Journal of the Nigerian Association of Mathematical Physics
***Volume* 16 (May, 2010), pp 395 - 400**

On Coefficient Bounds of A Subclass of Univalent Functions

[1]Oladipo A.T. and [2]Opoola T.O.
[1]Department of pure and Applied Mathematics,
Ladoke Akintola University of Technology, Ogbomoso.
P.M.B. 4000, Ogbomoso, Nigeria.
[2]Department of Mathematics,
University of Ilorin,
P.M.B. 1515, Ilorin, Nigeria.

Corresponding authors: **e-mail: atlab_3@yahoo.com; opoola_stc@yahoo.com Tel. +2348033929030**

Abstract

Let $A(\omega)$ ***denote the class of functions*** $f(z) = (z-\omega) + \sum_{k=2}^{\infty} a_k (z-\omega)^k$ ***analytic and univalent in the unit disk*** $E = \{z : |z| < 1\}$***, and*** ω ***is a fixed point in*** E***, and let*** $T_n^{\alpha}(\alpha, \beta)$ ***denote the subclass of*** $A(\omega)$ ***whose functions satisfy the inequality.*** $\mathrm{Re}\left(\frac{D^n f(z)^{\alpha}}{\alpha^n (z-\omega)^{\alpha}}\right) > \beta, z \in E, n \in 0 \cup N, 0 \le \beta < 1, \alpha > 0$ ***is real and*** D^n ***is the modified salagean derivative operator in terms of*** ω***. In this paper, the authors establish some coefficient bounds for functions of the class*** $T_n^{\alpha}(\alpha, \beta)$***. We also briefly discuss the Fekete-Szego theorem concerning our results.***

Keywords: analytic functions, univalent functions, coefficient bounds and modified Salagean operator.

2000 Mathematics Subject Classification: Primary 30C45

1.0 Introduction

Let A denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \qquad (1.1)$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. Also let $A(\omega)$ denote the class of functions of the form

$$f(z)=(z-\omega)+\sum_{k=2}^{\infty}a_k(z-\omega)^k \tag{1.2}$$

which are also analytic in the unit disk $E=\{z:|z|<1\}$ and normalized by $f(\omega)=0$ and $f'(\omega)-1=0$ and

ω is a fixed point in E see [1,2].

Let P denote the class of analytic function $p(z)$ of the form

$$p(z)=1+\sum_{k=1}^{\infty}p_k z^k \tag{1.3}$$

such that $\operatorname{Re} p(z)>0$, and let $P(\omega)$ denote the class of analytic function $p^{\omega}(z)$ of the form

$$p^{\omega}(z)=1+\sum_{k=1}^{\infty}B_k(z-\omega)^k \tag{1.4}$$

and that

$$|B_k|\le\frac{2}{(1+d)(1-d)^k},\ d=|\omega|, k\ge 1 \tag{1.5}$$

that are regular in E and satisfying $p^{\omega}(\omega)=1$ and $\operatorname{Re} p(z)>0$. We note here that $A(0)\equiv A$ and $P(0)\equiv P.$

Furthermore, let $T_n^{\alpha}(\beta)$ denote the subclass of A defined in (1) whose functions satisfy

$$\operatorname{Re}\left(\frac{D^n f(z)^{\alpha}}{\alpha^n z^{\alpha}}\right)>\beta, z\in E \tag{1.6}$$

$n=0,1,2,\ldots$ $0\le\beta<1$, $\alpha>0$ is real and D^n is the Salagean derivative operator defined as

$$D^0 f(z)=f(z), D^1 f(z)=zf'(z),\ldots, D^n f(z)=z(D^{n-1}f(z))' \tag{1.7}$$

The above class of functions was studied in [3,4,5,6], some interesting properties of the class were established. The class $T_n^{\alpha}(\beta)$ is a class of Bazilevic functions of type α.

The aim of the present paper is to derive coefficient bounds for the extention of the class defined in (1.6) and briefly point out the relationship of our results with the classical Fekete-Szego theorem.

The extension shall be defined as follows.

Definition 1.1: A normalized analytic function given by (1.2) belongs to the class $T_n^{\alpha}(\omega,\beta)$ if and only if

$$\operatorname{Re}\left(\frac{D^n f(z)^{\alpha}}{\alpha^n (z-\omega)^{\alpha}}\right)>\beta, z\in E \tag{1.8}$$

where n,α,β are as earlier defined and D^n in this class shall be modified as follows

$$D^0 f(z)=f(z), D^1 f(z)=(z-\omega)f'(z),\ldots, D^n f(z)=(z-\omega)(D^{n-1}f(z))' \tag{1.9}$$

and ω is a fixed point in E. This extension serves as a new generalization to the class of functions defined in (1.6).

2.0 Main Result

In this section we state and prove the following

Theorem 2.1. Let $f(z)$ defined by (1.2) belongs to $T_n^{\alpha}(\omega,\beta)$, then

$$|a_2| \le \frac{2\alpha^{n-1}(1-\beta)}{(\alpha+1)^n(1-d^2)}, \alpha > 0,\ d = |\omega|, \tag{2.1}$$

$$|a_3| \le \frac{2\alpha^{n-1}(1-\beta)[(\alpha+1)^{2n}(1+d) - \alpha^{n-1}(1-\beta)(\alpha-1)(\alpha+2)^n]}{(\alpha+1)^{2n}(\alpha+2)^n(1-d^2)^2}, 0 < \alpha < 1, \tag{2.2}$$

$$\frac{2(1-\beta)(1+d)\alpha^{n-1}}{(\alpha+2)^n(1-d^2)^2}, \alpha \ge 1$$

$$|a_4| \le \begin{cases} \Omega_1 + \Omega_2 + \Omega_3 + \Omega_4, \alpha \in (0,1) \\ \Omega_1 + \Omega_3 + \Omega_4, \alpha \in [1,2) \\ \Omega_1 + \Omega_3, \alpha \in [2,\infty) \end{cases} \tag{2.3}$$

where

$$\Omega_1 = \frac{2\alpha^{n-1}(1-\beta)}{(\alpha+3)^n(1-d^2)(1-d)^2}$$

$$\Omega_2 = \frac{4\alpha^{2n-2}(1-\alpha)(1-\beta)^2}{(\alpha+1)^n(\alpha+2)^n(1-d^2)^2(1-d)}$$

$$\Omega_3 = \frac{4\alpha^{3n-3}(\alpha-1)^2(1-\beta)^3}{(\alpha+1)^{3n}\ (1-d^2)^3}$$

$$\Omega_4 = \frac{4\alpha^{3n-3}(\alpha-1)(2-\alpha)(1-\beta)^3}{3(\alpha+1)^{3n}(1-d^2)^3}$$

and

$$|a_5| \le \begin{cases} \Omega_1 + \Omega_2 + \Omega_4 + \Omega_5 + \Omega_6 + \Omega_8, \alpha \in (0,1) \\ \Omega_1 + \Omega_2 + \Omega_7, \alpha \in [1,2) \\ \Omega_1 + \Omega_2 + \Omega_3 + \Omega_8, \alpha \in [2,3) \\ \Omega_1 + \Omega_2 + \Omega_8, \alpha \in [3,\infty) \end{cases} \tag{2.4}$$

Where

$$\Omega_1 = \frac{2\alpha^{n-1}(1-\beta)}{(\alpha+4)^n(1-d^2)(1-d)^3}$$

$$\Omega_2 = \frac{12\alpha^{3n-3}(\alpha-1)^2(1-\beta)^3}{(\alpha+1)^{2n}(\alpha+2)^n(1-d^2)^3(1-d)}$$

$$\Omega_3 = \frac{20\alpha^{4n-4}(\alpha-1)^2(\alpha-2)(1-\beta)^4}{3(\alpha+1)^{4n}\ (1-d^2)^4}$$

$$\Omega_4 = \frac{10\alpha^{4n-4}(\alpha-1)^3(1-\beta)^4}{(\alpha+1)^{4n}(1-d^2)^4}$$

$$\Omega_5 = \frac{4\alpha^{2n-2}(1-\alpha)\ (1-\beta)^2}{(\alpha+1)^n(\alpha+3)^n(1-d^2)^2(1-d)^2}$$

$$\Omega_6 = \frac{2\alpha^{2n-2}(1-\alpha)\ (1-\beta)^2}{(\alpha+2)^{2n}(1-d^2)^2(1-d)^2}$$

$$\Omega_7 = \frac{4\alpha^{3n-3}(\alpha-1)\ (\alpha-2)(1-\beta)^3}{(\alpha+1)^{2n}(\alpha+2)^n(1-d^2)^3(1-d)}$$

$$\Omega_8 = \frac{4\alpha^{4n-4}(\alpha-1)\ (\alpha-2)(3-\alpha)(1-\beta)^4}{3(\alpha+1)^{4n}(1-d^2)^4}$$

Proof: Let $f \in T_n^{\alpha}(\omega,\beta)$, we define

$$\frac{\dfrac{D^n f(z)^{\alpha}}{\alpha^n (z-\omega)^{\alpha}} - \beta}{1-\beta} = p^{\omega}(z) \qquad (2.5)$$

from (2.5) we have that

$$D^n f(z)^{\alpha} = \alpha^n (z-\omega)^{\alpha} + \alpha^n (1-\beta)(z-\omega)^{\alpha} \sum_{k=1}^{\infty} B_k (z-\omega)^k \qquad (2.6)$$

and from (2.6) we have that

$$D^n f(z)^{\alpha} = (z-\omega)^{\alpha}[\alpha^n + \sum_{j=1}^{\infty}(\alpha+j)^n \alpha_j (a_2(z-\omega) + a_3(z-\omega)^2 + a_4(z-\omega)^3 + \ldots)^j] \qquad (2.7)$$

On putting (2.7) in (2.6) we have

$$(z-\omega)[\alpha^n + \sum_{j=1}^{\infty}(\alpha+j)^n \alpha_j (a_2(z-\omega) + a_3(z-\omega)^2 + a_4(z-\omega)^3 + \ldots)^j] =$$

$$(z-\omega)^{\alpha}[\alpha^n + \alpha^n(1-\beta)\sum_{k=1}^{\infty} B_k (z-\omega)^k]. \qquad (2.8)$$

Let $\Psi_j = (\alpha+j)^n$, $(j = 1,2,3,\ldots)$ and $L = \alpha^n(1-\beta)$ in (2.8) hence we have

$$L[B_1(z-\omega)+B_2(z-\omega)^2+B_3(z-\omega)^3+B_4(z-\omega)^4+...]=\alpha_1\frac{\Psi_1}{L}a_2(z-\omega)+$$

$$[\alpha_1 a_3+\alpha_2 a_2^2]\frac{\Psi_2}{L}(z-\omega)^2+[\alpha_1 a_4+2\alpha_2 a_2 a_3+\alpha_3 a_2^3]\frac{\Psi_3}{L}(z-\omega)^3+$$

$$[\alpha_1 a_5+\alpha_2(2a_2a_4+a_3^2)+3\alpha_3 a_2^2 a_3+\alpha_4 a_2^4]\frac{\Psi_4}{L}(z-\omega)^4+... \qquad (2.9)$$

On comparing the coefficient in (2.9) we obtain

$$a_2=\frac{LB_1}{\alpha_1} \qquad (2.10)$$

$$a_3=\frac{LB_2}{\alpha_1\Psi_2}-\frac{\alpha_2 L^2 B_1^2}{\alpha_1^3\Psi_1^2} \qquad (2.11)$$

$$a_4=\frac{LB_3}{\alpha_1\Psi_3}-\frac{\alpha_2 L^2 B_1 B_2}{\alpha_1^2\Psi_1\Psi_2}+\frac{\alpha_2^2 L^3 B_1^3}{\alpha_1^3\Psi_1^3}-\frac{\alpha_3 L^3 B_1^3}{\alpha_1^4\Psi_1^3} \qquad (2.12)$$

$$a_5=\frac{LB_4}{\alpha_1\Psi_4}+\frac{6\alpha_2^2 L^3 B_1^2 B_2}{\alpha_1^5\Psi_1^2\Psi_2}+\frac{5\alpha_2\alpha_3 L^4 B_1^4}{\alpha_1^6\Psi_1^4}-\frac{2\alpha_2 L^2 B_1 B_3}{\alpha_1^3\Psi_1\Psi_3}-\frac{\alpha_2 L^2 B_2^2}{\alpha_1^3\Psi_2^2}$$

$$-\frac{3\alpha_3 L^3 B_1^2 B_2}{\alpha_1^4\Psi_1^2\Psi_2}-\frac{\alpha_4 L^4 B_1^4}{\alpha_1^5\Psi_1^4} \qquad (2.13)$$

It is known that if $P^{\omega}\in P(\omega)$ then (1.5), that is,

$\left|B_k\right|\le\frac{2}{(1+d)(1-d)^k}$, $k=1,2,3,...$ Hence we have the results.

The classical Fekete-Szego Theorem states that for $f\in S$ by (1.1)

$$\left|a_3-\mu a_2^2\right|\le\begin{cases}3-4\mu, & \mu\le 0\\ 1+2\exp\left(\frac{-2\mu}{1-\mu}\right), & 0\le\mu\le 1\\ 4\mu-3, & \mu\ge 1\end{cases}$$

and that this is sharp (see[8]). Several authors attempted the related problems for either μ is complex or μ is real as they appeared in literatures. Our next result shows the connection between our class and Fekete-Szego classical theorem.

Theorem 2.2: Let $f\in T_n^{\alpha}(\omega,\beta)$. Then

$$\left|a_3-\mu a_2^2\right|\le\frac{2\alpha^{n-1}(1-\beta)}{\Psi_2(1-d^2)(1-d)}-\frac{2\alpha^{2n-2}(\alpha-1)(1-\beta)^2}{\Psi_1^2(1-d^2)^2}-\mu\frac{4\alpha^{2n-2}(1-\beta)^2}{\Psi_1^2(1-d^2)^2},\ \mu\le 0 \qquad (2.14)$$

$$\left|a_2a_4-a_3^2\right|\le\frac{4\alpha^{2n-2}(1-\beta)^2}{\Psi_1\Psi_3(1-d^2)^2(1-d)^2}-\frac{4\alpha^{2n-2}(1-\beta)^2}{\Psi_2^2(1-d^2)^2(1-d)^2}+\frac{4\alpha^{4n-4}(\alpha-1)^2(1-\beta)^4}{\Psi_1^4(1-d^2)^4}$$

$$-\frac{8\alpha^{4n-4}(\alpha-1)(\alpha-2)(1-\beta)^4}{3\Psi_1^4(1-d^2)^4} \qquad (2.15)$$

Proof: The proof follows from Theorem 2.1. Theorem 2.1 connects the class of functions discussed in this paper directly to Fekete-Szego classical theorem.

3.0 Corrollaries

We consider some special cases of equivalent classes of $T_n^\alpha(\omega,\beta)$.

3.1 The class $T_n^\alpha(0,\beta) \equiv T_n^\alpha(\beta)$. Let $f \in T_n^\alpha(0,\beta)$, then $|a_k| \le \frac{2(1-\beta)}{(\alpha+k)^n}, k = 2,3,4,5$

3.2 The class $T_0^1(0,\beta) \equiv B(\beta)$. Let $f \in T_0^1(0,\beta)$, then $|a_k| \le 2(1-\beta), k = 2,3,4,5$

3.3 The class $T_0^1(\omega,\beta)$. Let $f \in T_0^1(\omega,\beta)$ then $|a_k| \le \frac{2(1-\beta)}{(1+d)(1-d)^{k-1}}, k = 2,3,4,5$

3.4 The class $T_0^1(0,0) \equiv S_0 \equiv G_0$. Let $f \in T_0^1(0,0)$, then $|a_k| \le 2$

3.5 The class $T_0^1(\omega,0)$. Let $f \in T_0^1(\omega,0)$ then $|a_k| \le \frac{2}{(1+d)(1-d)^{k-1}}, k = 2,3,4,5$

3.6 The class $B_1(\alpha)$ of Bazilevic functions which are equivalent to $T_1^\alpha(0,0)$, for $k = 2.$ Let $f \in T_1^\alpha(0,0)$, then $|a_2| \le \frac{2}{(\alpha+1)}$.

3.7 The class $T_0^1(0,\beta) \equiv \delta(\beta)$. Let $f \in T_1^1(0,\beta)$, then $|a_k| \le \frac{2(1-\beta)}{k}, k = 2,3,4,5$

3.8 The class $T_1^1(\omega,\beta)$. Let $f \in T_1^1(\omega,\beta)$, then $|a_k| \le \frac{2(1-\beta)}{k(1+d)(1-d)^{k-1}}, k = 2,3,4,5$

References

[1] Kanas, S. and Ronning, F. (1999): *Uniformly starlike and convex functions and other related classes of univalent functions.* Ann. Univ. Mariae Curie-skldowka Section A, 53, 95-105.

[2] Mugur Acu and Shigeyoshi Owa (2005): *On some subclasses of univalent functions.* Journal of Inequalities in Pure and Applied Mathematics, Vol. 6, Issue 3, Article 70, 1-14.

[3] Opoola, T.O. (1994): *On a new subclass of univalent functions.* Mathematical Cluj, 36, 59, (2) 195-200

[4] Babalola, K.O. and Opoola,T.O. (2006): *Iterated integral transforms of Caratheodory functions and their applications to analytic and univalent functions.* Tamkang Journal of Mathematics, 37 (4) 355-366.

[5] Babalola, K.O. and Opoola,T.O. (2008): *On the coefficients of certain analytic and univalent functions.* Advances in Inequalities for Series Nova Science Publishers (Edited by S.S. Dragmoir and A. Sofo) 5-17.

[6] Opoola, T.O., Babalola, K.O., Fadipe-Joseph A.O.,and Rauf, K. (2004): *On coefficient bounds of a subclass of univalent functions.* Journal of the Nigeria Mathematical Society, Vol.24, 87-92.

[7] Keogh, F.R. and Merkes, E.P. (1969): *A coefficient inequality for certain classes of analytic functions.* Proc. Amer. Math. Soc. 20, 8-12.

[8] Maslina Darus (2002): *The Fekete-Szego theorem for close-to-convex functions of the class* $K_{sh}(\alpha,\beta)$. Acta Mathematica Academiae Paedagogicae Nyiregyhaziensis, 18, 13-18.