Journal of the Nigerian Association of Mathematical Physics
Volume **16 (May, 2010), pp 407 - 412**
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A Generalization of Sufficient Conditions for the Convexity of Twice Differentiable Real-valued Single-Variable Functions

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Abstract

In [3] real-valued convex functions of single variable were characterized using the geometric chord property. It centred on the first order condition, integral and monotonicity of the derivative of convex functions. In this work we major on extending this characterization to the second derivative and the epigraph of these functions which is the link between convex sets and convex functions.

Keyword: **Convex Function, Epigraph of a Function, Twice Differentiable Function**

1.0 Introduction

Linear functions are very appealing because they are easy to manipulate and their graphs are especially simple (lines in the plane for one independent variable, planes in space for two independent variables and so on). In this work we will consider a class of functions called convex functions which includes the class of linear functions but which has a much wider range of applications than that of linear functions. Convex functions have been characterized using any of the derivative, the integral and the epigraph. In particular [3] gives a characterization which combines the first order condition, monotonicity and the integral of the derivative of convex functions. We shall flavour this characterization with the second derivative and the epigraph of convex functions by extension. We will begin with the definition of convex functions.

Definition 1.1 Let $I \subseteq \mathbb{R}$ be a nonempty closed and bounded interval. A function $f: I \longrightarrow \mathbb{R}$ is said to be convex on I if for any $x_1, x_2 \in I$ and all $\alpha \in [0,1]$, we have that

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2) \qquad (1.1)$$

If strict inequality holds in (1) for all $x_1 \neq x_2$, then f is called a strictly convex function.

If the above definition holds with the inequality reversed then f is concave. Thus the negative of a convex function is concave, and vice versa.

Geometrically this means that the values of a convex function are below the corresponding chord, that is, the values of a convex function at points on the line segment $\alpha x_1 + (1-\alpha)x_2$ are less or equal to the height of the chord joining the points $(x_1, f(x_1))$ and $(x_2, f(x_2))$.

2.0 Jensen's Inequality

Theorem 2.1 *Let f be a convex function defined on an interval I. If $x_1, x_2, \dots, x_n \in I$ and $\alpha_1, \alpha_2, \dots, \alpha_n \geq 0$ with $\sum_{i=1}^{n} \alpha_i = 1$, then*

$$f\left(\sum_{i=1}^{n} \alpha_i x_i\right) \leq \sum_{i=1}^{n} \alpha_i f(x_i). \qquad (2.1)$$

Proof: We shall prove this by induction. For $n = 1$ the theorem is trivially true. Further, by convexity hypotheses

$$f(\alpha_1 x_1 + \alpha_2 x_2) \leq \alpha_1 f(x_1) + \alpha_2 f(x_2) \qquad (2.2)$$

Thus the statement is true for $n = 2$.

Suppose that the theorem is true for $n = k - 1$,

$$\sum_{i=1}^{k} \alpha_i f(x_i) = (1 - \alpha_k) \sum_{i=1}^{k-1} \frac{\alpha_i}{1-\alpha_k} f(x_i) + \alpha_k f(x_k)$$

$$\geq (1 - \alpha_k) f\left(\sum_{i=1}^{k-1} \frac{\alpha_i}{1-\alpha_k}\right) + \alpha_k f(x_k) \quad \text{(By inductive hypothesis)}$$

$$\geq f\left((1 - \alpha_k) \sum_{i=1}^{k-1} \frac{\alpha_i}{1-\alpha_k} x_i + \alpha_k x_k\right) \quad \text{(by (2.2))}$$

$$= f\left(\sum_{i=1}^{k-1} \alpha_i x_i + \alpha_k x_k\right)$$

$$= f\left(\sum_{i=1}^{k} \alpha_i x_i\right)$$

Observation 2.2 Clearly, Jensen's Inequality is an extension of (1.1) beyond two points which is a generalization of the notion of convexity to include the linear combination of the points. It plays a central role in many aspects of mathematics. A very important application of the above result is that the geometric mean of a set positive numbers does not exceed their arithmetic mean.

3. The Epigraph of a Convex Function

Definition 3.1 Let $I \subseteq \mathbb{R}$ be a nonempty closed and bounded interval and $f: I \longrightarrow \mathbb{R}$. The epigraph of f is given by

$$epi f = \{(x,t): f(x) \leq t, \ \ x \in I, \ \ t \in \mathbb{R}\}. \qquad (3.1)$$

The link between convex sets and convex functions is through the epigraph: *A function is convex if, and only if, its epigraph is a convex set.*

Definition 3.2 *A function is concave if and only if its hypograph, defined as*

$$hyp f = \{(x,t): f(x) \geq t, \ \ x \in S, \ \ t \in \mathbb{R}\} \qquad (3.2)$$

is a convex set.

4. Convexity, Differentiability and the Geometric Chord Property

We now consider some results from [1, 4, 5, 6, 7], which will be very useful in the proof of the main result.

Lemma 4.1 Let f be a convex function defined on some interval $I \subseteq \mathbb{R}$, and let $I_1 = [x_1, y_1]$ and $I_2 = [x_2, y_2]$ be nondegenerate subintervals of I. That is, $x_1 < y_1$ and $x_2 < y_2$. Assume that I_1 lies to the left of I_2. That is, $x_1 \leq x_2$ and $y_1 \leq y_2$ Then the slope of the chord over I_1 is greater than the slope of the chord over I_2. In particular,

$$\frac{f(y_1)-f(x_1)}{y_1-x_1} \leq \frac{f(y_2)-f(x_1)}{y_2-x_1} \leq \frac{f(y_2)-f(x_2)}{y_2-x_2}. \qquad (4.1)$$

Proof. Since $x_1 < y_1 \leq y_2$ we write y_1 as a convex combination of x_1 and y_2, namely

$$y_1 = \frac{y_2-y_1}{y_2-x_1} x_1 + \frac{y_1-x_1}{y_2-x_1} y_2$$

By convexity

$$f(y_1) \leq \frac{y_2-y_1}{y_2-x_1} f(x_1) + \frac{y_1-x_1}{y_2-x_1} f(y_2).$$

Subtracting $f(x_1)$ from both sides gives

$$f(y_1) - f(x_1) \leq \frac{x_1-y_1}{y_2-x_1} f(x_1) + \frac{y_1-x_1}{y_2-x_1} f(y_2).$$

Dividing by $y_1 - x_1$ gives

$$\frac{f(y_1)-f(x_1)}{y_1-x_1} \leq \frac{-1}{y_2-x_1} f(x_1) + \frac{1}{y_2-x_1} f(y_2) = \frac{f(y_2)-f(x_1)}{y_2-x_1} \qquad (4.2)$$

Similarly

$$x_2 = \frac{y_2-x_2}{y_2-x_1} x_1 + \frac{x_2-x_1}{y_2-x_1} y_2,$$

so

$$f(x_2) \le \frac{y_2-x_2}{y_2-x_1} f(x_1) + \frac{x_2-x_1}{y_2-x_1} f(y_2),$$
$$-f(x_2) \ge \frac{y_2-x_2}{y_2-x_1} f(x_1) + \frac{x_1-x_2}{y_2-x_1} f(y_2),$$
$$f(y_2) - f(x_2) \ge \frac{x_2-y_2}{y_2-x_1} f(x_1) + \frac{y_2-x_2}{y_2-x_1} f(y_2),$$
$$\frac{f(y_2)-f(x_2)}{y_2-x_2} \ge \frac{-1}{y_2-x_1} f(x_1) + \frac{1}{y_2-x_1} f(y_2) = \frac{f(y_2)-f(x_1)}{y_2-x_1} \qquad (4.3)$$

Combining (4.2) and (4.3) completes the proof.
This lemma has a number of consequences as we shall see in the remaining sections.

Lemma 4.2 *Suppose $f: I \to \mathbb{R}$ is convex. Then $x \in int\ (I)$ implies $f'_-(x)$ and $f'_+(x)$ exist and $f'_-(x) \le f'_+(x)$.* *Proof*:

Note that Lemma 4.1 implies that $\frac{f(x_3)-f(x_1)}{x_3-x_1}$ is nondecreasing in x_1 and x_3 for $x_1 \ne x_3$. Therefore for all $x_1 < x_2 < x_3$, we have

$$f'_+(x_2) \equiv \lim_{x_3 \to x_2} \frac{f(x_3)-f(x_2)}{x_3-x_2} \ge \frac{f(x_2)-f(x_1)}{x_2-x_1} \qquad (4.4)$$

and

$$f'_+(x_2) \ge \lim_{x_1 \to x_2} \frac{f(x_2)-f(x_1)}{x_2-x_1} f'_-(x_2) \qquad (4.5)$$

Lemma 4.3 (a) $f: I \to \mathbb{R}$ is convex if and only if for all $x_2 \in int\ I$ and all $x_1, x_3 \in I$ with $x_1 < x_2 < x_3$, we have

$$f(x_3) - f(x_2) \ge f'_+(x_2)(x_3 - x_2)$$

and

$$f(x_2) - f(x_1) \le f'_-(x_2)(x_2 - x_1)$$

(b) f is strictly convex if and only if the inequalities are strict for $x_1, x_3 \ne x_2$

Proof. Suppose f is convex. The statement is trivial if $x_1 = x_3$, so suppose that $x_1 \ne x_3$, then Lemma 4.1 implies

$$f'_-(x_3) \equiv \lim_{x_2 \to x_3} \frac{f(x_3)-f(x_2)}{x_3-x_2} \ge \frac{f(x_3)-f(x_1)}{x_3-x_1} \ge \lim_{x_2 \to x_1} \frac{f(x_2)-f(x_1)}{x_2-x_1} \equiv f'_+(x_1)$$

Multiplying through by $(x_3 - x_1)$ then yields the result.
If f is strictly convex, then Lemma 4.1 implies that the inequalities are strict.

Lemma 4.4 If $f: I \to \mathbb{R}$ is convex and differentiable, then f' is continuous on I.

Proof. If $[x-\delta, x+\delta] \subset (a, b)$, then for $0 < h < \delta$, we have that

$$f'(x) \le f'(x+h) \le \frac{f(x+\delta)-f(x+h)}{\delta - h}$$

Let $h \to 0_+$ then $\delta \to 0_+$, and we have that f' is continuous from the right. By similarly argument f' is continuous from the left.

Lemma 4.5 If f is convex on an open interval $[a, b]$, then

$$f(b) - f(a) = \int_a^b f'_\pm(x)dx. \qquad (4.6)$$

Proof. If $a = x_0 < x_1 < \cdots x_n = b$ is a partition, then

$$\begin{aligned}
\int_a^b f'_+(x)dx &= \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f'_+(x)dx. \\
&\ge \sum_{i=1}^n \int_{x_{i-1}}^{x_i} f'_+(x_{i-1})dx \\
&= \sum_{i=2}^n f'_+(x_{i-1})(x_i - x_{i-1}) + f'_+(x_0)(x_1 - x_0) \\
&\ge \sum_{i=2}^n \big(f(x_{i-1}) - f(x_{i-2})\big) + f'_+(x_0)(x_1 - x_0).
\end{aligned}$$

Taking the limits as $\Delta x_i \to 0$, we get $\int_a^b f'_+(x)dx \ge f(b) - f(a)$. By a similar argument $\int_a^b f'_+(x)dx \le f(b) - f(a)$ which completes the proof.

5. Characterizations of Convex Functions

We now present some characterizations of convex functions showing how they can be recognized.

Theorem 5.1 First Order Characterization of Convex Functions

Let $I \subseteq \mathbb{R}$ be an open interval and let $f: I \to \mathbb{R}$ then

(*i*) f *is convex if and only if for any* $x_1, x_2 \in I$

$$f(x_2) \geq f'(x_1)(x_2 - x_1) + f(x_1) \qquad (5.1)$$

(*ii*) f *is strictly convex if and only if for any* $x_1, x_2 \in I, x_1 \neq x_2$

$$f(x_2) > f'(x_1)(x_2 - x_1) + f(x_1) \qquad (5.2)$$

Theorem 5.2 Second Order Characterization of Convex Functions

*Let $I \subseteq \mathbb{R}$ be an open interval. S*uppose that $f: I \to \mathbb{R}$ is a twice differentiable function then f is convex if, and only if,

$$f''(x) \geq 0, \quad x \in I. \qquad (5.3)$$ See [2].

The next result shows that derived function of a convex function is a monotone increasing.

Theorem 5.3 Characterization of Convex Function with the Monotonicity of the Derivative

Let $f: I \to \mathbb{R}$ be differentiable over the open interval $I \subseteq \mathbb{R}$ then f is convex if and only if

$$f'(x_1) \leq f'(x_2), \ \ \forall x_1 < x_2 \in I \qquad (5.4)$$

Proof: Let $x_1, x_2, x_3 \in I$ with $x_1 < x_2 < x_3$. By the geometrical chord property

$$\frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_3)-f(x_1)}{x_3-x_1} \leq \frac{f(x_3)-f(x_2)}{x_3-x_2} \qquad (5.5)$$

Considering the first inequality; as $x \to x_1$ from the right the slopes decrease. This implies that the Newton quotients $\frac{f(x)-f(x_1)}{x-x_1}$ used to compute $f'_+(x_1)$ are increasing. Also the second inequality reveals that the ratios representing the slopes increase as $x \longrightarrow x_3$ from the left.

Now at x_2, these inequalities show that for any $x_1 < x_2$ the ratios $\frac{f(x_2)-f(x_1)}{x_2-x_1}$ are bounded above by $\frac{f(x_3)-f(x_1)}{x_3-x_1}$ which does not depend on x_2.

Thus the Newton quotients used to compute the left-hand derivative at x_2 are increasing and bounded above implying that $f'_-(x_2)$ exists. By similar argument $f'_+(x_2)$ also exists.

Furthermore, it follows that

$$f'_+(x_1) \leq \frac{f(x_2)-f(x_1)}{x_2-x_1} \leq \frac{f(x_3)-f(x_2)}{x_3-x_1} \leq f'_-(x_2) \qquad (5.6)$$

so that

$$f'_-(x_1) \leq f'_+(x_1) \leq f'_-(x_2) \leq f'_+(x_2) \qquad (5.7)$$

Thus f'_- and f'_+ are non-decreasing.

Theorem 5.4 Derivative and Integral Characterization of Convex Function Using the Geometric Chord Property

Let $f: I \to \mathbb{R}$ be differentiable over the open interval $I \subseteq \mathbb{R}$ then the following statements are equivalent.

(*i*) $f(x_2) \leq \lambda f(x_1) + (1-\lambda) f(x_3), \ \ \lambda \in [0,1], \ \ x_1, x_2 x_3 \in I$ (5.8)

(*ii*) $f'(x_1) \leq f'(x_2), \ \ \forall x_1 < x_2 \in I$ (5.9)

(*iii*) $f(x) - f(x_0) = \int_{x_0}^{x} f'(t)dt \quad x, x_0 \in I$ (5.10)

(*iv*) $f(x) \geq f'(x_0)(x - x_0) + f(x_0) \quad x, x_0 \in I$ (5.11)

This shows that a presentation with any of the statement above without a pre-information on the nature of the function does not only imply that the function is convex but also a presentation with all the others [3].

Theorem 5.5 Epigraph Characterization of Convex Function

Let $I \subseteq \mathbb{R}$ be an open interval and $f: I \to \mathbb{R}$. Then f is convex if, and only if, the epigraph of f is convex set [2].

Remark 5.6 Many results for convex functions can be proved (or interpreted) geometrically using epigraphs, and applying results for convex sets. As an example, consider the First Order Condition for convexity (as given in [1]):

$$f(x_2) \geq f(x_1) + f'(x_1)(x_2 - x_1)$$

where f is convex and $x_1, x_2 \in S$. We can interpret this basic inequality geometrically in terms of epi$f$: If $(x_2, t) \in \text{epi}f$, then

$$t \geq f(x_2) \geq f(x_1) + f'(x_1)(x_2 - x_1). \qquad (5.12)$$

6. Sufficient Conditions for the Convexity of Twice Differentiable Real-valued Single Variable Functions

Theorem 6.1 *Let $f: I \to \mathbb{R}$ be twice differentiable over the open interval $I \subseteq \mathbb{R}$ then the following statements are equivalent.*

(*i*) $f(x) \leq \alpha f(x_1) + (1-\alpha) f(x_2), \;\; \alpha \in (0,1), \;\; x_1, x_2 \in I.$ (6.1)

(*ii*) $(\alpha x_1 + (1-\alpha)x_2, \; \alpha\theta_1 + (1-\alpha)\theta_2) \in epif$ (6.2)

for $x_1, \; x_2 \in I$ *and* $(x_1, \theta_1), \; (x_2, \theta_2) \in epif$.

(*iii*) $f'(x_1) \leq f'(x_2), \;\; \forall x_1 < x_2 \in I.$ (6.3)

(*iv*) $f(x) - f(x_0) = \int_{x_0}^{x} f'(t)dt \quad x, x_0 \in I.$ (6.4)

(*v*) $f(x_2) \geq f(x_1) + (x_2 - x_1) f'(x_1) \quad x_1, x_2 \in I$ (6.5)

(*vi*) $f''(x) \geq 0, \; x \in I.$ (6.6)

Proof. We shall show that (*i*)⟹(*ii*)⟹(*iii*)⟹(*iv*)⟹(*v*)⟹(*vi*)⟹(*i*).

(*i*)⟹(*ii*). Assuming that f is convex and let $x_1, x_2 \in I$ and $(x_1, \theta_1), \; (x_2, \theta_2) \in epif$, then for any $\alpha \in (0,1)$ we have

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2) \leq \alpha\theta_1 + (1-\alpha)\theta_2 \qquad (6.7)$$

Since $x = \alpha x_1 + (1-\alpha)x_2 \in I$, we have that

$$(\alpha x_1 + (1-\alpha)x_2, \alpha\theta_1 + (1-\alpha)\theta_2) \in epif \qquad (6.8)$$

(*ii*)⟹(*iii*). Since $epif$ is convex $(\alpha x_2 + (1-\alpha)x_1, \; \alpha f(x_2) + (1-\alpha) f(x_1)) \in epif$

$$\Rightarrow f(\alpha x_2 + (1-\alpha)x_1) \leq \alpha f(x_2) + (1-\alpha) f(x_1) \qquad (6.9)$$

Similarly

$$f(\alpha x_1 + (1-\alpha)x_2) \leq \alpha f(x_1) + (1-\alpha) f(x_2) \qquad (6.10)$$

Rearranging we have

$$\frac{f(\alpha x_2 + (1-\alpha)x_1) - f(x_1)}{\alpha} + f(x_1) \leq f(x_2) \qquad (6.11)$$

and

$$\frac{f(\alpha x_1 + (1-\alpha)x_2) - f(x_2)}{\alpha} + f(x_2) \leq f(x_1) \qquad (6.12)$$

Summing (6.11) and (6.12) and letting $\alpha \longrightarrow 0$ we have

$$f'(x_1)(x_2 - x_1) \leq f'(x_2)(x_2 - x_1).$$

$$\Rightarrow (f'(x_2) - f'(x_1))(x_2 - x_1) \geq 0.$$

Since $x_1 \neq x_2$ it follows that $x_2 - x_1 > 0 \Rightarrow \;\; x_2 > x_1$. Thus

$$f'(x_2) \geq f'(x_1) \qquad (6.13)$$

(*iii*)⇒(*iv*): Since a function which is non-decreasing on an interval is integrable on that interval, it follows that f'_- and f'_+ are Riemann integrable

Suppose $x_0 < x \in I$ (the argument for $x < x_0$ is similar and omitted). For any partition $x_0 < x_1 < \cdots < x_n = x$, by (5.6) and (5.7)

$$f'_-(x_{k-1}) \leq f'_+(x_{k-1}) \leq \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}} \leq f'_-(x_k) \leq f'_+(x_k) \qquad (6.14)$$

Since

$$\sum_{k=1}^{n}[f(x_k) - f(x_{k-1})] = \sum_{k=1}^{n} \frac{f(x_k) - f(x_{k-1})}{x_k - x_{k-1}}(x_k - x_{k-1}) = f(x) - f(x_0) \qquad (6.15)$$

$$\Rightarrow \quad \int_{x_0}^{x} f'_-(t)dt = \int_{x_0}^{x} f'_+(t)dt = f(x) - f(x_0) \tag{6.16}$$

(*iv*)⇒(*v*): From (6.14), (6.15) and (6.16), we observe that

$$f'(x_0) \leq \frac{f(x) - f(x_0)}{x - x_0}, \text{ if } x_0 < x \tag{6.17}$$

and

$$f'(x_0) \geq \frac{f(x) - f(x_0)}{x - x_0} \text{ if } x_0 > x \tag{6.18}$$

In either case

$f(x) - f(x_0) \geq f'(x_0)(x - x_0)$.

(*v*)⇒(*vi*): By the mean value theorem

$$f(x_2) = f(x_1) + (x_2 - x_1)f'(x_1) + \frac{1}{2}(x_2 - x_1)^2 f''(x) \tag{6.19}$$

$$x = \alpha x_2 + (1 - \alpha)x_1 \in I$$

Since

$$f(x_2) \geq f(x_1) + (x_2 - x_1)f'(x_1) \tag{6.20}$$

It follows that

$$f''(x) \geq 0 \qquad \text{(Since } x_1 \neq x_2\text{)}$$

(*vi*)⇒(*i*): The Taylor-series expansion of f about the point x_0 is

$$f(x) = f(x_0) + (x - x_0)f'(x_0) + \frac{1}{2}(x - x_0)^2 f''(x_*) \tag{6.21}$$

If $f''(x) \geq 0$ then the last term is non-negative.

Let $x_0 = \alpha x_1 + (1 - \alpha)x_2$ and let $x = x_1$. Then

$$f(x_1) \geq f(x_0) + f'(x_0)[(1 - \alpha)(x_1 - x_2)]. \tag{6.22}$$

Now let $x = x_2$ and get

$$f(x_2) \geq f(x_0) + f'(x_0)[\alpha(x_1 - x_2)]. \tag{6.23}$$

Multiplying (6.22) by α and (6.23) by $1 - \alpha$ and adding gives (*i*).

7. Concluding Remarks

Theorem 6.1 gives the relationship between the second derivative and the epigraph of a convex function, and Theorem 7.1 of [3], suggesting that any of the properties in [3] and these leading to this extension is equivalent to all the properties in both results, in particular Theorem 6.1.

Thus if a given mathematical concern cannot incorporate a given property or definition of convexity we can resort to another as shown by result above. Hence this extension to the second derivative and the epigraph which leads to this result gives a wider definition of convexity.

Although these properties abound in optimization materials, a characterization which gives the relationship in Theorem 6.1 above has not been achieved.

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