Journal of the Nigerian Association of Mathematical Physics
***Volume* 16 (May, 2010), pp 391 - 394**

Integral Solutions To Heun's Differential Equation via Some Rational Polynomial of Degree 2, 3,4,5,6 Transformation

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Abstract

The present work determines the integral form solution derived from the transformation of Heun's equation to hypergeometric equation by rational substitution.

AMS Subject classification: 33CXX
Key words: Hypergeometric Functions, Heun's Functions Polynomial transform.

1.0 Introduction

Polynomial transformations from some properties of hypergeometric equation

Considering Hn (a, q, α, β, γ, δ, ϵ, x) as the analytic solution of GHE equation[2] around x = 0 and normalized by $H_n(0) = 1$, we seek to answer the following questions

(i) When is Hn (x) reducible to some hypergeometric equation ${}_2F_1$?

(ii) When is $DH_n(x)$ again a $H_n(x)$ for a good choice of parameters?

Maier [4] in 2005 solved the problem (i) in full generality from the following theorem, enlarging the work of Kuiken[1]

Theorem1.1 ***if the Heun's equation parameter value (a,q,$\alpha,\beta,\gamma,\delta$,) are such that the equation is non trivial (q $\neq$ 0 or $\alpha\beta \neq 0$), and all four of t = 0,1, a, ∞ are singular points, then there are only seven non composite no prefatory Heun- to-hypergeometric transformations, up to isomorphism. These seven transformations involve polynomial maps of degree 2,4,3,4,5,6 respectively. A representative list gives***

(1) $H_n\left(2,\alpha\beta,\alpha,\beta,\gamma,\alpha+\beta-2\gamma+1;t\right) = {}_2F_1\left(\frac{\alpha}{2},\frac{\beta}{2};\gamma;1-(1-t)^2\right).$

(1.1)

(2) $H_n\left(4,\alpha\beta;\alpha,\beta,\frac{1}{2},\frac{2(\alpha+\beta)}{2};t\right) = {}_2F_1\left(\frac{\alpha}{3},\frac{\beta}{3};\frac{1}{2};1-(1-t)^2(1-\frac{t}{4})\right).$

(1.2)

(3) $H_n\left(2,\alpha\beta;\alpha,\beta\frac{\alpha+\beta+2}{4},\frac{\alpha+\beta}{2};t = {}_2F_1\left(\frac{\alpha}{4},\frac{\beta}{4};\frac{\alpha+\beta 2}{4};1-4[t(2-t)-\frac{1}{2}]^2\right).\right.$

(1.3)

(4) $H_n\left(\frac{1}{2}+i\frac{\sqrt{3}}{2},\alpha\beta\left(\frac{1}{2}+i\frac{\sqrt{3}}{6}\right),\alpha,\beta,\frac{(\alpha+\beta+1}{3};\frac{\alpha+\beta+1}{3};t\right) = {}_2F_1\left(\frac{\alpha}{3},\frac{\beta}{3};\frac{\alpha+\beta+1}{3};1-\left[1-\frac{t}{\frac{1}{2}+\frac{i\sqrt{3}}{6}}\right]^3\right)$

(5) $H_n\left(\frac{1}{2}+i\frac{5\sqrt{2}}{4}, \alpha\left(\frac{1}{2}-\alpha\right)\left(\frac{1}{2}+i\frac{\sqrt{2}}{4}\right); \alpha, \frac{1}{3}-\alpha, \frac{1}{2}, ; t\right)=$

$${}_2F_1\left(\frac{\alpha}{4}, \frac{2-3\alpha}{12}; \frac{1}{2}; 1-\left(1-\frac{4t}{2+i\sqrt{2}}\right)^3\left(1-\frac{4t}{2+5i\sqrt{2}}\right)\right).$$

(1.4)

(6) $H_n\left(\frac{1}{2}+i\frac{11\sqrt{15}}{90}, \alpha\left(\frac{5}{6}-\alpha\right)\left(\frac{1}{2}+i\frac{\sqrt{15}}{18}\right); \alpha\frac{5}{6}-\alpha, \frac{2}{3}; \frac{2}{3}; t\right)=$

$${}_2F_1\left(\frac{\alpha}{5}, \frac{5-6\alpha}{30}; \frac{2}{3}; -\frac{2025}{64}i\sqrt{15}t\left(\frac{18t-9-i\sqrt{15}}{18}\right)^3\right)$$

(1.5)

(7) $H_n(\frac{1}{2}\underline{+}i\frac{\sqrt{3}}{2}, \alpha(1-\alpha)(\frac{1}{2}+i\frac{\sqrt{3}}{6}); \alpha; 1-\alpha, \frac{2}{3}, \frac{2}{3}; t) =$

$${}_2F_1\left(\frac{\alpha}{6}, \frac{1}{6}-\frac{\alpha}{6}; \frac{2}{3}; 1-4\{[1-\frac{t}{\frac{1}{2}+\frac{i\sqrt{3}}{6}}]^3-\frac{1}{2}\}^2\right)$$

Main Results

2.0 Integral solutions

In this section we shall apply the relations above in deriving the integral form of solution via these polynomial transformations. Let be $\int_c$ a integral operator defined over a compact interval *C*. Since $(a)_{n-1} = \frac{(a-1)n}{a-1}$, we have $\int_c {}_2F_1\ a, b; c; z = R(t) = \frac{c-1}{(a-1)(b-1}\ {}_2F_1\ (a-1, b-1; c-1; z = R(t)|c$ and through a push and pull- back processes we the following possible solutions;

$$\int_c {}_2F_1\left(\frac{\alpha}{2}, \frac{\beta}{2}; \gamma; 1-(1-t)^2\right)dt = \frac{4t(\gamma-1)}{3(\alpha-2)(\beta-2}\left[1+\frac{t}{2t-t^2}\right]{}_2F_1(\frac{\alpha-2}{2}, \frac{\beta-2}{2}; \gamma; 2t-t^2)c,$$

(2.1)

and the pull back operator gives

(1) $\int_c Hn(2, \alpha\beta; \alpha, \beta, \gamma, \alpha+\beta-2\gamma+1; t)dt = \frac{4t(\gamma-1}{3(\alpha-2)(\beta-2)}\left[1+\frac{t}{2t-t^2}\right]$
$x\ \ Hn\,(2, (\beta-2)(\alpha-2)/2; \alpha-2, \beta-2, \gamma+1, \alpha+\beta-2\gamma-1; t)|c.$

The other six transformations work in the same way:

(2) $\int_c H_n\left(4, \alpha\beta; \alpha, \beta, \frac{1}{2}, \frac{2(\alpha+\beta)}{3}; t\right)dt =$

$$\frac{t}{4}\left[1+\frac{2t^2+9t}{4(1-(1-t)^2(1-\frac{t}{4}))}\right]\frac{1}{(\alpha-3)(\beta-3)}$$

$$x\,Hn\left(4,(\alpha-3)(\beta-3);\alpha-3,\beta-3,\tfrac{1}{2};\tfrac{2}{3}(\alpha+\beta-6);t\right)|c \tag{2.2}$$

(3) $\int_c Hn\left(2,\alpha\beta;\alpha,\beta,\frac{\alpha+\beta+2}{4},\frac{(\alpha+\beta)}{2};t\right)dt$

$$=\frac{64(\alpha+\beta-2)}{(\alpha-4)(\beta-4)^2}x\frac{t}{15}\left[1+\frac{52t-80t^2+56t^2-8t^2}{1-4(t(2-t)-\frac{1}{2})^2}\right]$$

$$x\;\; H_n\left(2,(\alpha-4)(\beta-4),\alpha-4,\beta-4,\frac{\alpha+\beta-6}{4},\frac{(\alpha-8)}{2};t\right)|c$$

(2.3)

(4) $\int_c H_n\ (\frac{1}{2}+i\frac{\sqrt{3}}{2},\alpha\beta\left(\frac{1}{2}+i\frac{\sqrt{3}}{6}\right),\alpha,\beta,\frac{(\alpha+\beta+1}{3};\frac{\alpha+\beta+1}{3};t)$dt

$$=3\,\frac{(\beta+\alpha-2}{4(\alpha-3)(\beta-3)}\frac{t}{8}\left[1+\frac{t^3+9t\left(\frac{1}{2}+i\frac{\sqrt{3}}{6}\right)^2+2t^2\left(\frac{1}{2}+i\frac{\sqrt{3}}{6}\right)}{\left(\frac{1}{2}+i\frac{\sqrt{3}}{6}\right)^3\left(1-\left(1-\frac{t}{\left(\frac{1}{2}+i\frac{\sqrt{3}}{6}\right)^3}\right)\right)}\right]$$

$$xH_n(\frac{1}{2}+i\frac{\sqrt{3}}{2},(\beta-3)(\alpha-3)(\frac{1}{2}+i\frac{\sqrt{3}}{6};$$

$$\alpha-3,\beta-3,\frac{\alpha+\beta-5}{3},\frac{\alpha+b-5}{3};t)|c \tag{2.4}$$

(5)

$$\int_c H_n\left(\frac{1}{2}+i\frac{5\sqrt{2}}{4},\alpha\left(\frac{2}{3}-\alpha\right)\left(\frac{1}{2}+i\frac{\sqrt{2}}{4}\right);\alpha,\frac{2}{3}-\alpha,\frac{1}{2},\frac{1}{2};t\right)dt$$

$$=\frac{12}{(\alpha-4)(10+3\alpha)}$$

$$x\frac{t}{20}\left[1+27\left(\frac{4t}{2+i\sqrt{2}}+9\frac{4t}{2+5i\sqrt{2}}\right)t-17\left(\left(\frac{4t}{2+i\sqrt{2}}\right)^2+\left(\frac{4t}{2+i\sqrt{2}}\right)\frac{4t}{2+5i\sqrt{2}}\right)t^2\right]$$

$$xHn(\frac{1}{2}+i\frac{5\sqrt{2}}{4},(\alpha-4)(\frac{14}{2}-\alpha)(\frac{1}{2}+i\frac{\sqrt{2}}{4};$$

$$\alpha-4,\frac{14}{3}-\alpha,\frac{1}{2},\frac{1}{2};t)|c \tag{2.5}$$

(6)

$$\int_c H_n\left(\frac{1}{2}+i\frac{11\sqrt{15}}{90},\alpha\left(\frac{5}{6}-\alpha\right)\left(\frac{1}{2}+i\frac{\sqrt{15}}{18}\right);\alpha,\frac{5}{6}-\alpha,\frac{2}{3},\frac{2}{3};t\right)dt\ =$$

$$\frac{(18t-9-i\sqrt{15}}{18})^4$$

$$((t^2-t)+1/360(18t-9-i\sqrt{15)}-t/180(18t-9-i\sqrt{15}+$$

$$1/60((18t-9-i\sqrt{15})/18)^2\ \text{x}$$

$$H_n(\frac{1}{2}+i\frac{11\sqrt{15}}{90},(\alpha-5)\left(\frac{35}{6}-\alpha\right)\left(\frac{1}{2}+i\frac{\sqrt{15}}{18}\right);$$

$$\alpha-5,\frac{35}{6}-\alpha,\frac{2}{3},\frac{2}{3};t)|c$$

(7) Putting $r = 1/2 \pm i\sqrt{3}/6$, we obtain

$$\int_c H_n\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}, \alpha(1-\alpha)(1-\alpha)\left(\frac{1}{2}+i\frac{\sqrt{3}}{6}\right); \alpha, 1-\alpha, \frac{2}{3}, \frac{2}{3}; t\right) dt = \frac{-12(-\frac{4}{7r^6r^7}+r^5t^6-12r^4t^5+19r^3t^4-16r^2t^3+6rt^2}{(\alpha-6)(12-\alpha)}$$

$$Hx\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}, (\alpha-6)(7-\alpha\left(\frac{1}{2}+i\frac{\sqrt{3}}{6}\right); \alpha-6, 7-\alpha, \frac{2}{3}, \frac{2}{3}; t\right)|c \quad (2.6)$$

Equivalent results for the transformation involving polynomial map of degree 2 proposed in [1] has been obtained in the same way as above.

References

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