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On The Stability Of Equilibrium Points In The Perturbed Restricted Three- Body Problem With Variable Masses

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Abstract

The stability of five equilibrium points is investigated in the restricted three-body problem under the influence of small perturbations in the Coriolis and centrifugal forces, in which the masses of the main bodies vary isotropically in accordance with the combined Meshcherskii's law. For the autonomized system, it is found that collinear points remain unstable despite the introduction of perturbations, while triangular points are stable with respect to a constant κ of a particular integral of the Gylden-Meshcherskii problem. It is further observed in the triangular case that the presence of this constant make the Coriolis and the centrifugal forces possess both stabilizing and destabilizing behaviors. The region of stability of triangular points depend on κ, and does not exist for $\kappa \leq 0.714531$, $\kappa = 1.333333$ and $\kappa \geq 9.952$. The equilibrium points of the non autonomous dynamical system are found to be generally unstable using the Lyapunov Characteristic Numbers.

Keyword: **celestial mechanics; mass variation**

1.0 Introduction

The restricted three-body problem describes the motion of an infinitesimal mass moving under the gravitational effects of the two finite masses, called primaries, which move in circular orbits around their center of mass on account of their mutual attraction and the infinitesimal mass not influencing the motion of the primaries. The approximate circular motion of the planets around the sun, and the small masses of the asteroids and the satellites of the planets compared to the planet's masses, originally suggested the formulation of the above restricted problem.

Poincare [18] introduced the idea of qualitative understanding of motion in the three body problem by investigating the flow associated with the governing differential equations. He in particular tried to characterize the stability of motion of three mutually gravitationally attracted bodies. He quickly discovered the notable fact that a small variation in initial conditions could cause drastically different dynamical behaviors.

Routh [19] established the condition for linear stability of the triangular libration points. When this condition is satisfied, all the roots of the characteristic equation are pure imaginary, which leads to pure oscillatory solution.

The first investigation of the existence of the libration points for variable masses in the absence of reactive forces was performed by [17], in which the plane problem of three bodies with finite variable masses was considered, and the existence of five analogous particular solutions was established. Winter

[27] showed that the stability of the triangular points is due to the existence of the Coriolis terms in the equations of motion when they are written in a rotating coordinate system. The absence of the Coriolis force renders the triangular solution unstable according to Wintner, so that the oscillatory solution of the linearized equations of motion is replaced by exponential terms with real characteristic exponents.

The effect of a small perturbation of the Coriolis force on the stability of the equilibrium points, keeping the centrifugal force constant, was studied by [15]. He maintained that the collinear points remain unstable and obtained for the stability of the triangular points a relation between the critical value of the mass parameter μ_c and the change $\in$ in the Coriolis force:

$$\mu_c = \mu_0 + \frac{16\in}{3\sqrt{69}}.$$

He concluded that the Coriolis force is a stabilizing force. This work was also extended by [5], by considering the effect of perturbations $\in$ and $\in'$ in the Coriolis and centrifugal forces, respectively and found that collinear points remain unstable; for the triangular points they obtained the relation

$$\mu_c = \mu_0 + \frac{4\left(36\in - 19\in'\right)}{27\sqrt{69}}.$$

They inferred that the range of stability increases or decreases depending on whether the points $(\in,\in')$ lies in one or the other of the two parts in which the $(\in,\in')$ plane is divided by the line $36\in - 19\in' = 0$. [20] and [21], later investigated the effect of small perturbations in the Coriolis and centrifugal forces in the restricted three-body problem with variable mass under the assumption that the infinitesimal mass is variable and the primaries are spherical with constant masses. The combined effect of perturbations, radiation and oblateness on the stability of equilibrium points in the restricted three-body problem was studied by [1]. They found that the collinear points remain unstable, while the triangular are stable for $0 < \mu \le \mu_c$ and unstable for $\mu_c < \mu \le \frac{1}{2}$. They observed further that the Coriolis force has a stabilizing tendency, while the centrifugal force, radiation and oblateness of the primaries have destabilizing effects; consequently the overall effect is that the range of stability of the triangular points decreases.

Gasanov [7] investigated the libration points and the general case in the problem of the motion of a star inside a layered inhomogeneous elliptical galaxy with variable mass and established seven liberation points of the autonomized equations, located (except for one) outside the gravitating galaxy. He examined the stability of these points using the Lyapunov Characteristic Number (L.C.N), and concluded that solutions with negative exponents are stable. [22] examined the effects of perturbations on the nonlinear stability of triangular points in the restricted three-body problem with variable mass. [23] studied the stability of seven equilibrium points in the photogravitational restricted three-body problem with variable masses. They found that the collinear and coplanar points of the autonomized system are unstable and the triangular points conditionally stable while the stability of these solutions for the non autonomous dynamical system were studied based on the concept of the Lyapunov Characteristic Numbers (LCN) and they found that solutions with negative exponents consequently having positive LCN are stable, those with positive exponents having negative LCN are unstable, while the stability or instability of constant solutions and solutions with pure imaginary exponents, with zero LCN's cannot be determined. They concluded in general that motion around the equilibrium points $L_i\,(i = 1, 2...7)$ for the restricted three-body problem with variable masses is unstable.

A study of the motion and the existence of libration points in the restricted problem, under the condition that the motion of the variable-mass main bodies is within the framework of the Gylden-Meshcherskii problem([9] and [11]) with isotropic mass variation of the primaries varying in proportion to each other in accordance with the combined Meshcherskii law, have been studied by [2], [3], [4], [6], [7], [8] and [16].

Our aim in this paper is to study the effects of perturbations in the Coriolis and centrifugal forces on the locations and stability of five equilibrium points, when the masses of the primaries vary according to the unified law [11].

2.0 Equations Of Motion

Let (x, y, z) be the coordinates of the infinitesimal body in the orbital plane.
The equations of motion of the infinitesimal mass m in the gravitational field of the luminous primary of variable masses m_1 and m_2 in a barycentric coordinate system Oxyz, rotating with an angular velocity $\omega(t)$ about the z-axis perpendicular to the plane of motion of the primaries, while the x-axis always passes through these points have the form [2] and [8]

$$\begin{aligned}
\ddot{x} - 2\omega\dot{y} &= \omega^2 x + \dot{\omega} y - \mu_1 \frac{(x-x_1)}{r_1^3} - \mu_2 \frac{(x-x_2)}{r_2^3}, \\
\ddot{y} + 2\omega\dot{x} &= \omega^2 y - \dot{\omega} x - \mu_1 \frac{y}{r_1^3} - \mu_2 \frac{y}{r_2^3}, \qquad (2.1) \\
\ddot{z} &= -\mu_1 \frac{z}{r_1^3} - \mu_2 \frac{z}{r_2^3},
\end{aligned}$$

with
$$r_1^2 = (x-x_1)^2 + y^2 + z^2, \quad r_2^2 = (x-x_2) + y^2 + z^2$$
$$x_1 = -\frac{\mu_2 r}{\mu_1 + \mu_2}, \quad x_2 = \frac{\mu_1 r}{\mu_1 + \mu_2}$$

where r_1 and r_2 are distances of the infinitesimal mass from these primaries positioned at $(x_1, 0, 0)$ and $(x_2, 0, 0)$. μ_1 and μ_2 are the product of the masses of the primaries and gravitational constant f and a dot denotes differentiation with respect to time t.
Next, we introduce small perturbations in the Coriolis and centrifugal forces with the help of the parameters φ and ψ respectively such that

$$\varphi = 1 + \in; \quad \in = 1, \quad \psi = 1 + \in'; \quad \in' = 1$$

Hence, equations of motion of the infinitesimal mass m in the perturbed gravitational field of the variable primaries in a coordinate system (x, y) has the form:

$$\begin{aligned}
\ddot{x} - 2\omega\dot{y}\varphi - \dot{\omega}\varphi y &= \omega^2 x\psi - \frac{\mu_1 (x-x_1)}{r_1^3} - \frac{\mu_2 (x-x_2)}{r_2^3} \qquad (2.2) \\
\ddot{y} + 2\omega\dot{x}\varphi + \dot{\omega}\varphi x &= \omega^2 y\psi - \frac{\mu_1 y}{r_1^3} - \frac{\mu_2 y}{r_2^3}
\end{aligned}$$

where $\quad r_i^2 = (x - x_i)^2 + y^2, \quad \varphi = 1 + \in \quad \psi = 1 + \in', \quad i = 1, 2$

Here $\in, \in'$ represent the perturbation in the Coriolis and centrifugal forces .The third expression of system (2.2) does not appear has we consider motion in the $xy-$ plane only.

3.0 Autonomization of the Equations of Motion

The equations of motion of system (2.2) are non-integrable differential equations with variable coefficients. We transform from (x, y, t) to (ξ, η, τ) with the help of a Meshcherskii's transformation

$$x=\xi R(t),\ \ y=\eta R(t),\ \frac{dt}{d\tau}=R^2(t)\,, \tag{3.1}$$

$$r_i=\rho_i R(t),(i=1,2)\,,\ r=\rho_{12}R(t)\,,$$

the particular solutions of the Gylden-Meshcherskii problem

$$\omega(t)=\frac{\omega_0}{R^2(t)},\quad x_1=\xi_1 R(t),\quad x_2=\xi_2 R(t)\,,\ C=\rho_{12}^2\omega_0 \tag{3.2}$$

and the unified Meshcherskii's law (1902),

$$\mu(t)=\frac{\mu_0}{R(t)}\,,\ \mu_1(t)=\frac{\mu_{10}}{R(t)}\,,\mu_2(t)=\frac{\mu_{20}}{R(t)}\,,\mu(t)=\mu_1(t)+\mu_2(t)$$

where $R(t)=\sqrt{\alpha t^2+2\beta t+\gamma}\,:,\alpha,\beta,\gamma,\mu_0,\mu_{10}$ and μ_{20} are constants, (3.3)

The system (2.2) in the autonomized form becomes

$$\xi''-2\omega_0\varphi\eta'=\frac{\partial\Omega}{\partial\xi},\quad \eta''+2\omega_0\varphi\xi'=\frac{\partial\Omega}{\partial\eta} \tag{3.4}$$

where
$$\Omega=\frac{(\xi^2+\eta^2)(\omega_0^2\psi+\Delta)}{2}+\frac{\mu_{10}}{\rho_1}+\frac{\mu_{20}}{\rho_2}$$

$$\rho_i^2=(\xi-\xi_i)^2+\eta^2\,,\quad \xi_1=\frac{-\mu_{20}}{\mu_0}\rho_{12}\,,\quad \xi_2=\frac{\mu_{10}}{\mu_0}\rho_{12}\,,\ \Delta=\beta^2-\alpha\gamma\,,\quad i=1,2$$

and dashes denote differentiation with respect to τ .

We make choices of units at initial time t_0 such that

$$\mu_0=f\,,\rho_{12}=1,\ \omega_0=1,\ \Delta=\kappa-1\,,$$

where $\kappa=\dfrac{(\beta^2-\alpha\gamma+\omega_0^2)}{\omega_0^2}$ is a constant of integration of a particular integral [8]

$$r\mu=\kappa C^2\,, \tag{3.5}$$

of the Gylden-Meshcherskii problem. $C\neq 0$ is a constant of the area integral, from which we have

$$\mu_0=\kappa$$

The ranges of variation of the parameter κ are; i. If $\Delta=0$, we would have $\kappa=1$
ii. If $\Delta>0$, this implies $1<\kappa<\infty$ and iii. If $\Delta<0$, this implies $0<\kappa<1$.

Introducing the mass parameter υ, expressed as

$$\frac{\mu_{10}}{\mu_0}=1-\upsilon\,,\ \frac{\mu_{20}}{\mu_0}=\upsilon\,,\quad \text{where } 0<\upsilon\leq\frac{1}{2}\,.$$

where υ, is the ratio of the mass of the smaller primary to the total mass of the primaries.
With the choice of these constants, system (3.4) takes the form

$$\xi''-2\varphi\eta'=\frac{\partial\Omega}{\partial\xi},\quad \eta''+2\varphi\xi'=\frac{\partial\Omega}{\partial\eta} \tag{3.6}$$

where
$$\Omega=\frac{(\xi^2+\eta^2)(\psi+\kappa-1)}{2}+\frac{\kappa(1-\upsilon)}{\rho_1}+\frac{\kappa\upsilon}{\rho_2}$$

$$\rho_1=\sqrt{(\xi+\upsilon)^2+\eta^2}\ ,\rho_2=\sqrt{(\xi+\upsilon-1)^2+\eta^2}\ ,\ 0<\kappa<\infty$$

4.0 Locations Of Equilibrium Points

It is well known that the infinitesimal mass can be at rest in a rotating coordinate frame, at five libration points (three collinear $L_{1,2,3}$ and two triangular $L_{4,5}$), where the gravitational and centrifugal forces just balance each other. The particular solutions of the restricted three-body problem for the isotropic case of mass variation have been considered in different respects by [2], [4], [8], [13], and [23].

4.1 Locations of Triangular Points

The triangular points are the solutions of the equations

$$\frac{\partial\Omega}{\partial\xi}=0,\ \frac{\partial\Omega}{\partial\eta}=0$$

i.e. $$(\psi+\kappa-1)\xi-\frac{\kappa(1-\upsilon)(\xi+\upsilon)}{\rho_1^3}-\frac{\kappa\upsilon(\xi+\upsilon-1)}{\rho_2^3}=0$$

and $$\left(\psi+\kappa-1-\frac{\kappa(1-\upsilon)}{\rho_1^3}-\frac{\kappa\upsilon}{\rho_2^3}\right)\eta=0,\ \eta\neq0 \tag{4.1}$$

Hence the exact coordinate of triangular points corresponding to L_4 and L_5 are

$$\xi=\frac{\rho_1^2-\rho_2^2}{2}-\upsilon+\frac{1}{2},\ \eta=\pm\left\{\frac{\rho_1^2+\rho_2^2}{2}-\left(\frac{\rho_1^2-\rho_2^2}{2}\right)^2-\frac{1}{4}\right\}^{\frac{1}{2}} \tag{4.2}$$

from (4.1), we have

$$\rho_1=\left(\frac{\kappa}{\psi+\kappa-1}\right)^{\frac{1}{3}}\text{ and }\rho_2=\left(\frac{\kappa}{\psi+\kappa-1}\right)^{\frac{1}{3}}$$

Substituting expressions above in (4.2), the coordinates of the triangular points are;

$$\xi=\frac{1}{2}[1-2\upsilon],\qquad \eta=\pm\left\{\frac{\kappa^{2/3}}{(\psi+\kappa-1)^{2/3}}-\frac{1}{4}\right\}^{\frac{1}{2}} \tag{4.3}$$

where the positive sign corresponds to L_4 and the negative to L_5 .These points form simple triangles with the line joining the primaries, different from the classical problem where these points make equilateral triangles. It is obvious that the positions of the triangular points L_4 and L_5 are affected by the factors which appear due to perturbation in the centrifugal force. If this is ignored, i.e., $\psi=1$ the points L_4 and L_5 of the autonomized system will be fully analogous to the classical case.

4.2 Locations of Collinear Points

The collinear points are the solutions of the equations

$$\Omega_\xi=0,\ \eta=0$$

That is, the collinear points lie on the line joining the two primaries. To obtain the abscissa, we denote the expression $\left(\Omega_{\xi}\right)_{\eta=\zeta=0}$ by $f\left(\xi\right)$. There are only three roots, $\varepsilon_1, \varepsilon_2$ and ε_3 of the equation $f\left(\xi\right)=0$, with one lying in each of the interval $\left(-\upsilon-1,-\upsilon\right),\left(-\upsilon,1-\upsilon\right)$ and $\left(1-\upsilon,2-\upsilon\right)$. These three roots correspond to the three collinear points L_1, L_2 and L_3. Their respective abscissae are

$$\xi^1=-\upsilon-\varepsilon_1, \quad \xi^2=1-\upsilon-\varepsilon_2, \quad \xi^3=1-\upsilon+\varepsilon_3, \tag{4.4}$$

where $\varepsilon_i>0, i=1,2,3$ are roots of the equation $f\left(\xi\right)=\left(\Omega_{\xi}\right)_{\eta=\zeta=0}=0$.

For the system of equations with variable coefficients, the equilibrium points are determined from the transformation (3.1) in the form [14]

$$x^{(i)}=\xi^{(i)}R\left(t\right), y^{(i)}=\eta^{(i)}R\left(t\right), \; i=4,5 \tag{4.5}$$

where $\xi^{(i)}\left(\tau\right),\eta^{(i)}\left(\tau\right)$ and $\zeta^{(i)}\left(\tau\right),\left(i=1,2,...5\right)$ are the libration points of the system with constant coefficients. Consequently, the triangular points of system (2.2), have the form

$$x^{(4,5)}=\left[\frac{1}{2}-\upsilon\right]R\left(t\right), \; y^{(4,5)}=\pm\left\{\frac{\kappa^{2/3}}{(\psi+\kappa-1)^{2/3}}-\frac{1}{4}\right\}^{\frac{1}{2}}R\left(t\right) \tag{4.6}$$

and collinear points are represented as

$$\begin{aligned} x^{(1)}\left(t\right)&=\left(-\upsilon-\varepsilon_1\right)R\left(t\right)\\ x^{(2)}\left(t\right)&=\left(1-\upsilon-\varepsilon_2\right)R\left(t\right)\\ x^{(3)}\left(t\right)&=\left(1-\upsilon+\varepsilon_3\right)R\left(t\right)\end{aligned} \tag{4.7}$$

The libration points of the system of equations (2.2) with variable coefficients and those of the autonomized systems (3.6) differ only by the term $R\left(t\right)$.

5.0 Stability Of Equilibrium Points Of The Autonomized System

The stability of constant coefficients linear systems of ordinary differential equations is determined completely by the Eigen values of the coefficient matrix. Due to the small perturbations in the Coriolis and centrifugal forces of the primaries and the parameter κ due to variation in masses of the primaries, the position of the infinitesimal body would be displaced a little from the equilibrium point

We denote the equilibrium points and their positions as $L\left(\xi_0,\eta_0\right)$.Let a small displacement in $\left(\xi_0,\eta_0\right)$ be $\left(u,v\right)$. Then we write

$$\xi=\xi_0+u, \quad \eta=\eta_0+v \tag{5.1}$$

Substituting these values in equations of system (8) we obtain the variational equations,

$$\begin{aligned} u''-2\varphi v'&=\left(\Omega^0_{\xi\xi}\right)u+\left(\Omega^0_{\xi\eta}\right)v,\\ v''+2\varphi u'&=\left(\Omega^0_{\xi\eta}\right)u+\left(\Omega^0_{\eta\eta}\right)v,\end{aligned} \tag{5.2}$$

The characteristic equation corresponding to (17), is

$$\lambda^4-\left(\Omega^0_{\xi\xi}+\Omega^0_{\eta\eta}-4\varphi^2\right)\lambda^2+\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}-\left(\Omega^0_{\xi\eta}\right)^2=0 \tag{5.3}$$

Where the superscript 0 indicates that the partial derivatives are evaluated at the equilibrium points $\left(\xi_0,\eta_0\right)$. In computation of these derivatives, we substitute, $\varphi=1+\in, \psi=1+\in'$, $i=1,2$ and neglect the second and higher order terms in $\in,\in'$ and their products.

5.1 The Triangular Points

In the case of triangular points, these partial derivatives are given by

$$\Omega^o_{\xi\xi} = \frac{1}{4}\left(3\kappa + 5\in'\right) \tag{5.4}$$

$$\Omega^o_{\eta\eta} = \frac{1}{4}\left(9\kappa + 7\in'\right) \tag{5.5}$$

$$\Omega^o_{\xi\eta} = \left(\frac{3\kappa - 6\kappa\upsilon + 5\in' - 10\upsilon\in'}{4}\right)\left(\frac{9\kappa - 8\in'}{3\kappa}\right)^{1/2} \tag{5.6}$$

Substituting equations (5.4), (5.5) and (5.6) in the characteristic equation (18), we have

$$\lambda^4 - \left(3\kappa - 4 + 3\in' - 8\in\right)\lambda^2 + \frac{3}{4}\kappa\left(9\kappa + 22\in'\right)\upsilon\left(1-\upsilon\right) = 0 \tag{5.7}$$

The roots of (5.7) are given by

$$\lambda^2_{1,2} = \frac{-P \pm \sqrt{D}}{2} \tag{5.8}$$

where $P = 4 - 3\kappa + 8\in - 3\in', \quad D = P^2 - 4Q$

$$Q = \frac{3\kappa}{4}\left(9\kappa + 22\in'\right)\upsilon\left(1-\upsilon\right), \ a = 1 \tag{5.9}$$

Consequently, the roots of the characteristic equation depend, on the mass parameter υ, perturbations $\in, \in'$ and the parameter κ. So the nature of these roots is controlled by κ and the sign of the discriminant D, which is given by

$$D = 3\kappa\left(9\kappa + 22\in'\right)\upsilon^2 - 3\kappa\left(9\kappa + 22\in' +\right)\upsilon + 9\kappa^2 - 24\kappa + 16 + \left(4 - 3\kappa\right)\left(16\in - 6\in'\right)$$

Since D is a monotonous function of υ in the interval $(0, 1/2]$ and has values opposite in signs at endpoints, there is only one value of υ, say υ_{c_κ} in the interval $0 < \upsilon \le \frac{1}{2}$ for which the discriminant vanishes. Since the nature of the roots depend on the nature of the discriminant, the following cases are possible:

1. When $0 < \upsilon < \upsilon_{c_\kappa}$, $D > 0$ and $P > 0$, in this case all $\lambda_i\left(i = 1, 2, 3, 4\right)$ are pure imaginary and given as

$$\lambda_{1,2,3,4} = \pm i\Lambda_n \qquad \left(n = 1, 2\right)$$

where
$$\Lambda_i = \left[\frac{1}{2}\left(-P \mp \sqrt{D}\right)\right]^{1/2}, \ i = 1, 2 \tag{5.10}$$

Consequently, the triangular point is stable in this case.
The general solution is written [24], as

$$\begin{aligned} u &= A_1 \cos\Lambda_1\tau + C_1 \sin\Lambda_1\tau + A_2 \cos\Lambda_2\tau + C_2 \sin\Lambda_2\tau \\ v &= \bar{A}_1 \cos\Lambda_1\tau + \bar{C}_1 \sin\Lambda_1\tau + \bar{A}_2 \cos\Lambda_2\tau + \bar{C}_2 \sin\Lambda_2\tau \end{aligned} \tag{5.11}$$

where, $A_i, \bar{A}_i, C_i$ *and* $\bar{C}_i$ ($i = 1, 2$) are constants.

2. For $P < 0$, $0 < \upsilon < \upsilon_{c_\kappa}$, the discriminant $D > 0$, in this case the roots of the characteristic equation (5.7) are real and distinct and written as

$$\lambda_{1,2} = \pm U_1, \lambda_{34} = \pm U_2$$

where $$U_{1,2} = \left(\frac{1}{2}\left(P \pm \sqrt{D}\right)\right)^{\frac{1}{2}} \tag{5.12}$$

The positive root induces instability at the triangular points.

The general solution for real roots with the condition $P < 0, D > 0$ is written as

$$\begin{aligned} u &= A_1 e^{U_1\tau} + A_2 e^{-U_1\tau} + A_3 e^{U_2\tau} + A_4 e^{-U_2\tau} \\ v &= c_1 A_1 e^{U_1\tau} + c_1 A_2 e^{-U_1\tau} + c_2 A_3 e^{U_2\tau} + c_2 A_4 e^{-U_2\tau} \end{aligned} \tag{5.13}$$

where c_1, c_2, A_1 and A_2 are constants.

3. When $\upsilon_{c_\kappa} < \upsilon \leq \frac{1}{2}$, $D < 0$, hence either $P < 0, P = 0$, or $P > 0$. The real parts of two of the values of λ are positive and equal. Therefore, the triangular point is unstable.

4. When $\upsilon = \upsilon_{C_\kappa}$, $D = 0$. The following cases are possible.

(i) If $P < 0$, two roots are real and equal, while the other two are negative and also equal. In this case, the triangular points are unstable.

(ii) If $P = 0$, here all the roots are zero, and the triangular point is stable [16]

5.1.1 Critical Mass

The critical values of the mass parameter are the values of the mass ratio υ when the discriminant vanishes. This is different from the restricted problem with constant masses in which there exists only one value of the mass parameter for which the discriminant is zero because in our problem these values depend on the parameter κ. The values of the critical mass parameter υ_{c_κ} are given by

$$\upsilon_{C_\kappa} = \upsilon_{0_\kappa} + \upsilon_{p_{1_\kappa}} + \upsilon_{p_{2_\kappa}} \tag{5.14}$$

Where $\upsilon_{0_\kappa} = \frac{1}{2} - \frac{1}{6\kappa\sqrt{3}}\sqrt{96\kappa - 9\kappa^2 - 64}$, $\upsilon_{p_{1_\kappa}} = \frac{16(4-3\kappa)}{3\kappa\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\in$

$$\upsilon_{p_{2_\kappa}} = \frac{4(78\kappa - 9\kappa^2 - 88)}{27\kappa^2\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\in' \tag{5.15}$$

Clearly, υ_{c_κ} represents the effects of the constant (κ) of a particular integral of the Gylden-Meshcherskii problem and perturbations on the critical mass value of the restricted problem. Though κ take values between zero and infinity, we consider only values in the range $0.714532 \leq \kappa \leq 9.9532$, for values of κ outside this are not physically meaningful. When $\kappa = 1, 2$ the value of υ_o coincides with the classical Routhian value $\mu_{o_{1,2}} = 0.038521$ but differs for $\kappa > 2$ and does not exist for $\kappa > 9.9532$. If there is no perturbation in the centrifugal force, i.e.$\in' = 0$,

$$\upsilon_{c_\kappa} = \upsilon_{0_\kappa} + \frac{16(4-3\kappa)}{3\kappa\sqrt{3}\sqrt{96 - 9\kappa^2 - 64}}\in \tag{5.16}$$

From (5.16), we find that $\upsilon_{c_\kappa} > \upsilon_{0_\kappa}$. Thus, keeping the centrifugal force constant, and $0 < \kappa < \frac{4}{3}$, the Coriolis force remains a stabilizing force, which agrees with the result of Szebehely [25], but becomes a destabilizing force, once $\frac{4}{3} < \kappa < 10$ and does not exist for $\kappa > 9.9532$.

If there is no perturbation in the Coriolis force, equation (5.14), becomes

$$\upsilon_{c_\kappa} = \upsilon_{0_\kappa} + \frac{4\left(78\kappa - 9\kappa^2 - 88\right)}{27\kappa^2\sqrt{3}\sqrt{96\kappa - 9\kappa^2 - 64}}\in' \qquad (5.17)$$

Here, we find that $\upsilon_{c_\kappa} < \upsilon_{0_\kappa}$, for $0 < \kappa < \frac{4}{3}$. So, keeping the Coriolis force constant and $\kappa < \frac{4}{3}$, the centrifugal force is always a destabilizing force, but becomes stabilizing once κ is in the range $\frac{4}{3} < \kappa \le 7$, and again becomes destabilizing when $7 < \kappa < 10$ and these effects are (stabilizing or destabilizing) void for $\kappa > 9.9532$. We note that, the value $\kappa = \frac{4}{3}$, corresponds to ignoring the effects of the Coriolis and centrifugal forces and the Routhian value $\upsilon_{0_{4/3}} = 0$. Consequently, for $\kappa = \frac{4}{3}$, the critical value of the mass parameter υ_{C_κ} is zero. For $\kappa = 1, 2$, the critical mass ratio (5.14) is respectively

$$\upsilon_{c_1} = 0.038521 + \frac{4\left[36\in - 19\in'\right]}{27\sqrt{69}} \qquad (5.18)$$

and

$$\upsilon_{c_2} = 0.038521 + \frac{4\left[4\in' - 18\in\right]}{27\sqrt{69}} \qquad (5.19)$$

Equation (5.18) is same as that worked out by [5]. If further we keep the centrifugal force constant (i.e.$\in' = 0$), the relation fully coincides with that of [25] Ignoring perturbations in the Coriolis and centrifugal forces of the primaries, in either equations above, the critical mass corresponds to the classical case of [24].

Hence, the region of stability increases, decreases or does not exist depends on the constant κ of a particular integral of the Glyden-Mescherskii problem, we conclude that the triangular point of the autonomized system is stable for $0 < \upsilon < \upsilon_{c_\kappa}$, and unstable for $\upsilon_c \le \upsilon \le \frac{1}{2}$ due to the constant κ of a particular integral of the Gylden-Meshcherskii problem.

5.2 Stability of Collinear Points

In order to study the stability of the collinear libration points, we first compute the partial derivatives at the collinear libration points of the points L_1, L_2 and L_3. Let us consider the point corresponding to L_1 with coordinate $\left(-\upsilon - \varepsilon, 0\right)$.

Using $\rho_1 = \varepsilon_1 < 1$ and $\rho_2 = 1 + \varepsilon_1 > 1$ (5.20)

We get $\Omega^0_{\xi\xi} = \psi + \kappa - 1 + 2\kappa f_1,$ (5.21)

Now, $\varepsilon_1 > 0,\ ,\ 0 < \upsilon \le \frac{1}{2}$ and for all κ, $\Omega^0_{\xi\xi} > 0$

Similarly, $\Omega^0_{\eta\eta} = \psi - 1 + \kappa(1 - f_1)$ and $\Omega^0_{\eta\eta} < 0$. (5.22)

where $$f_1 = \frac{(1-\upsilon)}{\varepsilon_1^3} + \frac{\upsilon}{\left(1+\varepsilon_1\right)^3}$$

Finally $$\Omega^0_{\xi\eta}=\Omega_{\eta\xi}=0 \tag{5.23}$$

Substituting (5.21), (5.22) and (5.23) in characteristic equation (5.3), we get

$$\lambda^4+\lambda^2\left(4+8\in-2\kappa-2\in'-\kappa f_1\right)+\kappa\in'\left(2+f_1\right)+\kappa^2\left(1+f_1-2f_1^2\right)=0 \tag{5.24}$$

Because $\Omega^0_{\xi\xi}\Omega^0_{\eta\eta}<0$, the discriminant is positive, and the four roots of equation (5.25) can be written as

$$\lambda_1=\sigma_1\,,\lambda_2=-\sigma_1\,,\ \lambda_3=i\sigma_2\,,\ \lambda_4=-i\sigma_2 \tag{5.25}$$

The general solution is

$$\begin{aligned}u&=A_1e^{\sigma_1\tau}+A_2e^{-\sigma_1\tau}+A_3e^{i\sigma_2\tau}+A_4e^{-i\sigma_2\tau}\\ v&=B_1e^{\sigma_1\tau}+B_2e^{-\sigma\,\tau}+B_3e^{i\sigma_2\tau}+B_4e^{-i\sigma_2\tau}\end{aligned} \tag{5.26}$$

Here, $A_{j's}$ and $B_{j's}$ $\left(j=1,2,3,4,\right)$ are constant.

$$\sigma_{1,2}=\pm\left(\frac{-P}{2}+\frac{\sqrt{D}}{2}\right)^{\frac{1}{2}},\sigma_{3,4}=\pm\left(\frac{-P}{2}-\frac{\sqrt{D}}{2}\right)^{\frac{1}{2}}$$

$$P=\left(4+8\in-2\kappa-2\in'-\kappa f_1\right),\ Q=\kappa\in'\left(2+f_1\right)+\kappa^2\left(1+f_1-2f_1^2\right)$$

where $\lambda_i,i=1,2,3,4$ are real. Our investigations show that not all the roots of the characteristic equation are pure imaginary numbers, so the solution is unstable. The same procedure shows that L_2 and L_3 are unstable. Therefore, we can conclude that the stability behavior of the collinear points does not change due to perturbations in the Coriolis and centrifugal forces and the constant κ of a particular integral of the motion of the variable primaries. Hence, they remain unstable.

6.0 Stability of Equilibrium Points of Equations with Variable Coefficients

Stability of non-autonomous solutions is related to the Lyapunov Characteristic Numbers which governs the long-time asymptotic exponential behaviors of solutions.

The investigation of stability for system (2.2) with variable coefficients is difficult to establish for two reasons; first we must know a particular solution to this system of equations and second, it contains an unknown function-the angular velocity $\omega(t)$. However the function $\omega(t)$ is determined using the particular solutions (4) of the Gylden-Meshcherskii problem. The analysis of the stability of the libration points $L_i\left(i=1,2,...5\right)$ of the equations of motion with coefficients varying with time would solely depend on the methods applied, since these libration points are themselves time dependent, which means that a change in time would result in a change in the locations of the libration points. For example, using the definition of a Lyapunov stable solution [10], we have in the triangular case

$$\lim_{t\to\infty}x^{(4,5)}=\frac{1}{2}\lim_{t\to\infty}\left[1-2\upsilon\right]R(t)$$

Hence, $$\lim_{t\to\infty}x(t)=\infty$$

This at once proves the instability of the solutions $x(t)$, and similarly for $y(t)$, according to the Lyapunov theorem and verifies the result of [10].

The system of equations (3.6) with constant coefficients and the reducible systems are regular. By regular we mean that, there exists a generalized Lyapunov transformation carrying the system to another system with constant matrix [26]. The system (2.2) of equations with variable coefficients is reducible systems due to the transformation (3.1). The reducible systems are regular because the characteristic numbers are invariant with respect to transformation, consequently we can apply the theorem of Lyapunov, using the Lyapunov Characteristic Number (LCN) on the stability of the perturbed motion to the particular steady-state solutions of the system (2.2). The calculations of the Lyapunov characteristic numbers here are limited to finding the maximum LCN. This produces an easily computed value that can be used as a metric to give qualitative indication of how stability varies over the solutions. The Calculation of the Lyapunov characteristic numbers as defined by [6] and [[15] is used here.

6.1 Stability of Triangular Points

Calculating the LCN of the triangular solutions (4.6) varying with time with the consideration that as $t \to \infty,\ \tau$ is approaching a finite value, we have,

$$L_{4,5}\left[x(t)\right] = -\lim_{t\to\infty}\frac{1}{t}\ln\left|\frac{1}{2}(-2\upsilon+1)R(t)\right| = 0$$

similarly, $$L_{4,5}\left[y(t)\right] = 0 \qquad (6.1)$$

Thus, the Lyapunov characteristic number is zero for triangular solutions, therefore, the stability or instability of the perturbed motion cannot be determined directly from the triangular equilibrium solutions.

Using equation (5.1), the particular solutions of the system of equations with variable coefficients (2.2) can be represented, given transformation (3.1) and solutions (5.11) as

$$\begin{aligned}
&x_1 = A_1\cos\Lambda_1\tau R(t),\quad x_2 = C_1\sin\Lambda_1\tau R(t)\\
&x_3 = A_2\cos\Lambda_2\tau R(t),\quad x_4 = C_2\sin\Lambda_2\tau R(t),\quad x_5 = \xi_0 R(t)\\
&y_1 = \bar{A}_1\cos\Lambda_1\tau R(t),\quad y_2 = \bar{C}_1\sin\Lambda_1\tau R(t),\quad y_3 = \bar{A}_2\cos\Lambda_2\tau R(t)\\
&y_4 = \bar{C}_2\sin\Lambda_2\tau R(t),\quad y_5 = \eta_0 R(t)
\end{aligned} \qquad (6.2)$$

where ξ_0, η_0 are coordinates of the infinitesimal mass.

These solutions correspond to the region where $0 < \upsilon < \upsilon_{c_\kappa}, P > 0$ i.e. $0 < \kappa < \frac{4}{3}\in' + \frac{8}{3}\in$.

Using equation (4.6), the particular solutions to the system of equations with variable coefficients (1) can be represented, given transformation (3.1) and solutions (5.13) as

$$\begin{aligned}
&x_4 = e^{\pm U_1\tau}R(t), x_5 = e^{\pm U_2\tau}R(t), x_6 = \xi_0 R(t)\\
&y_4 = c_1e^{\pm U_1\tau}R(t), y_5 = c_2e^{\pm U_2\tau}R(t), y_6 = \eta_0 R(t)
\end{aligned} \qquad (6.3)$$

The solutions of system (6.3) correspond to the region where $0 < \upsilon < \upsilon_{c_\kappa}$, but $P < 0$, i.e. $\frac{4}{3} < \kappa(1+\in') - \frac{8}{3}\in < \infty$ .we have chosen these regions, since it contains region where the triangular points for

the autonomized equations are stable, as well as region where they are unstable. These regions are determined by the

constant of integration (κ) of the Glyden-Mescherskii problem, the Coriolis and centrifugal forces. Since in both cases $0 < \upsilon < \upsilon_{c_\kappa}$, $(D > 0)$

For the solutions (43), their LCN's are

$$L(x_1) = -\lim_{t\to\infty}\frac{1}{t}\ln\left|\cos\Lambda_1\tau R(t)\right| = 0$$

Hence $$L(x_1) = L(x_2) = L(x_3) = L(y_1) = L(y_2) = L(y_3) = 0 \qquad (6.4)$$

By [18] angular velocity representation $0 < \frac{\omega^2(t)}{2\pi f\rho(t)} < 1$, we found [23] a relationship between the old and new independent variables t and τ given as

$$\lim_{t\to\infty}\frac{\tau}{t}=\Gamma<S^2 \qquad (6.5)$$

where $S=2\pi\kappa\rho_0$ is finite and τ always tends to a finite value as t is always approaching infinity.
So that in view of the particular solutions (6.3) and considering (6.5), the LCN are

$$L(x_4)=-\lim_{t\to\infty}\frac{1}{t}\ln\left|e^{\pm U_1\tau}R(t)\right|=\mp U_1\Gamma$$

Thus

$$L(x_4)=\mp U_1\Gamma,\quad L(x_5)=\mp U_2\Gamma\quad L(x_6)=0 \qquad (6.6)$$

Similarly

$$L(y_4)=\mp U_1\Gamma\quad L(y_5)=\mp U_2\Gamma\quad L(y_6)=0 \qquad (6.7)$$

Hence the LCN are positive for solutions with negative exponents, negative for solutions with positive exponents and, zero for solutions with imaginary exponents and constant solutions. We conclude that solutions with negative exponent, consequently having $LCN>0$ are stable; solutions with positive exponents are unstable according to the Lyapunov theorem, while constant solutions and solution with imaginary exponents give no information about the stability or instability of the solutions
We make the following remarks;

1. If roots of characteristic equation corresponding to triangular solutions of the autonomized equations are positive, then the LCN of the solutions varying with time is negative, and consequently the solutions are unstable according to the Lyapunov theorem.
2. If roots of characteristic equation corresponding to triangular solutions of the autonomized equations are pure imaginary numbers, then the LCN of the corresponding solution varying with time is zero, in this case the stability or instability of the solutions can not be determined.
3. If roots of characteristic equation corresponding to triangular solutions of the autonomized equations are negative, then the LCN of corresponding solutions varying with time is positive and consequently the solutions are stable.

The same stability analysis done for the triangular solutions, shows for the collinear that, solutions with positive exponents are unstable, those for negative exponents are stable, while the stability or instability of oscillatory and constant solutions can not be determined.

7.0 Discussion

The equations of perturbed motion (2.2) are different from that obtained by [14] due to the presence of perturbations in the Coriolis and centrifugal forces. However if this are ignored, the equation (2.2) will fully coincide with those obtained by [2], [8], [13], and [14].

Equation (5.14) gives the critical value υ_{c_κ} of the mass parameter. It shows the effects of, perturbations in the Coriolis and centrifugal forces and the constant of a particular integral κ on the critical mass value. The critical mass value (5.18) verifies the results of [5]. By keeping the centrifugal force constant, equation (5.16) gives the relationship of the critical mass value, to the change $\in$ in the Coriolis force. Here for $0<\kappa<\frac{4}{3}$ the Coriolis force remains a stabilizing force, but becomes destabilizing for $\frac{4}{3}<\kappa\le 9.952$, and does not exist for $\kappa\ge 10$.

Also, if the Coriolis force is kept constant, equation (5.17) provides the relationship of the critical mass value to the change $\in'$ in the centrifugal force. So, keeping the Coriolis force constant and $0<\kappa<\frac{4}{3}$, leads to the fact that the centrifugal force is always a destabilizing force, but becomes stabilizing and again destabilizing due to the constant κ. The overall effect is that the increase, decrease or non existence in the range of stability of the triangular points would solely depend on the constant κ of a particular integral of

the Glyden-Mescherskii problem. The collinear points of the autonomized system are different from those of the classical problem due to the introduction of perturbation in the centrifugal force and the parameter due to mass variation. However, despite the introduction of these parameters, the collinear libration points remain unstable.
For the stability of the non autonomous system, system (6.4) coincides with the case when the mass ratio of the autonomous system is in the range $0 < \upsilon < \upsilon_{c_\kappa}$ but $\kappa < \frac{4}{3} - \in' + \frac{8}{3}\in$. In this case the all LCN's are zero, and so the region of stability does not exist. This confirms the result of [12] that the region of stability of the triangular points does not exist when the restricted problem with constant masses evolves into one with variable mass. For the case when $0 < \upsilon < \upsilon_{c_\kappa}$ but $\kappa > \frac{4}{3} - \in' + \frac{8}{3}\in$, the LCN are positive for negative roots and negative for positive roots. Here the region of stability or instability depends solely on the constant κ. Our generalization of the stability of the non autonomous system is in agreement with [14] and [23].

7.0 Concluding Remarks

The system of equations (2.2) derived are non-integrable differential equations with variable coefficients, and since functional differential equations of motion for classical field theory are generally difficult, often impossible, to express in a form that is amenable to analysis. Thus, in order to obtain useful dynamical predictions from realistic models, it is frequently required to replace the functional differential equations of motion by approximations that are ordinary or partial differential equations. Thus, using the particular solutions of the Gylden-Meshcherskii problem, unified Meshcherskii's law and a Meshcherskii transformation, the system (2.2) is reduced to a system (3.6) of perturbed equations with constant coefficients in a coordinate system rotating with a constant angular velocity, then the search for the particular solutions of the system with variable coefficients comes down to the search for triangular steady-state solutions of (3.6). Analogous particular steady-state solutions (libration points) for the system of equations with variable coefficients are obtained with the help of the equilibrium solutions of the autonomized system (3.6) and the transformation (3.1).
The stability of collinear equilibrium points of the autonomized system under the influence of a constant κ of a particular integral of the motion of the variable primary bodies, together with the effects of the perturbations in Coriolis and centrifugal, does not change despite the introduction of perturbations. Hence, they remain unstable. The stability of triangular equilibrium points of the autonomized system is stable is seen to be stable for $0 < \upsilon < \upsilon_{c_\kappa}$, and unstable for $\upsilon_{c_\kappa} \le \upsilon \le \frac{1}{2}$ due to the constant κ of a particular integral of the Gylden-Meshcherskii problem, where υ_{c_κ} is the critical mass value, which depends on the joint effects of the parameters. The range of stability increases, decreases, remains unchanged or does not exist according to the constant κ of a particular integral of the Glyden-Meshcherskii problem.

The stability of libration points varying with time, for some initial conditions, we find according to the Lyapunov theorem, that solutions with negative exponents consequently having positive LCN are stable, those with positive exponents having negative LCN are unstable, while the stability or instability of constant and pure oscillatory solutions having zero LCN's, cannot be determined.
The range of stability or instability depends on the overall on the parameter (κ), since its choice determines the stabilizing or destabilizing ability of the presence of the Coriolis and centrifugal forces. Hence as κ increases, the LCN of solutions with negative exponents increases. On the other hand, for solutions with positive exponents the range of instability of these solutions increases.
We conclude that since the stability of the triangular equilibrium solutions of the non autonomous system cannot be determined when $0 < \upsilon < \upsilon_{c_\kappa}$, $0 < \kappa < \frac{4}{3}\in' + \frac{8}{3}\in$, but has unstable solutions in the same range but with change in the range of the constant. We conclude that motion around the equilibrium points

for the perturbed restricted three-body problem with variable masses is in general unstable according to the theorem of Lyapunov.

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