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On The Theory Of Filtration At A Decreasing Rate

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Abstract

We investigate a one – dimensional filtration model based on [4]. The resulting equation, together with initial and boundary conditions were solved by analytical method. Various cases of the model were considered to obtain the concentration of impurities suspended in the liquid. Hence we employed the Laplace transform, asymptotic techniques, separation of variable to suite each of the cases to be considered. The result obtained, shows that filtration occurs at a decreasing rate in accordance with practical condition of filter use.

1.0 Introduction

Several scientists have made contributions in various ways to the theory of filtration, leading to the formulation of mathematical models to interpret and solve physical filtration problems. Far back 1856, Darcy in the study of John J. and Jonathan K. [6], worked on the theory of cake filtration through a porous media. He considered a fluid flowing through the spaces between solid particles making up the porous media. The amount of solids present was considered to be responsible for the resistance to fluid flow in the form of friction. The resistance was found to cause a pressure loss which was directly proportional to the thickness of the cake.
Demchik examined the theory of filtration at a decreasing rate based on the Mints model [4]. Agbonrofo [1] re-examined the Demchik model and made several generalizations. Agbonrofo and Ayeni [2] worked on one of these generalizations. They considered the effect of pressure in the cause of filtration, their result showed the aging process of a filter as a result of blockage of the pores by impurities. In this paper we considered other cases under the generalizations made by [1]

2.0 Mathematical Formulation

Following Demchik [4] the system of model equation with initial and boundary conclusions corresponding to the mints model for $\gamma = 0$ has the form

$$\rho_t + v(t)C_x, \quad \rho_t = \beta C - a(t)\rho \tag{2.1}$$

$$v(t) = v_0/(1+\gamma t), \; a(t) = a_0 v(t): \quad v_0, \gamma, a_0 = \text{Constant}$$

$$\rho(x,0) = 0, \; C(0,t) = C_0, \; C_0 = \text{Constant} \tag{2.2}$$

Where x is the coordinate along the thickness of the filter

$v(t)$ is the filtration rate

$C(t)$ is the concentration of impurities suspended in the liquid

$\rho(x,t)$ is the sediment

β is the kinetic coefficient assumed to be a constant

C_0 is the impurity concentration in the liquid at the filter inlet.

In this paper, of a particular interest is to determine the concentration of impurity in the liquid. In order to achieve this, we eliminate the function ρ from equation (2.1) and putting $C = U(1+\gamma t)^{-z}$, where $z = a_0 v_0 \gamma^{-1}$, we obtain the hyperbolic equation.

$$U_{xt} + b(t)U_t - PU = 0$$

$$b(t) = \beta v_0^{-1}(1+\gamma t), \quad P = \beta v_0^{-1}(z-1) \tag{2.3}$$

We obtain the following from equations (2.1) and (2.2)

$$U(x,0) = C_0 1^{-\beta v_0^{-1} x} \qquad U(0,t) = C_0(1+\gamma t)^z, \tag{2.4}$$

Equation (2.3) and (2.4) are given in terms of the concentration of impurities suspended in the liquid

3.0 Method of Solution

We employed analytical method [7] to determine the concentration of impurities suspended in the liquid. The particular method of solution to be used will be determined by the nature of the case to be considered.

Case 3.1

Considering equation (2.3) and (2.4)

$$U_{xt} + b(t)U_t - PU = 0 \tag{3.1}$$

$$b(t) = \beta v_0^{-1}(1+\gamma t), \quad P = \beta v_0^{-1}(z-1)$$

$$U(x,0) = C_0 \lambda^{-\beta v_0^{-1} x} \qquad U(0,t) = C_0(1+\gamma t)^z, \tag{3.2}$$

First we assume the constant γ to be zero ($\gamma = 0$), this implies that;

$$\beta(t) = \beta v_0^{-1}, \qquad P = 0$$

Equation (3.1) then becomes

$$U_{xt} + \beta v_0^{-1} U_t = 0 \tag{3.3}$$

Then we seek Laplace transform for equation (3.3)

$$L\{U_{xt}\} + L\{\beta v_0^{-1} U_t\} = 0$$

This implies that

$$\frac{d}{dx} L\{U_t\} + \beta v_0^{-1} L\{U_t\} = 0$$

$$\frac{dL\{U_t\}}{L\{U_t\}} = -\beta v_0^{-1} dx$$

Integrating

$$\log_e L\{U_t\} = -\beta v_0^{-1} x + K_0(s)$$

then, $$L\{U_t\} = K_1(s)\lambda^{-\beta v_0 x} \qquad (3.4)$$

Considering LHS, by definition

$$L\{U_t\} = \int_0^\infty \lambda^{-\alpha} U_t dt \qquad (3.5)$$

Using integration by part we have

$$L\{U_t\} = s\overline{U} - U(x,0) \qquad (3.6)$$

$$s\overline{U} - U(x,0) = K_1(s)\lambda^{-\beta v_0^{-1} x} \qquad (3.7)$$

From the initial condition

$$U(x,0) = C_0 \lambda^{-\beta v_0^{-1} x}$$

then equation (3.7) becomes

$$s\overline{U} - C_0 \lambda^{-\beta v_0^{-1} x} = K_1(s)\lambda^{-\beta v_0^{-1} x}$$

so we have

$$\overline{U} = L\{U\} = \frac{k_2(s)\lambda^{-\beta v_0^{-1} x}}{s} \qquad (3.8)$$

where, $$K_2(s) = K_1(s) + C_0$$

to obtain $K_2(s)$ recall equation (3.7)

$$s\overline{U} - U(x,0) = K_1(s)\lambda^{-\beta v_0^{-1} x}$$

using the boundary condition

$$U(0,t) = C_0$$

$$K_2(s) = sU(0,s) \qquad (3.9)$$

From equation (3.8) substituting for $k_2(s)$

$$\overline{U} = \frac{sU(0,s)}{s}\lambda^{-Bvo^{-1}x} \quad = U(0,s)\lambda^{-Bvo^{-1}x} \qquad (3.10)$$

Taking the inverse transform equation (3.10) becomes

$$U(x,t) = L^{-1}\left\{\overline{U}\{0,s\}\lambda^{-Bvo^{-1}x}\right\} \qquad (3.11)$$

To obtain $\overline{U}\{0,s\}$, we transform the boundary condition

$$U\{0,t\} = C_0$$

$$\Rightarrow L\{U(0,t)\} = \frac{C_0}{s}$$

$$\overline{U}(0,s) = \frac{C_0}{s} \qquad (3.12)$$

Equation (3.11) gives

$$U(x,t) = L^{-1}\left\{\frac{C_0}{s}\lambda^{-Bvo^{-1}x}\right\}$$

$$= C_0 L^{-1}\left\{\frac{1}{s}\right\}\lambda^{-Bvo^{-1}x}$$

Since $L^{-1}\left\{\frac{1}{s}\right\} = 1$

$$\therefore \quad U(x,t) = C_0 \lambda^{-\beta v_0^{-1} x}$$

(3.13)

Where $U(x,t)$ is the concentration of impurities suspended in the liquid.

Case 3.2

Here we introduce the variable t (time) to case 3.1 to obtain the following

$$U_{xt} + b(t)U_t = 0$$

(3.14)

Where,

$$b(t) = \beta v_0^{-1}\lambda^{-\alpha t}$$

$$U(x,0) = C_0\lambda^{-\beta v_0^{-1}x} \qquad U(0,t) = C_0(1+t)$$

(3.15)

Using asymptotic expansion we have,

$$\beta v_0^{-1}\lambda^{-\alpha t} = \beta v_0^{-1} - \alpha t \beta v_0^{-1} + \frac{\alpha^2 t^2}{2!}\beta v_0^{-1} -$$

(3.16)

let $U = U_0 + \alpha U_1 + \alpha^2 U_2 +$

(3.17)

Equation (3.14) becomes

$$\left(\frac{\partial^2 U_0}{\partial x \partial t} + \frac{\alpha \partial^2 U_1}{\partial x \partial t} + \frac{\alpha^2 \partial^2 U_2}{\partial x \partial t}\right) = -\left(\frac{\beta}{v_0} - \frac{\alpha t \beta}{v_0} + \frac{\alpha^2 t^2 \beta}{2v_0} - .....\right)\left(\frac{\partial U_0}{\partial t} + \frac{\alpha \partial U_1}{\partial t} + \frac{\alpha^2 \partial U_2}{\partial t}\right)$$

$$\left(\frac{\partial^2 U_0}{\partial x \partial t} + \frac{\alpha \partial^2 U_1}{\partial x \partial t} + \frac{\alpha^2 \partial^2 U_2}{\partial x \partial t}\right) = -\left(\begin{array}{l} \frac{\beta}{v_0}\frac{\partial U_0}{\partial t} + \frac{\alpha\beta}{v_0}\frac{\partial U_1}{\partial t} + \frac{\alpha^2 t\beta}{v_0}\frac{\partial U_2}{\partial t} - \frac{\alpha t\beta}{v_0}\frac{\partial U_0}{\partial t} - \frac{\alpha^2 t\beta}{v_0}\frac{\partial U_1}{\partial t} \\ -\frac{\alpha^3 t\beta}{v_0}\frac{\partial U_2}{\partial t} + \frac{\alpha^2 t^2 \beta}{2v_0}\frac{\partial U_0}{\partial t} + \frac{\alpha^3 t^2\beta}{2v_0}\frac{\partial U_1}{\partial t} + \frac{\alpha^4 t^2 \beta}{2v_0}\frac{\partial U_2}{\partial t} \end{array}\right)$$

(3.18)

We now equate terms in power of α

For $\alpha^0 : \frac{\partial^2 U_0}{\partial x \partial t} = \frac{\beta}{v_0}\frac{\partial U_0}{\partial t}$ (3.19)

For $\alpha^1 : \frac{\partial^2 U_1}{\partial x \partial t} = \frac{\beta}{v_0}\left(\frac{t \partial U_0}{\partial t} - \frac{\partial U_1}{\partial t}\right)$

(3.20)

For α^2: $\frac{\alpha \partial^2 U_2}{\partial x \partial t} = \frac{\beta}{v_0}\left(\frac{t\partial U_1}{\partial t} - \frac{t^2}{2}\frac{\partial U_0}{\partial t} - \frac{\partial U_2}{\partial t}\right)$

(3.21)

The initial and boundary condition for U_0, U_1 *and* U_2 are:

$$U_0(x,0)=C_0\lambda^{-\beta\nu_0^{-1}x},\quad U_0(0,t)=C_0(1+t)$$
(3.22)
$$U_1(x,0)=0,\quad U_1(0,t)=0$$
(3.23)
$$U_2(x,0)=0,\quad U_2(0,t)=0$$
(3.24)

Considering equation (3.19)

$$\frac{\partial^2 U_0}{\partial x\partial t}=\frac{\beta}{\nu_0}\frac{\partial U_0}{\partial t}$$
$$U_0(x,0)=C_0\lambda^{-\beta\nu_0^{-1}x},\quad U_0(0,t)=C_0(1+t) \qquad (3.25)$$

Using the method of separation of variables we assume a solution of form

$$U_0(x,t)=X(x)T(t)\neq 0$$

Where $X(x)$ is a function of x only and $T(t)$) a function of t only.

The result obtained is given as

$$U_0(x,t)=C_0(1+t)\lambda^{-\frac{\beta}{\nu_0}x} \qquad (3.26)$$

Obtaining $U_1(x,t)$ we recall equation

$$\frac{\partial^2 U_1}{\partial x\partial t}=\frac{\beta}{\nu_0}\left(\frac{t\partial U_0}{\partial t}-\frac{\partial U_1}{\partial t}\right) \qquad (3.27)$$
$$U_1(x,0)=0,\quad U_1(0,t)=0$$

From equation (3.26), equation (3.27) becomes

$$\frac{\partial^2 U_1}{\partial x\partial t}=\frac{\beta}{\nu_0}\left(C_0t\lambda^{-\frac{\beta}{\nu_0}x}\right)-\frac{\partial U_1}{\partial t} \qquad (3.28)$$

Using the same method of separation variables we have the following result for $U_1(x,t)$

$$U_1(x,t)=\frac{\beta C_0t^2x}{2\nu_0}\lambda^{-\frac{\beta}{\nu_0}x} \qquad (3.29)$$

To obtain $U_2(x,t)$, recall equation (3.21)

$$\frac{\alpha\partial^2 U_2}{\partial x\partial t}=\frac{\beta}{\nu_0}\left(\frac{t\partial U_1}{\partial t}-\frac{t^2}{2}\frac{\partial U_0}{\partial t}-\frac{\partial U_2}{\partial t}\right) \qquad (3.30)$$
$$U_1(x,0)=0,\quad U_1(0,t)=0$$

From the solution of $U_1(x,t)$

$$\frac{\partial U_1}{\partial t}=\frac{\beta C_0tx}{2\nu_0}\lambda^{-\frac{\beta}{\nu_0}x}$$

then equation (3.30) gives

$$\frac{\partial^2 U_2}{\partial x\partial t}=\frac{\beta}{\nu_0}\left(\frac{C_0\beta t^2x}{\nu_0}\lambda^{-\frac{\beta}{\nu_0}x}-\frac{t^2}{2}C_0\lambda^{-\frac{\beta}{\nu_0}x}-\frac{\partial U_2}{\partial t}\right) \qquad (3.31)$$

The result obtained from the same method is given below

$$U_2(x,t)=\frac{\beta C_0t^3x}{6\nu_0}\left(\frac{2\beta x-\nu_0}{\nu_0}\right)\lambda^{-\frac{\beta}{\nu_0}x} \qquad (3.32)$$

From equation (3.17)

$$U = U_0 + \alpha U_1 + \alpha^2 U_2 + \ldots$$

Putting α = 0.01 we have

$$U = U_0 + 0.01U_1$$

Therefore,

$$U(x,t) = \left((1+t) + \frac{0.01\beta t^2 x}{2\nu_0} \right) C_0 \lambda^{-\frac{\beta}{\nu_0}x} \tag{3.33}$$

Case 3.3

We then consider the inhomogeneous form of case 3.2, and this gives us the equation

$$U_{xt} + b(t)U_t = f(x,t) \tag{3.34}$$

$$b(t) = \beta \nu_0^{-1} \lambda^{-\alpha t}$$

$$f(x,t) = \alpha\lambda^{-(x+t)} \tag{3.35}$$

$$U(x,0) = C_0 \lambda^{-\beta\nu_0^{-1}x} \qquad U(0,t) = C_0(1+t) \tag{3.36}$$

Following the same method of asymptotic expansion of case 3.2 and integration, we arrived at the following result

$$U(x,t) = C_0(1+t)\lambda^{-\frac{\beta}{\nu_0}x} + 0.01\left[\frac{\beta}{\nu_0}\frac{t^2 x}{2} C_0 \lambda^{-\frac{\beta}{\nu_0}x} + \frac{\nu_0}{\beta - \nu_0}\left(\lambda^{-\left(\frac{\beta x}{\nu_0}+t\right)} - \lambda^{-(x+t)} - \lambda^{-\frac{\beta}{\nu_0}x} + \lambda^{-x} \right) \right] \tag{3.37}$$

Case 3.4

Let us now generalize the filtration problem to

$$a(t)U_{xt} + b(t)U_t + c(t) = 0 \quad 0 < x < 1, \;\; t > 0 \tag{3.38}$$

and then determine the criteria for $a(t)$, $b(t)$ *and* $c(t)$ by seeking similarity transformation [1] so that,

$$U = f(\eta) \quad where\ \eta = \frac{x}{t^\alpha} \text{ (the similarity variable)}$$

$$\frac{\partial U}{\partial t} = \frac{dt}{d\eta}\frac{\partial \eta}{\partial t} \tag{3.39}$$

But $\eta = xt^{-\alpha}$ (3.40)

$$\frac{\partial \eta}{\partial t} = \alpha\, x\, t^{-(\alpha+1)} = \frac{-\alpha\eta}{t}$$

$$\Rightarrow Ut = \frac{\partial U}{\partial t} = \frac{-\alpha\eta}{t}\frac{df}{d\eta} \tag{3.41}$$

Similarly

$$\frac{\partial U}{\partial x} = \frac{df}{d\eta}\frac{\partial \eta}{dx} \tag{3.42}$$

$$\Rightarrow \frac{\partial u}{\partial x} = \frac{1}{t^\alpha}\frac{df}{d\eta} \tag{3.43}$$

Also,

$$U_{xt} = \frac{\partial^2 U}{\partial x \partial t} = \frac{\partial}{\partial x}\left(\frac{\partial U}{\partial t}\right)$$

$$= \frac{\partial}{\partial x}\left(\frac{-\alpha\eta}{t}\frac{df}{d\eta}\right)$$

$$= -\frac{\alpha}{t}\left(\frac{\partial\eta}{\partial x}\frac{df}{d\eta} + \eta\frac{d^2 f}{d\eta^2}\frac{\partial\eta}{dx}\right)$$

$$= -\frac{\alpha}{t}\left(t^{-\alpha}\frac{df}{d\eta} + \eta t^{-\alpha}\frac{d^2 f}{d\eta^2}\right)$$

Thus,

$$\frac{\partial^2 U}{\partial x \partial t} = -\alpha t^{-(\alpha+1)}\left(\frac{\partial f}{\partial\eta} + \eta\frac{d^2 f}{d\eta^2}\right) \tag{3.44}$$

Substituting equation (3.44) and (3.42) in (3.38) we have

$$-\frac{a(t)}{t^{\alpha+1}}\alpha\left(\frac{df}{d\eta} + \frac{d^2 f}{d\eta^2}\right) - b(t)\frac{\alpha\eta}{t}\ \frac{df}{d\eta} + c(t)f = 0$$

$$\Rightarrow -\frac{a(t)}{t^{\alpha}}\alpha\left(\frac{df}{d\eta} + \eta\frac{d^2 f}{d\eta^2}\right) - b(t)\alpha t\frac{df}{d\eta} + c(t)f = 0 \tag{3.45}$$

Hence for similarity to exist

$$a(t) = t^{\alpha},\ b(t) = \frac{1}{\alpha},\ c(t) = 1$$

Such that equation (3.8) becomes

$$\alpha\eta\frac{d^2 f}{d\eta^2} + \alpha\frac{df}{d\eta} + \eta\frac{df}{d\eta} + f = 0$$

$$\Rightarrow \alpha\eta f^{11} + \alpha f^{1} + \eta f^{1} + f = 0 \tag{3.46}$$

We now seek the Frobenius series solution to resolve (3.46) by assuming a Solution of the form

$$f = \sum_{r=0}^{\infty} a_r \eta^{r+c} \tag{3.47}$$

$$f^{'} = \sum_{r=0}^{\infty} (r+c) a_r \eta^{r+c-1} \tag{3.48}$$

$$f^{''} = \sum_{r=0}^{\infty} (r+c)(r+c) a_r \eta^{r+c-2} \tag{3.49}$$

Then equation (3.46) yields

$$\alpha\sum_{r=0}^{\infty}(r+c)(r+c-1)a_r\eta^{r+c-1} + \alpha\sum_{r=0}^{\infty}(r+c)a_r\eta^{r+c-1} + \sum_{r=0}^{\infty}(r+c)a_r\eta^{r+c} + = \sum_{r=0}^{\infty}a_r\eta^{r+c} = 0 \tag{3.50}$$

This implies that,

$$\sum_{r=0}^{\infty}\{(r+c)(r+c-1)+(r+c)\}\,a_r\eta^{r+c-1}+\frac{1}{\alpha}\sum_{r=0}^{\infty}(r+c+1)a_r\eta^{r+c}=0$$

$$\Rightarrow (c(c-1)+c)a_0\eta^{c-1}+\sum_{r=1}^{\infty}\{(r+c)(r+c-1)+(r+c)\}a_r\eta^{r+c-1}+\frac{1}{\alpha}\sum_{r=1}^{\infty}(r+c+1)a_r\eta^{r+c} \qquad (3.51)$$

Let $r=r+1$ in the second term such that,

$$(c(c-1)+c)a_0\eta^{c-1}+\sum_{r=0}^{\infty}\{(r+c+1)(r+c)(r+c+1)\}a_{r+1}\eta^{r+c}+\frac{1}{\alpha}\sum_{r=0}^{\infty}(r+c+1)a_r\eta^{r+c}$$

$$\therefore\Rightarrow (c(c-1)+c)a_0\eta^{c-1}+\sum_{r=0}^{\infty}\{(r+c+1)(r+c)+(r+c+1)a_{r+1}+\}\eta^{r+c}+\frac{1}{\alpha}(r+c+1)a_r\}\eta^{r+c}$$

$$(3.52)$$

Finding the indicial and the recurrence relation we obtain the following

$$f=\sum_{r=0}^{\infty}a_r\eta^{r+c}=a_0\eta^c+a_1\eta^{c+1}+a_2\eta^{c+2}+a_3\eta^{c+3}+a_4\eta^{c+4}+.....$$

$$\Rightarrow f=\eta^c\left\{a_0-\frac{a_{o\eta}}{\alpha(c+1)}+\frac{a_o\eta^2}{\alpha^2(c+1)(c+2)}-\frac{a_o\eta^3}{\alpha^3(c+1(c+2)(c+3))}+....\right\} \qquad (3.53)$$

$$f|_{c=0}=V=a_0\left\{1-\frac{\eta}{\alpha}+\frac{\eta^2}{2\alpha^2}-\frac{\eta^3}{6\alpha^3}+.....\right\} \qquad (3.54)$$

$$\therefore V=Ae^{\frac{-\eta}{\alpha}}$$

To obtain the second independent solution we differentiate equation (3.53) with respect to c, then we have

$$\frac{\partial f}{\partial c}=a_0\eta^c 1n\eta\left\{1-\frac{\eta}{\alpha(c+1)}+\frac{\eta^2}{\alpha^2(c+1(c+2))}+\frac{\eta^3}{\alpha^3(c+1)(c+2)(c+3)}+.....\right\}$$

$$+a_0\eta^c\frac{\partial}{\partial c}\left\{1-\frac{\eta}{\alpha(c+1)}+\frac{\eta^2}{\alpha^2(c+1(c+2))}\right\}-\frac{\eta^3}{\alpha^3(c+1)(c+2)(c+3)}+.....$$

$$\frac{\partial f}{\partial c}=a_0\eta^c 1n\eta\left\{1-\frac{\eta}{\alpha(c+1)}+\frac{\eta^2}{\alpha^2(c+1)(c+2)}\right\}-\frac{\eta^3}{\alpha^3(c+1)(c+2)(c+3)}+.....$$

$$+a_0\eta^c\left\{\frac{\eta}{\alpha}\left(\frac{1}{(c+1)^2}\right)-\frac{\eta^2}{\alpha^2}\left(\frac{2c+3}{(c+2)^2(c+2)^2}\right)+\frac{\eta^3}{\alpha^3}\left(\frac{3c^2+12c+11}{(c+1)^2(c+2)^2(c+3)^2}\right)\right\}+..... \qquad (3.55)$$

$$\frac{\partial f}{\partial c}|_{c=0}=W=a_0\left\{1n\eta\left(1-\frac{\eta}{\alpha}+\frac{\eta^2}{2\alpha^2}-\frac{\eta^3}{6\alpha^3}\right)+\frac{\eta}{\alpha}-\frac{3\eta^2}{2^2\alpha^2}+\frac{11\eta^3}{2^2.3^2.\alpha^3}-.......\right\} \qquad (3.56)$$

$$f=V+W$$

$$\Rightarrow f=Ae^{\frac{-\eta}{\alpha}}+B\left\{1n\eta^{\frac{-\eta}{\alpha}}+\frac{\eta}{\alpha}-\frac{3\eta^2}{2^2\alpha^2}+\frac{11}{2^2.3^2}\left(\frac{\eta}{\alpha}\right)^3\right\}$$

$$\therefore f = Ae^{\frac{-\eta}{\alpha}} + B\left\{1n\eta e^{\frac{-\eta}{\alpha}} + \frac{\eta}{\alpha} - \frac{3}{2^2} + \left(\frac{\eta}{\alpha}\right)^2 \frac{11}{2^2.3^2}\left(\frac{\eta}{\alpha}\right)^3\right\} \tag{3.57}$$

But $U(x,t) = f(\eta)$

Where $\eta = \dfrac{x}{t^a}$

So equation (3.20) becomes

$$U(x,t) = Ae^{\frac{-x}{\alpha t^\alpha}} + B\left\{\ln\frac{x}{t^\alpha}e^{\frac{-x}{\alpha t^\alpha}} + \frac{x}{\alpha t^\alpha} - \frac{3}{2^2}\left(\frac{x}{\alpha t^\alpha}\right)^2 + \frac{11}{2^2.3^2}\left(\frac{x}{\alpha t^\alpha}\right)^3 - \ldots\right\} \tag{3.58}$$

From physical point of view, require $f(0)$ to be finite, thus $B \equiv 0$ and

$$U(x,t) = Ae^{\frac{-x}{\alpha}}$$

If we impose the condition $U(0,t) = 1$

Then $A = 1$ which implies that

$$U(x,t) = e^{\frac{-x}{\alpha}} \tag{3.59}$$

Therefore we have the graph of the results in figures 1 - 4

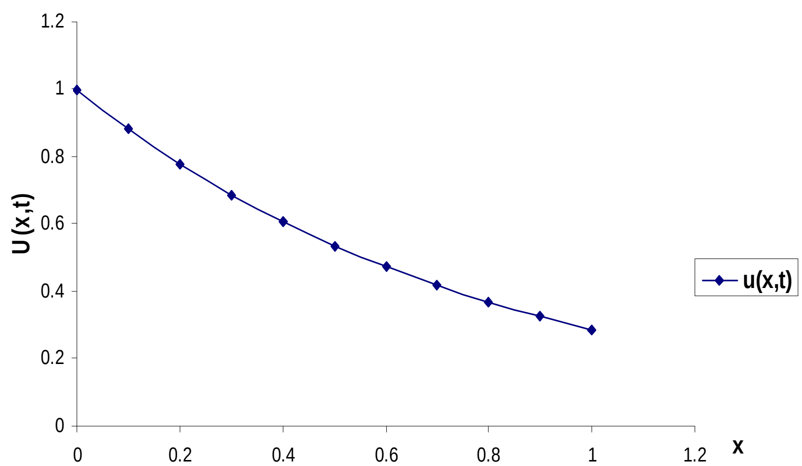

Figure 1: The graph of U(x,t) against x

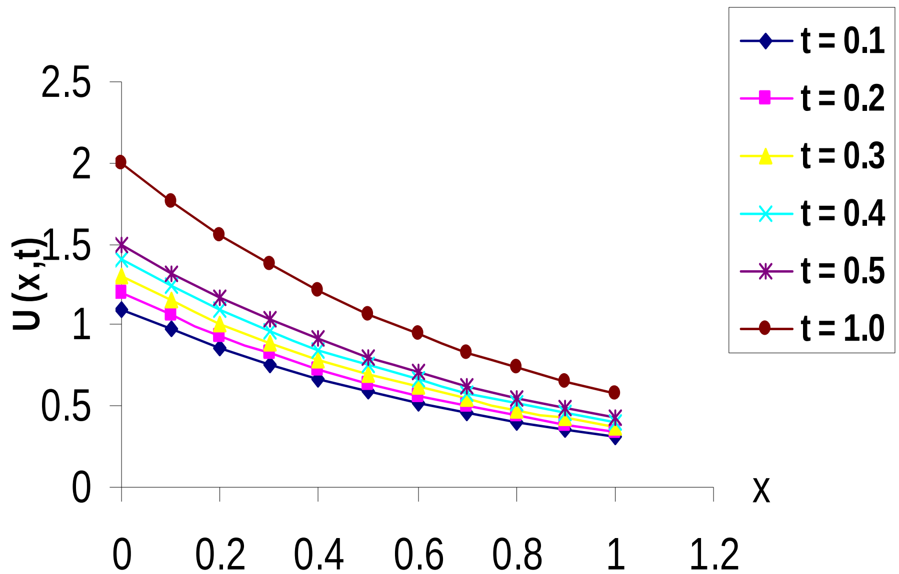

Figure 2: The graph of U(x,t) against x

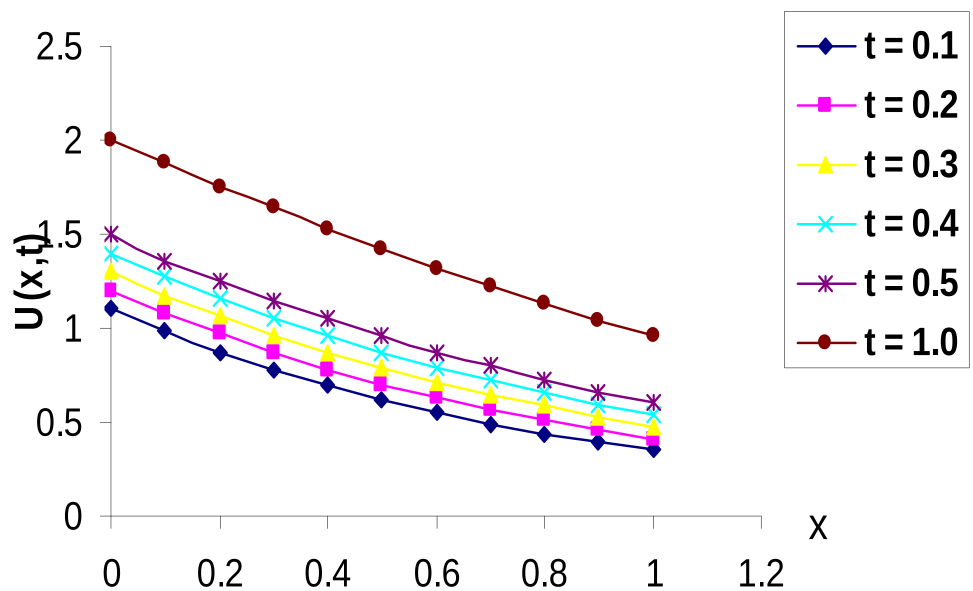

Figure 3: The graph of U(x,t) against x

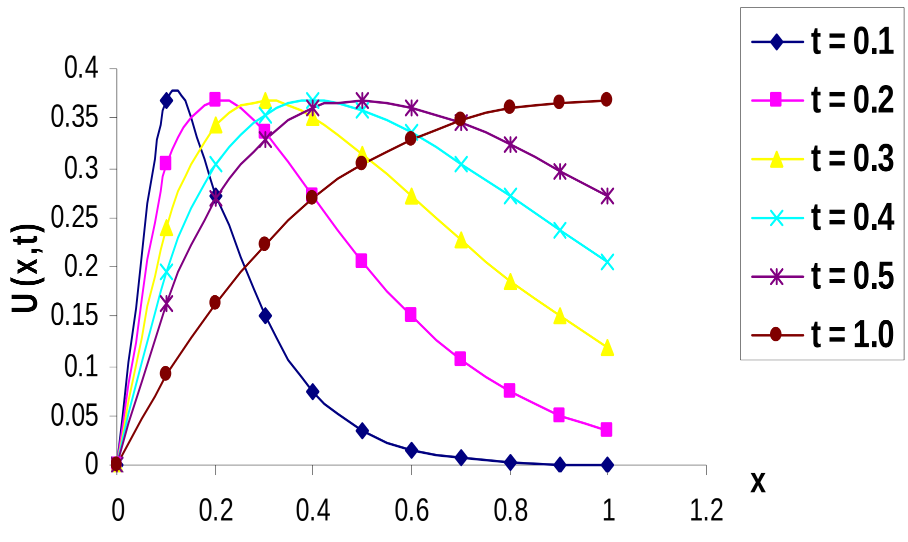

Figure 4: The graph of the U(x,t) against x

4.0 Discussion of Results and Conclusion

A one-dimension filtration model was investigated. The resulting equation, together with initial and boundary conditions were solved by analytical method. We employed Laplace transform for case 3.1 and the result obtained (see figure 1) shows that filtration occurs at a non-increasing rate.

The variable t (time) was introduced to case 3.1 and the boundary condition (to obtain case 3.2), in order to determine what happens to filtration as the time increases. This was solved using asymptotic expansion and method of separation of variable. It was discovered that filtration occurs at a decreasing rate, though the rate increases with time.

In case 3.3, the inhomogeneous form of case 3.2 was considered. The same method for case 3.2 was employed. And the result was the same as that of problem 3.2. We then went further to consider the generalized filtration problem by employing the similarity transformation with the similarity variable $\eta = \frac{x}{t^a}$ to case 3.4. Here, we seek the Frobenius series solution to obtain the result, and observed that filtration occurs at a decreasing rate for all finite time. Although, the concentration increases with time when t is sufficiently large.

Finally, in this paper we have been able to make the following contributions.

1. With the generalization of the particular solution of [4], we still obtained results in accordance with practical condition of filter use.
2. We showed that concentration of impurities decreases along the thickness of the filter.
3. We showed that filtration occurs at a decreasing rate for various values of time (t).

REFERENCES

[1] C. O. Agbonrofo (2003), *A Mathematical model for filtration*. M. Tech thesis. Ladoke akintola University of technology, Ogbomoso, Nigeria.

[2] C. O. Agbonrofo & R. O. Ayeni (2004), *A Mathematical model for filtration*. Journal of the Nigeria Association of Mathematical Physics, Vol. 8, PP 293-294

[3] G. I. Barenblatt & Vazquez (1998), *A new free boundary problem for unsteady flow in porous media*. Euro Journal of applied Mathematics Vol. 9, PP 37-54

[4] I .I.Demchik (1998), *The Theory of filtration at a decreasing rate.* Journal of Applied Maths Mechs, Vol. 62, PP 479-481

[5] H.H. Ngo, et al (1996), *Mathematical modeling of down flow floating medium Filter with in-line flocculation Arrangement.* Water Science and Technology. Vol. 34 PP 355-362

[6] J. John & K. Jonathan (1999), *Literature Review on Cake Filtration.* SIAM Journal on Mathematical Analysis. Vol. 31, PP. 251-273.

[7] U.T. Myint and D. Lokeneth (1997), *Partial differential equation for Scientist and Engineers (3rd Edition).* Simon & Schuster Company, pp 137 – 145, 447 – 457.