential energy has a maximum rather than a minimum at $L_{4,5}$. The stability is actually achieved through the influence of the Coriolis force, because the coordinate system is rotating [4] and [26]. For the stability of the triangular points [25] asserted that the Coriolis force is a stabilizing force. The effects of small perturbations in the restricted three-body problem have also been studied by [1], [3], [6] and [23].

Further more, the classical model assumes that the masses of the bodies are constant, but there are various practical problems where the mass does not remain constant. It has been shown that the mass of Jupiter is increasing [20]. The masses of celestial bodies are changing during evolution. A satellite moving around a radiating star surrounded by cloud varies its mass due to particles of this cloud. Comets loose part of their mass as traveling around the Sun (or other stars) due to their interaction with the solar wind which blows off particles from their surfaces. Thus, the classical three-body problem is not suited to discuss such practical important dynamical systems. The problem of two bodies with variable masses came into science practically following the work of [10], who for the relative motion of one mass point m about the other mass point M under the action of mutual gravitational force represented the sum of the masses of these points as varying with time by a certain law $M+m=\mu(t)$. [11] showed that the Gylden problem is a particular case of the problem of two bodies with variable masses under the condition that the laws of variation of the masses vary isotropically. The restricted problem dealing with variable mass of one or more bodies under different respects has been investigated by [2], [7], [9], [12], [20] and [22].

[15] and [16] formulated the photogravitational restricted three-body problem involving the sun, a planet and a dust particle and found that an allowance for direct solar radiation pressure forces result in a change in the positions of the libration points and to the appearance of new libration points (coplanar points). The restricted problem dealing with one or both radiating bodies under different aspects have also been studied by [5], [13], [17], [18], [19], [21] e.t.c.

In this paper we study the effects of a small perturbation in the Coriolis force on the locations and stability of autonomized triangular libration points in the photogravitational restricted problem of three bodies with variable masses, under the condition that the motion of the variable-mass main bodies is determined by the Gylden-Meshcherskii problem with isotropic mass variation of the primaries varying in proportion to each other in accordance with the unified Meshcherskii law, keeping the centrifugal force constant and placing some restrictions on the constant κ of a particular integral of the Gylden-Meshcherskii problem. The problem is photogravitational in the sense that the smaller primary is taken to be an intense emitter of radiation.

2.0 Equations of motion

Let m_1 and m_2 be the masses of the more and less massive primaries respectively, and let m be the mass of the infinitesimal body. We assume that the less massive primary is a source of radiation. We introduce a synodic coordinate system $(0, X, Y, Z)$ with the origin at the center of mass of the primaries, in which the axes rotate relative to the inertial space with angular velocity ω about the Z-axis. Let $(x_1, 0, 0)$ and $(x_2, 0, 0)$ be the coordinates of m_1 and m_2 respectively, and let (x, y, z) be the coordinates of the infinitesimal body in the orbital plane. The distances between m and m_1, m and m_2, m_1 and m_2 are r_1, r_2 and r, respectively. As the motion of the primaries is assumed to move within the framework of the Gylden-Meshcherskii problem, we only need to find the motion of the infinitesimal mass m.

The equations of motion of the infinitesimal body in the barycentric coordinate system $(0, X, Y, Z)$ have the form [2] and [9]:

$$\ddot{x} - 2\omega\dot{y} = \omega^2 x + \dot{\omega} y - \frac{\mu_1 (x - x_1)}{r_1^3} - \frac{\mu_2 (x - x_2)}{r_2^3}$$

$$\ddot{y} + 2\omega\dot{x} = \omega^2 y - \dot{\omega} x - \frac{\mu_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3} \qquad (2.1)$$

$$\ddot{z} = -\frac{\mu_1 z}{r_1^3} - \frac{\mu_2 z}{r_2^3}$$

where, $r_i^2 = (x - x_i)^2 + y^2 + z^2$, $\mu_i(t) = fm_i$, $i = 1, 2$

.

where f the gravitational constant. and dot denotes differentiation with respect to time t.
We introduce a small perturbation in the Coriolis force with the help of the parameter φ such that

$$\varphi = 1 + \in; \quad \in = 1.$$

The radiation repulsive force F_p exerted on a particle can be represented in terms of the gravitational attraction F_g ([15]) as

$$F_p = F_g (1 - q)$$

Here

$$q = 1 - (F_p / F_g)$$

Hence, the equations of motion of the perturbed photogravitational restricted three-body problem with variable masses now takes the form:

$$\ddot{x} - 2\omega\dot{y}\varphi - \omega^2 x\psi = \dot{\omega} y - \frac{\mu_1 (x - x_1)}{r_1^3} - \frac{\mu_2 q (x - x_2)}{r_2^3}$$

$$\ddot{y} + 2\omega\dot{x}\varphi - \omega^2 y\psi = -\dot{\omega} x - \frac{\mu_1 y}{r_1^3} - \frac{\mu_2 q y}{r_2^3} \qquad (2.2)$$

$$\ddot{z} = -\frac{\mu_1 z}{r_1^3} - \frac{\mu_2 q z}{r_2^3}$$

where

$$r_i^2 = (x - x_i)^2 + y^2 + \zeta^2,$$

$$\varphi = 1 + \in; \ \in = 1, \quad q = 1 - \in''; \quad \in'' = 1,$$

q is radiation factors of the less massive primary.
The system of equations (2.2) is non-integrable differential equations with variable coefficients. Thus, to obtain useful dynamical predictions, we perform autonomization process by transforming (2.2) to a dynamical system with constant coefficients.

Following [12], we transform (2.2) from (x, y, t) to the autonomized form (ξ, η, τ), using a Meshcherskii's transformation

$$x = \xi R(t), \ y = \eta R(t), \ \frac{dt}{d\tau} = R^2(t), \ r_i = \rho_i R(t), (i = 1, 2), \qquad (2.3)$$

the particular solutions of the Gylden-Meshcherskii problem

$$\omega(t) = \frac{\omega_0}{R(t)}, \quad x_1 = \xi_1 R(t), \quad x_2 = \xi_2 R(t), \ r = \rho_{12} R(t) \qquad (2.4)$$

and the unified Meshcherskii's law,

$$\mu(t)=\frac{\mu_0}{R(t)},\quad \mu_1(t)=\frac{\mu_{10}}{R(t)},\quad \mu_2(t)=\frac{\mu_{20}}{R(t)},\quad \mu(t)=\mu_1(t)+\mu_2(t)$$

where

$R(t)=\sqrt{\alpha t^2+2\beta t+\gamma}:\alpha,\beta,\gamma,\mu_0,\mu_{10}$ and μ_{20} are constants, (2.5)

$$\mu_1(t)=fm_1(t),\quad \mu_2(t)=fm_2(t),$$

System (2.2) now takes the form

$$\xi''-2\omega_0\varphi\eta'=\frac{\partial\Omega}{\partial\xi},\quad \eta''+2\omega_0\varphi\xi'=\frac{\partial\Omega}{\partial\eta} \tag{2.6}$$

$$\Omega=\frac{(\omega_0^2+\Delta)}{2}\left(\xi^2+\eta^2\right)+\frac{\Delta\left(\xi^2+\eta^2\right)}{2}+\frac{\mu_{10}}{\rho_1}+q\frac{\mu_{20}}{\rho_2}$$

where

$$\rho_1^2=\left(\xi-\xi_1\right)^2+\eta^2,\quad \rho_2^2=\left(\xi-\xi_2\right)^2+\eta^2$$

$$\xi_1=\frac{-\mu_{20}}{\mu_0}\rho_{12},\quad \xi_2=\frac{\mu_{10}}{\mu_0}\rho_{12},\quad \Delta=\beta^2-\alpha\gamma,$$

and dashes denote differentiation with respect to τ.

Choosing measurement units and introducing the mass parameter υ as in [22], the autonomized system (2.6) becomes

$$\xi''-2\varphi\eta'=\frac{\partial\Omega}{\partial\xi},\quad \eta''+2\varphi\xi'=\frac{\partial\Omega}{\partial\eta} \tag{2.7}$$

$$\Omega=\frac{\kappa\left(\xi^2+\eta^2\right)}{2}+\kappa\frac{(1-\upsilon)}{\rho_1}+q\kappa\frac{\upsilon}{\rho_2}$$

where

$$\rho_1^2=\left(\xi+\upsilon\right)^2+\eta^2,\quad \rho_2^2=\left(\xi+\upsilon-1\right)^2+\eta^2$$

$\kappa=\beta^2-\alpha\gamma+1$ is a constant of integration of a particular integral [9]

$$r\mu=\kappa C^2, \tag{2.8}$$

of the Gylden-Meshcherskii problem. $C\neq 0$ is a constant of the area integral

The ranges of variation of the parameter κ are;

i. If $\Delta=0$, we would have $\kappa=1$

ii. If $\Delta>0$, this implies $1<\kappa<\infty$ (2.9)

iii. If $\Delta<0$, this implies $0<\kappa<1$.

Though κ can take values between zero and infinity, in our problem we consider only values in the range $0.714532\leq\kappa\leq 1$, for large values of κ are not physically meaningful.

3.0 Positions and linear stability of the triangular points

The triangular equilibrium points of the autonomized system are the solutions of the equations

$$\frac{\partial\Omega}{\partial\xi}=0,\quad \frac{\partial\Omega}{\partial\eta}=0$$

i.e. $$\kappa\xi-\frac{\kappa(1-\upsilon)(\xi+\upsilon)}{\rho_1^3}-\frac{q\kappa\upsilon(\xi+\upsilon-1)}{\rho_2^3}=0 \tag{3.1}$$

and $$\left(\kappa-\frac{\kappa(1-\upsilon)}{\rho_1^3}-\frac{q\kappa\upsilon}{\rho_2^3}\right)\eta=0,\ \eta\neq 0 \tag{3.2}$$

from these, we have

$$\rho_1=1 \text{ and } \rho_2=q_2^{1/3} \tag{3.3}$$

Substituting expressions above in (3.1) and (3.2), the coordinates of the triangular points are;

$$\xi=\frac{1}{2}(1-q^{2/3}-2\upsilon+1), \qquad \eta=\pm\left\{\frac{1+q^{2/3}}{2}-\left(\frac{1-q^{2/3}}{2}\right)^2-\frac{1}{4}\right\}^{\frac{1}{2}} \tag{3.4}$$

where the positive sign corresponds to L_4 and the negative to L_5 .These points form simple triangles with the line joining the primaries. It is obvious that the positions of the triangular points are affected by the factors which appear due to radiation pressure of the less massive primary but are not affected due to the introduction of perturbation in the Coriolis force. If the radiation factor of the smaller primary are ignored i.e. $q_2=1$, the points L_4 and L_5 of the autonomized system will fully analogous to the classical case.

Next, we study the linear stability around the equilibrium points. Due to a small perturbation in the Coriolis force of the primaries and perturbations induced by the radiation pressure of the less massive primary, the position of the infinitesimal body would be displaced a little from the equilibrium point. If the resultant motion of the infinitesimal mass is a rapid departure from the vicinity of the point, we can call such a position of equilibrium point an "unstable one", if however the body merely oscillates about the equilibrium point, it is said to be a "stable position" (in the sense of Lyapunov). We denote the equilibrium points and their positions as $L(\xi_0,\eta_0)$. Let a small displacement in (ξ_0,η_0) be (u,v). Then we can write

$$\xi=\xi_0+u, \qquad \eta=\eta_0+v, \tag{3.5}$$

Substituting these values in equations (2.8), we obtain the variational equations,

$$\begin{aligned} u''-2\varphi v' &= \left(\Omega^0_{\xi\xi}\right)u+\left(\Omega^0_{\xi\eta}\right)v \\ v''+2\varphi u' &= \left(\Omega^0_{\xi\eta}\right)u+\left(\Omega^0_{\eta\eta}\right)v \end{aligned} \tag{3.6}$$

The characteristic equation corresponding to (3.6), is

$$\lambda^4-\left(\Omega^0_{\xi\xi}+\Omega^0_{\eta\eta}-4\varphi^2\right)\lambda^2+\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}-\left(\Omega^0_{\xi\eta}\right)^2=0 \tag{3.7}$$

where the superscript 0 indicates that the partial derivatives are evaluated at the equilibrium points (ξ_0,η_0). In a computation of these derivatives, we will substitute, $\varphi=1+\in$, $q=1-\in'$, where $\in$ and $\in'$ are very small positive quantities, and neglect their second and higher order terms and also their products. Then we have,

$$\Omega^o_{\xi\xi}=\frac{\kappa}{4}\left(3+4\in-6\upsilon\in\right) \tag{3.8}$$

$$\Omega^o_{\eta\eta}=\frac{\kappa}{4}\left(9-4\in+6\upsilon\in\right) \tag{3.9}$$

$$\Omega^{o}_{\xi\eta} = \frac{\kappa}{4}\left(3-6\upsilon+2\in-2\upsilon\in\right)\left(\frac{9-4\in}{3}\right)^{1/2} \tag{3.10}$$

Substituting equations (3.8), (3.9) and (3.10) in the characteristic equation (16), yields

$$\lambda^4 - \left(3\kappa-4-8\in'\right)\lambda^2 + \frac{3}{4}\kappa^2\left(9+2\in\right)\upsilon\left(1-\upsilon\right)=0 \tag{3.11}$$

The roots of (3.11), are given by

$$\lambda^2_{1,2} = \frac{-P \pm \sqrt{D}}{2} \tag{3.12}$$

where $P = 4-3\kappa+8\in'$, $D = P^2-4Q$

$$Q = \frac{3\kappa^2}{4}\left(9+2\in\right)\upsilon\left(1-\upsilon\right) > 0 \tag{3.13}$$

Hence, the roots of the characteristic equation depend, on the mass parameter υ, the radiation parameter $\in$, perturbation in the Coriolis force $\in'$. So the nature of these roots is controlled by $\in'$, κ and the sign of the discriminant D, given by

$$D = 3\kappa^2\left(9+2\in\right)\upsilon^2 - 3\kappa^2\left(9+2\in\right)\upsilon + 9\kappa^2 - 24\kappa + 16 + 16\in'\left(4-3\kappa\right)$$

Since D is a monotonous function of υ in the interval $(0, 1/2]$ and has values opposite in signs at endpoints $(D)_{\upsilon=0}$ and $(D)_{\upsilon=1/2}$, there are several values of υ, say υ_{c_κ} in the interval $0 < \upsilon \le \frac{1}{2}$ for which the discriminant vanishes. Since the nature of the roots depend on the nature of the discriminant, three cases are possible:

1. When $0 < \upsilon < \upsilon_{c_\kappa}$, $D > 0$, $P > 0$ always as $\kappa \le 1$, in this case, all λ_i $(i=1,2,3,4)$ are pure imaginary and given as

   $$\lambda_{1,2,3,4} = \pm i\Lambda_n \qquad (n=1,2)$$

where

$$\Lambda_1 = \sqrt{\frac{1}{2}\left(-P+\sqrt{D}\right)} \text{ and } \Lambda_2 = \sqrt{\frac{1}{2}\left(-P-\sqrt{D}\right)} \tag{3.14}$$

Consequently, the triangular point is stable in this case.
The general solution is written [24], as

$$\begin{aligned} u &= A_1\cos\Lambda_1\tau + C_1\sin\Lambda_1\tau + A_2\cos\Lambda_2\tau + C_2\sin\Lambda_2\tau \\ v &= \bar{A}_1\cos\Lambda_1\tau + \bar{C}_1\sin\Lambda_1\tau + \bar{A}_2\cos\Lambda_2\tau + \bar{C}_2\sin\Lambda_2\tau \end{aligned} \tag{3.15}$$

where, $A_i, \bar{A}_i, C_i$ *and* $\bar{C}_i$ ($i=1,2$) are constants.

2. When $\upsilon_{c_\kappa} < \upsilon \le \frac{1}{2}$, $D < 0$ with $P = 0$ or $P > 0$, the discriminant is negative. The real parts of two of the values of λ are positive and equal. Therefore, the triangular point is unstable.

3. When $\upsilon = \upsilon_{c_\kappa}$, D is zero. The double roots give secular terms in the solutions of the variational equations of motion. Therefore, the triangular point is unstable.

3.1 Critical Mass Ratio

The critical value of the mass parameter is the value of the mass ratio υ when the discriminant vanishes. In our problem there are several values of the critical mass parameters and are given by

$$\upsilon_{c_\kappa} = \upsilon_{o_\kappa} + \upsilon_{r_\kappa} + \upsilon_{p_\kappa} \qquad (3.16)$$

where $\upsilon_{o_\kappa} = \frac{1}{2} - \frac{1}{6\kappa\sqrt{3}}\sqrt{96\kappa - 9\kappa^2 - 64}$, $\upsilon_{r_\kappa} = \frac{2(24\kappa - 9\kappa^2 - 16)}{27\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\in$,

$$\upsilon_{p_\kappa} = \frac{4(144 - 108\kappa)}{27\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\in$$

Clearly, υ_{c_κ} represents the effects of the constant (κ) of a particular integral of the Gylden-Mescherskii problem, perturbations and radiation, on the critical mass value of the restricted problem. However, for $\kappa = 1$, the value of

υ_{o_1} coincides with the classical Routhian value $\mu_o = 0.038521$ In addition in the absence of radiation of the smaller primary and perturbation in Coriolis force i.e.$\in = \in' = 0$ and $\kappa = 1$, the value υ_{c_κ} will fully coincide with the classical case given by [24]. When the smaller primary is non radiating (i.e. $\upsilon_{r_\kappa} = 0$) and $\kappa = 1$ the critical mass value verifies the results of [25]. For any $0.714532 \le \kappa \le 1$
The critical mass values, υ_{C_κ} , for the values of κ in the interval $0.714532 \le \kappa \le 1$ are

$$\begin{aligned}
\upsilon_{C_{0.714532}} &= 0.498889 - 24.9788500\in_2 + 322.9320\in \\
\upsilon_{C_{0.720000}} &= 0.409910 - 0.29832496\in_2 + 11.67358\in \\
\upsilon_{C_{0.750000}} &= 0.280104 - 0.10188998\in_2 + 4.1920451\in \\
\upsilon_{C_{0.999900}} &= 0.038553 - 0.00892522\in_2 + 0.6465822\in \\
\upsilon_{C_1} &= 0.038520 - 0.00891747\in_2 + 0.6420578\in
\end{aligned} \qquad (3.17)$$

It is seen form the system of equations above that the Coriolis force has stronger stabilizing behavior that can counter the destabilizing tendency of the radiation pressure of the less massive primary. So that the overall effect is that the region of stability in this case is always increasing. We conclude that the triangular point is stable for $0 < \upsilon < \upsilon_{c_\kappa}$ and unstable for $\upsilon_{c_\kappa} \le \upsilon \le \frac{1}{2}$ due to the constant of a particular integral of the Gylden-Meshcherskii problem..

4.0 Discussion

The positions of the triangular points of the autonomized system different from that as worked out by [25] due to the introduction of the radiation pressure of the smaller primary. If this is ignored the points will fully coincide. These positions are different from that of [6] due to the radiating effects of the less massive primary and absence of the centrifugal force. We observe that the constant of a particular integral κ of the Gylden-Meshcherskii problem does not enter into the location of the triangular points, a reason of which is due to keeping the centrifugal force constant.

Equation (3.17) gives the critical values υ_{c_κ} of the mass parameter. It shows the effects of perturbation in the Coriolis force and radiation pressure force of the less massive primary on the critical mass value. Putting $\kappa = 1$ and ignoring the radiation coefficient of the less massive primary i.e. $\upsilon_{r_\kappa} = 0$ in equation (3.17) gives the critical value obtained by [25]. If further we ignore the effects of the Coriolis force, υ_{c_1} will fully coincide with the classical Routhian value given by [24].
It was further seen that radiation coefficient has strong destabilizing tendencies in the presence of the parameter κ. The overall effect is that the range of stability of the triangular points decreases.

5.0 Concluding remarks

The stability of triangular equilibrium points of the autonomous equations of motion under the influence of a constant κ of a particular integral of the motion of the variable primary bodies, together with the effects of the luminous less massive primary and perturbation in Coriolis, is seen to be stable for all mass ratio when $0 < \upsilon < \upsilon_{c_\kappa}$ and unstable when $\upsilon_{c_\kappa} \le \upsilon \le \frac{1}{2}$, where υ_{c_κ} are the critical mass value, which depends on the joint effect of the parameters.

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