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Onset of thermal instability in a low Prandtl number fluid with internal heat source in a porous medium

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Abstract

This paper investigates the condition leading to the onset of stationary convection in a low Prandtl number horizontal fluid layer in a porous medium heated from below with internal heat source. The internal heat source is taken as directly proportional to the temperature leading to a sinusoidal temperature gradient in the fluid layer. The effects of heat generation, porosity parameter and different Prandtl numbers, Pr are presented. The results show that the onset of stationary instability is hastened by increasing values of the internal heat generation as well as increments in the Prandtl number. Further, increases in the porosity parameter delayed the onset of stationary instability.

Keywords: low Prandtl number, heat source, porous medium, Grashof number, onset of instability.

1.0 Introduction

Buoyancy effect can become a major mechanism during a possible convective instability for a horizontal fluid layer in a porous medium heated from below and cooled from above. The instability of convection driven by buoyancy is referred to as the Rayleigh-Bernard convection. This phenomenon has wide range of applications in geophysics, engineering, and astrophysics. Among its engineering applications include material processing, crystal growth, cooling systems in nuclear reactors, cooling of electronic equipments and solar energy collections. In order to understand the physics of the onset of convection in a fluid layer heated from below, numerous theoretical and experimental studies have been carried out since Bernard demonstrated the onset of thermal instability in his early experiment in 1900. Comprehensive account of the various aspects of the determination of the criterion for the onset of Rayleigh-Bernard instability of this problem and its extensions could be found in [2], [4], [11] and [12].

Equally, the problem of thermal instability induced by internal heat sources has been extensively investigated due to its importance in atmospheric studies and convection of the mantle, to mention but a few. The presence of internal heat generation leads to nonlinear temperature distribution in the system and hence convection may occur whether the top boundary temperature is lower or higher than the bottom temperature as long as a negative temperature gradient of sufficient magnitude is maintained somewhere within the fluid. Various aspects of the problem of thermal convection in horizontal fluid layer have been carried out for different boundary conditions with uniform internal heat sources by [5], [7], [14], [15], [16], [17], [19] and [20]. For example, [17] studied analytically the problem of thermal instability of an internally heated fluid as well as heated from below, with various boundary conditions and showed that with increasing heat generation rate the fluid is prone to instability; while [16] studied the same problem and found that the critical wave number for the onset of convection decreases as the internal Rayleigh number increases. [6] reported the conceptual design of a

downward convecting solar pond filled with water – saturated porous medium in which the internal heat source varied exponentially with depth. [13] studied the same problem with the effect of varying gravity in the vertical direction using the energy method. Their study revealed the possibility of subcritical convection.

However, in most of these studies the conditions which lead to the onset of instability in a porous layer depend on two nondimensional parameters, namely the external Rayleigh number and the internal Rayleigh number. Recently, an impressive progress has been made in the understanding of thermal convection in Rayleigh-Bernard problem, namely the Rayleigh number, Ra and Prandtl number, Pr. This is particularly important in astrophysical environments where the Prandtl number of stars could be as low as 10^{-8} as pointed out by [3]; and in the flow of liquid metals which have been used for rapid cooling of nuclear reactors $\mathrm{Pr} << 1$ ([10] and [18]). The stability of buoyant – thermocapillary – driven flows have been studied for low Prandtl number fluids by [9] and, the characteristics and stability of buoyant – thermocapillary – driven flows in finite shallow cavities by[1]. Israel – Cookey [8] considered the problem of the effect of radiation absorption on thermal convection of a low Prandtl number horizontal fluid layer in a porous medium heated from below.

Most of the previous work on the effect of internal heat sources on the onset of thermal instability has focused on the linear stability analysis of convective flows infinitely extended layers when the heat source is constant. The aim of this paper is to study the onset of thermal instability in a low Prandtl number fluid layer in a porous medium with internal heat source when the heat source is proportional to the temperature. We focus on the situation in which the basic state temperature and the basic temperature gradient in the fluid are sinusoidal with respect to the vertical displacement. The Prandtl numbers in the range $0.001 \le \mathrm{Pr} \le 0.1$ are used for the case of two free-free boundaries.

2.0 Mathematical formulation

We consider a radiative absorbing porous layer of height h heated from below and confined between two horizontal parallel surfaces located at $z^* = -\frac{h}{2}$ and $z^* = \frac{h}{2}$. The lower and upper surfaces are maintained at temperatures T_1 and T_2 respectively. The fluid is assumed to be Newtonian with constant physical properties (kinematic viscosity, ν , thermal diffusivity, κ_T , and density, ρ) except for the density in the buoyancy term, which in the Boussinesq approximation, depends linearly on the temperature as follows

$$\rho = \rho_0[1 - \beta(T^* - T_0)] \tag{2.1}$$

where β is the thermal expansion coefficient, T^* the temperature, $T_0 = (T_1 + T_2)/2$ and ρ_0 is the density of the fluid at temperature T_0, such that $T_1 > T_2$ is the reference temperature. Further, the internal heat source is represented by the introduction of the term $Q_0(T^* - T_0)$ in the energy equation, where Q_0 is some constant of proportionality. Let us introduce a Cartesian rectangular coordinate system such that the z^*-axis is directed vertically upwards and $x^* -$ and $y^* -$ axes along the horizontal plane. The origin of the coordinates is located at equal distances from the horizontal boundaries of the layer. Thus, the convective motion is governed by the Navier-Stokes equation coupled to an energy equation. Using the scales $h, h^2/\nu, \nu/h, \rho_0(\nu/h)^2$ for length, time, velocity and pressure, respectively together with $T = (T^* - T_0)/(T_1 - T_2)$ for non-dimensionalization of the temperature, the governing equations governing take the form

$$\nabla.\mathbf{V} = 0$$
(2.2)
$$\frac{\partial \mathbf{V}}{\partial t} = -\nabla p + Gr\, T\, \mathbf{k} + \Delta \mathbf{V} - \chi^2\, \mathbf{V}$$
(2.3)
$$\frac{\partial T}{\partial t} + (\mathbf{V}.\nabla)T = \frac{1}{\Pr}\Delta T + QT$$
(2.4)

The dimensionless variables are the velocity vector $\mathbf{V} = (u, v, w)$, the pressure, *p*. The nondimensional parameters
arising from the scaling of the equations are the Grashof number, $Gr = g\beta h^3 (T_1 - T_2)/\nu^2$, the Prandtl number $\Pr = \nu / \kappa_T$, the internal heat source, $Q = h^2 Q_0 /(\rho_0 c_p)$, the porosity parameter $\chi^2 = h^2 / K$, K the permeability of the porous medium and **k is** the unit in the vertical direction and the direction of the acceleration due to gravity, *g*. Also, the operators $\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ and $\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ are the gradient and Laplace operators in Cartesian coordinates, respectively.

The boundary conditions are

$$w = 0,\ T = \pm 1/2 \qquad \text{at} \quad z = \mu 1/2$$
(2.5)

Equations (2.2) – (2.4) together with the boundary conditions (2.5) admits the following basic (stationary) solution

$$\mathbf{V} = 0,\ T_b = -\frac{1}{2Sin\frac{\alpha}{2}} Sin\alpha z, \quad p_b = \frac{Gr}{2\alpha Sin\frac{\alpha}{2}} Cos\alpha z,\ \alpha = \sqrt{Q\Pr}$$
(2.6)

where T_b and p_b are the basic temperature and pressure distribution of the system, respectively. It should be noted that the basic temperature distribution T_b is sinusoidal as a result of the fact that the internal heat source is proportional to temperature.

3.0 Basic State and Linear Stability Analysis

We now investigate the linear stability of the basic state with respect to infinitesimal disturbances which are periodic in the *x* and *y* directions. Following the standard normal mode procedures [2] and [4], we write the perturbed quantities as

$$\mathbf{V} = \mathbf{0} + (u, v, w),\ T_b + \theta,\ p_b + p$$
(3.1)

where $\theta << T_b,\ p << p_b$. Upon substituting these perturbations into the dimensionless equations (2.2) – (2.4), using the basic state solutions (2.6) and neglecting the products of the disturbances, we obtain the linearized perturbation equations

$$\nabla.\mathbf{v} = 0$$
(3.2)
$$\frac{\partial \mathbf{v}}{\partial t} = -\nabla p + Gr\theta \mathbf{k} + \nabla^2 \mathbf{v} - \chi^2 \mathbf{v}$$
(3.3)

$$\Pr\frac{\partial\theta}{\partial t}=\left(\nabla^2+\Pr Q\right)\theta+\Pr\alpha_0 w$$

(3.4)

with the boundary conditions

$$w=0,\quad \theta=0 \quad \text{at } z=\pm 1/2$$

(3.5)

$$\frac{\partial^2 w}{\partial z^2}=0 \text{ on a free surface}$$

(3.6)

where $\alpha_0=\left(\frac{\alpha}{2}\right)\frac{Cos(\alpha z)}{Sin(\alpha/2)}$ is the basic temperature gradient.

Taking the normal modes of the form [4]

$$w=W(z)f(x,y)e^{\sigma t},\quad \theta=\Theta(z)f(x,y)e^{\sigma t}$$

(3.7)

where $\sigma=\omega_R+i\omega_I$ is complex amplification rate and ω_R,ω_I are real numbers. Substitution of (3.7) into system (3.2) – (3.4) and the boundary conditions (3.5 and 3.6), eliminating the pressure and the velocity components *u* and *v* from the resulting equations, we obtain a system of linear differential equations with constant coefficients

$$(D^2-a^2)(D^2-a^2-\chi^2-\sigma)W=Gra^2\Theta$$

(3.8)

$$(D^2-a^2\Pr Q-\Pr\sigma)\Theta=-\alpha_0\Pr W$$

(3.9)

The above equations (3.8) – (3.9) are to be solved subject to the boundary conditions

$$W=0=\Theta \quad \text{at} \quad z=\pm 1/2$$
$$D^2W=D^4W=...=0 \quad \text{on a free surface}$$

(3.10)

In the above equations $D\equiv\frac{\partial}{\partial z},\ \nabla_h^2 f=-a^2 f$. Also, a^2 is a horizontal wave number arising from the separation variables and ∇_h^2 is the Laplace operator with respect to the horizontal coordinates. The above equations (3.8) – (3.10) constitute an eigenvalue problem for *Gr* with the parameters Pr, *Q*, *a*, χ and σ; and the critical value of *Gr* for the onset of instability is its minimum as Pr, *Q*, *a*, χ, σ, and *a* are varied. Next, we proceed with the study of all possible disturbances for all wave numbers by eliminating Θ from the eigenvalue problem (3.8) – (3.10) to obtain a sixth – order linear differential equation with constant coefficient

$$(D^2-a^2)(D^2-a^2-\chi^2)(D^2-a^2+\Pr Q-\Pr\sigma)W=-\alpha_0\Pr a^2 W$$

(3.11)

subject to the following boundary conditions

$$W=D^2W=D^4W=...=0 \quad \text{at } z=\pm 1/2$$

(3.12)

4.0 Results and Discussion

For an idealized free – free boundaries, we assume a trial solution for (3.11) characterizing the lowest mode of the form ([2] and [4])

$$W = w_0 Sin\pi z$$
(4.1)

where w_0 is a constant.

Substituting (4.1) into (3.11) and simplifying the relation for Grashof number as a function of Pr, Q, χ, a and σ:

$$Gr = \frac{(\pi^2 + a^2)(\pi^2 + a^2 + \chi^2 + \sigma)(\pi^2 + a^2 - \Pr Q + \Pr \sigma)}{\alpha_0 \Pr a^2}$$
(4.2)

For the case of the onset of stationary instability, we set $\sigma = 0$, $a = a_c$ and $Gr = Gr_c$ in (4.2), we obtain

$$Gr_c = \frac{(\pi^2 + a_c^2)(\pi^2 + a_c^2 + \chi^2)(\pi^2 + a_c^2 - \Pr Q)}{\alpha_0 \Pr a_c^2}$$
(4.3)

where Gr_c and a_c are the critical Grashof number and critical wave number, respectively. The critical wave number is obtained by minimizing (4.3). The modified Grashof number given in (4.3) reaches its minimum when $\partial Gr_c / \partial a_c^2 = 0$. Consequently, we obtain a sixth – order polynomial in a_c given by

$$2a_c^6 + (3\pi^2 + \chi^2 - \Pr Q)a_c^4 - (\pi^6 + \pi^4(\chi^2 - \Pr Q) - \chi^2\pi^2 \Pr Q) = 0 \qquad (4.4)$$

Using the solutions of (4.4) together with (4.3), we are able to determine the critical Grashof number, Gr_c, which will determine the stability of the system. According to the usual classification, the system is stable whenever $Gr < Gr_c$ and unstable whenever $Gr > Gr_c$. Since we are interested in computing the critical wave number, Now, solving the characteristic polynomial (4.4) numerically using the symbolic software "mathematica" we obtain six roots for which only one real positive root exist as the critical wave number, a_c. Using this root we obtain the critical Grashof number, Gr_c for the onset of instability. For the analysis that follows, we consider the fact that Gr_c attains its minimum at the centre of the channel and since $\sqrt{\Pr Q} << 1$ as $\alpha_0 \to 1$.

Numerically computed values of the critical wave numbers and critical Grashof numbers for various values of the parameters Pr, Q, and χ are summarized in Tables 1 – 2. It observed from these tables that in the absence of the internal heat parameter, Q and the porosity parameter, χ that is, when $\chi = 0$ and $Q = 0$, the critical Rayleigh number is given by $Ra_{cri} = Gr_{cri} \Pr = 657.511$ and $a_c = \pi / \sqrt{2}$. These results are in good agreement with those of [2]. Further, increases in the Prandtl number, Pr and the internal heat parameter, Q lead to decrease in the critical wave number and critical Grashof number. This in essence implies that increases in Pr and Q hasten the onset of instability. On the other hand onset of instability is delayed with increases in the porosity parameter, χ irrespective of the values of Pr and Q.

Tables 3 - 5 show the variation of Grashof number $Gr(a)$ on the wave number, a, for different values of the parameters Q, Pr and χ. Table 3 shows a slight decrease in the values of Gr as Q increases. This implies that increase in Q hastens the onset of instability in the system. In other words, the presence of the internal heat sources destabilizes the system. Further, increase in Pr hastened the onset of instability (Table 4); while the presence of χ delayed the onset of instability (Table 5).

Table 1: Computed values of critical wave number, a_c and critical Grashof number, Gr_c for Pr = 0.001 and various values of Q and χ

χ	a_c Q = 0	Gr_c Q = 0	a_c Q = 1	Gr_c Q = 1	a_c Q = 2	Gr_c Q = 2
0.0	2.22144	657511	2.22140	657440	2.22137	657368
0.2	2.22294	659287	2.22290	659216	2.22286	659144
0.4	2.22740	664611	2.22736	664539	2.22732	664466
0.6	2.23469	673468	2.23465	673395	2.23462	673322
0.8	2.24466	685837	2.24462	685763	2.24458	685688
1.0	2.25707	701689	2.25703	701613	2.25699	701536

Table 2: Computed values of critical wave number, a_c and critical Grashof number, Gr_c for Pr = 0.1 and various values of Q and χ

χ	a_c Q = 0	Gr_c Q = 0	a_c Q = 1	Gr_c Q = 1	a_c Q = 2	Gr_c Q = 2

0.0	2.22144	6575.11	2.21767	6503.50	2.22386	6432.27
0.2	2.22294	6592.87	2.21916	6521.09	2.21535	6449.68
0.4	2.22740	6646.11	2.22360	6573.80	2.21977	6501.88
0.6	2.23469	6734.68	2.23088	6661.51	2.22703	6588.73
0.8	2.24466	6858.37	2.24082	6783.99	2.23694	6710.01
1.0	2.25707	7016.89	2.25320	6940.96	2.24928	6865.44

Table 3: Variation of Grashof number, *Gr* with the wave number, *a* for $\chi = 0.2,\ \Pr = 0.001$ and various values of *Q*

a	*Gr* $Q = 0.5$	*Gr* $Q = 1.0$	*Gr* $Q = 3.0$

1.0	1288870	1288780	1288430
1.2	1008060	1007990	1007730
1.4	847409	847356	847142
1.6	752490	752444	752260
1.8	697463	697422	697257
2.0	668895	668857	668705
2.2	659346	659310	659165
2.4	664521	664486	664345
2.6	681899	681865	681726
2.8	710017	709983	709843
3.0	748075	748039	747898
3.2	795708	795672	795526
3.4	852855	852818	852667
3.6	919670	919630	919473
3.8	996466	996425	996260
4.0	1083680	1083640	1083470
4.2	1181870	1181820	1181640
4.4	1291640	1291590	1291390
4.6	1413700	1413650	1413440
4.8	1548810	1548750	1548530
5.0	1697790	1697730	1697490

Table 4: Variation of Grashof number, *Gr* with the wave number, *a* for $\chi = 0.2, Q = 1.0$ and various values of *Pr*

a	*Gr* *Pr* = 0.001	*Gr* *Pr* = 0.01	*Gr* *Pr* = 0.1
1.0	1288780	128723.0	12717.80

1.2	1007990	100681.0	9950.50
1.4	847356	84639.4	8368.02
1.6	752444	75161.7	7433.75
1.8	697422	69668.2	6893.03
2.0	668857	66817.3	6613.47
2.2	659310	65865.9	6521.74
2.4	664486	66385.4	6575.56
2.6	681865	68124.0	6750.13
2.8	709983	70935.6	7031.05
3.0	748039	74740.2	7410.49
3.2	795672	79501.7	7884.92
3.4	852818	85214.0	8453.80
3.6	919630	91892.3	9118.70
3.8	996425	99568.2	9882.78
4.0	1083640	108286.0	10750.50
4.2	1181820	118099.0	11727.20
4.4	1291590	129071.0	12819.10
4.6	1413650	141271.0	14033.30
4.8	1548750	154775.0	15377.30
5.0	1697730	169665.0	16859.30

Table 5: Variation of Grashof number, *Gr* with the wave number, *a* for $\Pr = 0.01, Q = 1.0$ and various values χ

a	*Gr* χ =0.2	*Gr* χ = 0.4	*Gr* χ = 0.8

1.0	128723.0	130139.0	135802.0
1.2	100681.0	101746.0	106004.0
1.4	84639.4	85495.1	88917.8
1.6	75161.7	75885.0	78778.3
1.8	69668.2	70304.0	72847.1
2.0	66817.3	67393.7	69699.5
2.2	65865.9	66401.8	68545.3
2.4	66385.4	66893.8	68927.3
2.6	68124.0	68614.4	70576.0
2.8	70935.6	71415.1	73333.4
3.0	74740.2	75214.5	77111.7
3.2	79501.7	79975.2	81869.1
3.4	85214.0	85690.3	87595.4
3.6	91892.3	92374.5	94303.2
3.8	99568.2	100059.0	102022.0
4.0	108286.0	108787.0	110793.0
4.2	118099.0	118614.0	120671.0
4.4	129071.0	129600.0	131717.0
4.6	141271.0	141816.0	143999.0
4.8	154775.0	155338.0	157593.0
5.0	169665.0	170249.0	172582.0

5.0 Conclusion

The obtained numerical results illustrate the onset of thermal instability in a horizontal low Prandtl number fluid layer with internal heat source heated from below using the linear stability analysis for idealized free boundaries. The results illustrate that increases in the internal heat source parameter and Prandtl number lead to destabilization of the system; whereas increase in the porosity parameter led to stabilization of the system.

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