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A three–step discretization scheme for direct numerical solution of second-order ordinary differential equations

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Abstract

In this paper, a three-step discretization (numerical) formula is developed for direct integration of second-order initial value problems in ordinary differential equations. The development of the method and analysis of its basic properties adopt Taylor series expansion and Dahlquist stability test methods. The results show that the method is consistent, zero-stable and convergent. The computer implementation of the method with two sample problems shows that the method is quite suitable for direct solution of second-order differential equations without reformulation as first-order systems.

1.0 Introduction

Direct discretization of second-order initial value problems of ordinary differential equations in the form

$$y'' = f(x, y, y'),\ y(a) = y_o,\ y'(a) = y, a \le x \le b \ldots\ldots\ldots\ldots(1.1)$$

involves the determination of an approximation y_n to the theoretical solution $y(x_n)$ of equation (1.1) using some previously calculated estimates y_{n+j}'s, j = 0 (1) without reformulation as first-order systems. This approach plays an important role in the field of numerical analysis of real life problems such as celestical mechanics, electrical network, radio-active process, transonic, airflow and traverse motion just to mention a few.

Many techniques of this type exist in Literature, [9], [5] but because they involve reformulation of the differential equation as first-order systems, they are inefficient and cumbersome to implement.

These reasons perhaps motivated [6], [4], [2], [16] to adopt the direct discretization method for integration of special second-order differential equations of the form

$$y'' = f(y),\ y(o) = y_o,\ y'(o) = z_o \ldots\ldots\ldots\ldots\ldots\ldots\ (1.2)$$

In this paper we consider a three-step numerical formula of the form

$$y_{n+3} = a_o y_n + a_1 y_{n+1} + a_2\ y_{n+2} + h^2(\beta_o y''_n + \beta_1 y''_{n+1} + \beta_2 y''_{n+2} + \beta_3 y'''_{n+3}) \ldots (1.3)$$

for direct solution of equation (1.1). The parameters a_j's and β_j's, j = 0 (1)3 are determined as to ensure that the resultant formula is consistent, zero-stable and convergent. Taylor series expansion of variables,

$y_{n+3}, \ y_{n+2}, \ y_{n+1}$ and their derivatives are used to generate the system of algebraic equations from where the values of the unknown parameters $a_o, a_1, a_2, a_3, \beta_0, \beta_1, \beta_2, \beta_3$ are (generated) determined while the basic stability property of the method is examined using Dahlquist stability model equation.

$$y'' = \lambda y \tag{1.4}$$

The result showed that the formular is weakly p-stable but accurate. Numerical application shows that the method is accurate and is suitable for solution of ODES. The advantages of the method include good efficiency and accuracy.

2. Derivation of the Method

The unknown parameters a_o, a_1, a_2, β_o, β_1, β_2 and β_3 are determined from the system of algebraic equations generated from the adoption of Taylor series expansions of y_{n+1}, y_{n+2}, y_{n+3}, and their derivatives $y''_{n+1}, y''_{n+2}, y''_{n+3}$ as given in the local truncation error equation

$$T_{n+3} = y_{n+3} - a_o y_n - a_1 y_{n+1} - a_2 y_{n+2} - h^2\left(\beta_o y''_n + \beta_1 y''_{n+1} + \beta_2 y''_{n+2} + \beta_3 y''_{n+3}\right) \tag{2.1}$$

Thus, substituting

$$y_{n+3} = \sum_{r=o}^{\infty} \frac{(3h)^r}{r!} y^r(x_n)$$

$$y_{n+2} = \sum_{r=o}^{\infty} \frac{(2h)^r}{r!} y^r(x_n)$$

$$y_{n+1} = \sum_{r=o}^{\infty} \frac{(h)^r}{r!} y^r(x_n)$$

$$y''_{n+3} = \sum_{r=o}^{\infty} \frac{(3h)^r}{r!} y^{(r+2)}_{(x_n)}$$

$$y''_{n+2} = \sum_{r=o}^{\infty} \frac{(2h)^r}{r!} y^{(r+2)}_{(x_n)}$$

$$y''_{n+1} = \sum_{r=o}^{\infty} \frac{h^r}{r!} y^{r+2}_{(x_n)}$$

into (2.1) yields

$$T_{n+3} = \sum_{r=o}^{\infty} \frac{(3h)^r}{r!} y^r(x_n) - a_o\, y(x_n) - a_1 \sum_{r=o}^{\infty} \frac{h^r}{r!} y^r(x_n) - a_2 \sum_{r=o}^{\infty} \frac{(2h)^r}{r!} y^r(x_n) - \beta_o h^2 y''(x_n) -$$
$$\beta_1 h^2 \sum_{r=o}^{\infty} \frac{h^r}{r!} y^{r+2}(x_n) - \beta_2 h^2 \sum_{r=o}^{\infty} \frac{(2h)^r}{r!} y^{(r+2)}(x_n) - \beta_3 h^3 \sum_{r=o}^{\infty} \frac{(3h)^r}{r!} y^{r+2}(x_n) \ldots\ldots\ldots\ldots (2.2)$$

Combining terms in equal powers of h in (2.2) to obtain

$$T_{n+3} = (1 - a_o - a_2)\, y(x_n) + (3 - a_1 - 2a_2)\, h\, y'(x_n) + (\tfrac{a}{2} - \tfrac{1}{2}a_1 - 2a_2 - \beta_o - \beta_1 - \beta_2 - \beta_3) h^2 y''(x_n)$$
$$+ \left(\frac{27}{6} - \frac{a_1}{6} - \frac{8}{6} a_2 - \beta_1 - 2\beta_2 - 3\beta_3\right) h^3 y'''(x_n) + \left(\frac{81}{24} - \frac{1}{24} a_1 - \frac{16}{24} a_2 - \tfrac{1}{2}\beta_1 - \tfrac{4}{2}\beta_2 - \tfrac{9}{2}\beta_3\right) h^4 y^4(x_n)$$
$$+ \left(\frac{243}{120} - \frac{1}{120} a_1 - \frac{32}{120} a_2 - \tfrac{1}{6}\beta_1 - \tfrac{8}{6}\beta_2 - \tfrac{27}{6} 3\beta_3\right) h^5 y5(x_n)$$
$$+ \left(\frac{729}{720} - \frac{1}{720} a_1 - \frac{64}{720} a_2 - \tfrac{1}{24}\beta_1 - \tfrac{16}{24}\beta_2 - \tfrac{81}{24}\beta_3\right) h^6 y^6 + \ 0(h^7) \tag{2.3}$$

Imposing accuracy of order 5 on T_{n+3} to have the system of equation

$$a_o + {}_1 + a_2 - 1 = 0 \tag{i}$$
$$a_1 + 2a_2 - 3 = 0 \tag{ii}$$
$$\frac{a_1}{2} + \frac{4}{2} a_2 + \beta_o + \beta_1 + \beta_2 + \beta_3 - \frac{9}{2} = 0 \tag{iii}$$

$$\frac{a_1}{6}+\frac{8}{6}a_2+\beta_1+2\beta_2+3\beta_3-\frac{27}{6}=0 \qquad \text{(iv)}$$

$$\frac{a_1}{24}+\frac{16}{24}a_2+\tfrac{1}{2}\beta_1+\tfrac{4}{2}\beta_2+\frac{9}{2}\beta_3-\frac{81}{24}=0 \qquad \text{(v)}$$

$$\frac{a_1}{120}+\frac{32}{120}a_2+\tfrac{1}{6}\beta_1+\tfrac{8}{6}\beta_2+\frac{27}{6}\beta_3-\frac{243}{120}=0 \qquad \text{(vi)}$$

$$\frac{a_1}{720}+\frac{64}{720}a_2+\tfrac{1}{6}\frac{\beta_1}{24}+\frac{16}{24}\beta_2+\frac{81}{24}\beta_3-\frac{729}{720}=0 \qquad \text{(vii)}$$

In matrix notation, it becomes

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & \frac{4}{2} & 1 & 1 & 1 & 1 \\ 0 & \frac{1}{6} & \frac{8}{6} & 0 & 1 & 2 & 3 \\ 0 & \frac{1}{24} & \frac{16}{24} & 0 & \frac{1}{2} & \frac{4}{2} & \frac{9}{6} \\ 0 & \frac{1}{120} & \frac{32}{120} & 0 & \frac{1}{6} & \frac{8}{6} & \frac{22}{6} \\ 0 & \frac{1}{720} & \frac{64}{720} & 0 & \frac{1}{24} & \frac{16}{24} & \frac{81}{24} \end{bmatrix} \begin{bmatrix} a_o \\ a_1 \\ a_2 \\ \beta_o \\ \beta_1 \\ \beta_2 \\ \beta_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ \frac{9}{2} \\ \frac{29}{6} \\ \frac{81}{241} \\ \frac{243}{120} \\ \frac{729}{720} \end{bmatrix} \qquad (2.4)$$

By adoption of Gaussian method, the solution of the above equations is

$$(a_o, a_1, a_2, \beta_o\ \beta_1, \beta_2, \beta_3) = (\,1, -3, 3,\ {-\tfrac{1}{2}}, {-\tfrac{9}{12}}, \tfrac{9}{12}, \tfrac{1}{12}) \qquad (2.5)$$

Substituting (2.5) into (1.3) yields a three-step method of the form

$$y_{n+3} = 3y_{n+2} - 3y_{n+1} + y_n + \frac{h^2}{12}(y''_{n+3} + 9y''_{n+2} - 9y_{n+1} - y_n)$$

(2.6)

with order of accuracy 5 and error constant $c_{p+2} = -\frac{1}{240}$.

In order to use formula (2.6), it is necessary to know the previous values y_{n+j}; j = 0(1)2 of y and f at x_{n+j}, j = 0, 1, 2 and the step length h > o. These back values can be generated by the fifth order one step formula

$$y_{n+3} = \frac{y_{n+2}}{1 + y_{n+2}(2k_i + 3k_2 + 4k_3)}$$

(2.7)

developed in [19].

In (2.7), $k_1 = hg(x_n, z_n)$, $k_2 = hg(x_n + \tfrac{1}{2}h,\ z_n + \tfrac{1}{2}k_1)$

$k_3 = hg(x_n,\ z_n + \tfrac{3}{4}hk_3)$ with $g(x_n, z_n) = -Z_n^2 f(x_n, y_n)$ and

$z_n = \frac{1}{y_n}$

In order to use formula (2.6), four important problems arise, namely (i) the need to have the starts values y_{n+j}, j = 0(1)3 and their corresponding derivatives values $y''_{n+j,j}$, j = 0(1)3

(ii) the choice of appropriate stepsize h and

(iii) the need to solve the implicit system of equations (2.6), now

$$y_{n+3} = A + \frac{h^2}{12}g(y_{n+3})$$

(2.8)

Where

$$A = 3y_{n+2} - 5y_{n+1} + y_n$$

$$g(y_{n+3}) = y''_{n+j} ,+9y_{n+2} - 9y_{n+1} - y_n$$

(2.9)

and (iv) finally the accuracy of the approximation y_{n+3}.

The actual realization of the fomula (2.6) requires the solution of implicit equation (2.8) rewritten as;

$$F(y_{n+3}) = 0$$

(2.10)

Where $F(y_{n+3}) = y_{n+3} - A - \frac{h^2}{12} g(y_{n+3})$

This can be achieved by the adoption of quasi Newton iteration scheme

$$[y_{n+3}^{m+1} = y_{n+3}^{m}] - G[y_{n+3}^{m}] / \left(I - \frac{h^2}{12} A \right) \qquad (2.11)$$

Where $A = \frac{\partial g}{\partial y_{n+3}} \left(y_{n+3}^{(m)} \right)$, m = 0, 1, 2

The convergence condition is that

$$\theta = \frac{\left| y_{n+3} - y_{n+3}^{(m)} \right|}{\left| y_{n+3}^{m} - y_{n+3}^{m-1} \right|} \leq \text{Tolerance}$$

(2.12)

3. The Basic Properties of the Method

In view of the process of derivation of the formula and computer implementation it is obvious that the use of the method for solution of second-order ODEs is error proved. In order to be ascertain of the accuracy and suitability of the method, the analysis of its basic properties such as consistency, zero-stability and convergence are undertaken.

3.1 Order of Accuracy and Error Constant

The local truncation error T_{n+3} can be rewritten as

$$T_{n+3} = C_o + C_1 hy(x_n) + C_2 h^2 y''(x_n) + C_3 h^3 y^3(x_n) + C_4 h^4 y^{(4)}_{(x_n)} + C_5 h^5 y^{(5)}_{(x_n)} + C_6 h^6 y^{(6)}_{(x_n)}$$
$$+ 0(h^7)$$

Where

$C_o = a_o + a_1 + a_2 - 1$

$C_1 = a_1 + 2a_2 - 3$

$$C_2 = \frac{a_1}{2} + \frac{4}{2}a_2 + \beta_o + \beta_1 + \beta_2 + \beta_3 - \frac{9}{2}$$
$$C_3 = \frac{a_1}{6} + \frac{8}{6}a_2 + \beta_1 + 2\beta_2 + 3\beta_3 - \frac{27}{6}$$
$$C_4 = \frac{a_1}{24} + \frac{16}{24}a_2 + \tfrac{1}{2}\beta_1 + \frac{4}{2}\beta_2 + \frac{9}{2}\beta_3 - \frac{81}{24} \qquad \ldots\ldots\ldots\ldots\ldots.. \ (3.1.1)$$
$$C_5 = \frac{a_1}{120} + \frac{32}{24}a_2 + \frac{\beta_1}{6} + \frac{8}{6}\beta_2 + \frac{27}{6}\beta_3 - \frac{243}{120}$$
$$C_6 = \frac{a_1}{720} + \frac{64}{720}a_2 + \frac{\beta_1}{24} + \frac{16}{24}\beta_2 + \frac{81}{24}\beta_3 - \frac{729}{720}$$

Using the values of the parameters a_js and β_js as contained in (2.5) in (3.1.1), we have

$$C_o = 1 - 3 + 3 - 1 = 0$$
$$C_1 = -3 + 6 - 3 = 0$$
$$C_2 = \frac{-3}{2} + \frac{12}{2} - \frac{1}{12} - \frac{9}{12} + \frac{9}{12} + \frac{1}{12} - \frac{9}{2} = 0$$
$$C_3 = \frac{-3}{6} + \frac{8x3}{6} - \frac{9}{12} + 2\left(\frac{9}{12}\right) + \frac{3}{12} - \frac{27}{6} = 0$$
$$C_4 = \frac{-3}{24} + \frac{16}{24}(3) + \frac{1}{2}\left(\frac{-9}{12}\right) + \frac{4}{2}\left(\frac{-9}{12}\right) + \frac{9}{2}\left(\frac{1}{12}\right) - \frac{81}{24} = 0$$
$$C_5 = \frac{-3}{120} + \frac{96}{120} - \frac{9}{72} + \frac{72}{72} + \frac{27}{72} - \frac{243}{120} = 0$$
$$C_6 = \frac{-3}{720} + \frac{192}{288} - \frac{9}{72} + \frac{144}{288} + \frac{81}{288} - \frac{729}{720} = 0$$
$$C_7 = \frac{-3}{540} + \frac{384}{504} - \frac{9}{1440} + \frac{288}{1440} + \frac{243}{1440} - \frac{2187}{1440} = -\frac{1}{240}$$

Since $C_o = C_1 = C_2 = C_3 = C_4 = C_5 = C_6 = 0$ but $C_7 \neq 0$, then the order p of accuracy of the method is 5 while its error constant $C_{p+2} = -\frac{1}{240}$.

3.2 Symmetry of the method

A linear multistep method such as (2.6) is symmetric [10] if its parameters aj's and βj's satisfy conditions of the form

$a_j = a_{k-j}$

$\beta_j = \beta_{k-j}$, j = 0(1)k/2 for even K

and

$a_j = -a_{k-j}$

$\beta_j = \beta_{k-j}$, = 0 (1) k if k is odd.

Now k = 3 is odd and

$a_o = -1 = -a_3$.

$a_1 = 3 = -a_2$

$a_3 = 1 = -a_o$.

Hence, the proposed formula is symmetric.

3.3. Consistency

A linear multistep formula (2.6) is consistent [5] if, it has the following properties:

(i) its order $\rho \geq 1$

(ii) $\sum_{j=o}^{k} a_j = 0$

(iii) ρ (r) = ρ' (r) = 0 for r = 1

(iv) ρ'' (r) = 2! δ (r) = 0 at r = 1

Where ρ (r) and δ (r) are respectively called the first and second characteristics polynomials of the formula [15].

Since the order P of accuracy of the method is 5, then condition (i) is satisfied.

Also since

$$\sum_{j=o}^{k} a_j = a_o + a_1 + a_2 + a_3 = 1 - 3 + 3 - 1 = 0$$

then condition (ii) is satisfied.

Now, the characteristic polynomial equation of the method is

$$\Pi(r, h) = p(r) - \frac{h^2}{12}\delta(r) \quad (3.3.1)$$

where,

$$\rho(r) = r^3 - 3r^2 + 3r - 1 \quad (3.3.2)$$

$$\delta(r) = \frac{1}{12}\left(r^3 + 9r^2 - 9r - 1\right) \quad (3.3.3)$$

are respectively the first and second characteristic polynomial of the scheme.

Now,

ρ (r) = $3r^2 - 6r + 3$

At r = 1

$\rho(1) = 1^3 - 3(1)^2 + 3(1) - 1 = 0 = \rho^1(1) = 3(1)^2 + (1) + 3.$

Thus (1) = $\rho^1(1) = 0$ at r = 1

Similarly,

$$\delta(r) = \frac{1}{12}\left(r^3 + 9r^2 - 9r - 1\right)$$

$$\delta(1) = \frac{1}{12}\left((1)^3 + 9(1)^2 - 9(1) - 1\right) = 0$$

Therefore $\rho^1(1) = 2!\,\delta(1) = 0$

Hence, the method is consistent.

3.4. Zero-stability

A multistep discretization method like (2.6) is said to be zero-stable if no root r of its first characteristic polynomial

$\rho(r) = r^3 - 3r^2 + 3r - 1$

has modulus greater than one and every root with modulus one must be simple [5].

From the first characteristic equation (3.3.2), we have

$\rho(r) = (r - 1)^3 = 0.$

with roots r = 1, 1, 1

Showing that all the roots of the characteristic equation lie within a unit circle. Hence the method is zero-stable.

3.5 Convergence

Since the formula has been shown to be consistent and zero-stable, therefore it is convergent [5].

4.0 Implementation

As mentioned earlier, the actual realization of formula (2.8) requires the iterative solution of equation (2.6) rewritten in the compact form

$$F(y_{n+3}) = 0 \qquad (4.1)$$

where,

$$F(y_{n+3}) = y_{n+3} - 3y_{n+2} + 3y_{n+1} - y_n - \frac{h^2}{12}[f_{n+3} + 9f_{n+2} - 9f_{n+1} - f_n] \qquad (4.2)$$

and y_{n+3} is the approximation to the solution of initial value problems (1.1) evaluated at $x = x_{n+3}$ and F is an analytic function involving the values of y and y'' at the current and previous steps.

To solve equation (4.1) for y_{n+3} the values of y''_{n+j}, j = 0 (1) 3 are required and they are estimated by using the predicted iterates y_{n+j} j = 0 (1) 3 generated by formula (2.7). Thus equation (4.1) is solved by quasi-Newton iteration scheme.

$$W_{n+3}^{(s)} d_{n+3}^{(s+1)} = -F(y_{n+3}^{(s)}, \quad S = 0, 1, 2$$
(4.3)

Where

$$d_{n+3}^{(s+1)} = y_{n+3}^{(S+1)} - y_{n+3}^{(s)}$$

$$W_{n+3}^{(s)} = \left[I - \frac{h^2}{12}\frac{df}{\partial y}\left(y_{n+3}^{(s)}\right)\right]$$

$$F\left(y_{n+3}^{(S)}\right) - y_{n+3}^{(s)} - 3y_{n+2} + 3y_{n+1} - y_n - \frac{h^2}{12}\left[F(y_{n+3}^{(s)} + 9F(y_{n+2}) - 9f(y_{n+1}) - f(y_n)\right]$$

The algorithm is achieved by adoption of predictor corrector mode denoted by PEC meaning Predict, Evaluate and Correct.

The values may be sensitive to error in the past values; hence the current approximation requires correction.

Thus, the idea is to first calculate the estimates of y and its derivative at the points

$x_{n+j} = x_n + j h$, j = 0 (1) 3.

Using the predictor formula (2.7) to serve as points from where the iteration will begin.

There are two basic ways of implementing the predictor – corrector algorithm, namely, repeate stages two and three until a prescribed error condition is met. The ultimate solution is the approximation y_{n+1} to the exact solution $y(x_{n+1})$ of equation (1.1). The mode is called iteration to convergence ([14]; [15]; [5] and [1]). Its stability is essentially that of the corrector formula alone. The second option is to repeat stages two and three for fixed numbers m times to yield approximation $y_{n+3}^{(m)}$ where $2 \leq m \leq 5$. The entire process is implemented in $P(EC)^m$ mode.

In this paper, the first approach, that is, PEC mode is adopted because it is cheaper to realize . The mode is described as

P: y_{n+j}, j = 0(1)3, using formula (2.7)

E: $y''_{n+1} = y''(x_{n+j}, y_{n+j}), \quad j = 0\,(1)\,3$

$$C: y_{n+3} = 3y_{n+2} - 3y_{n+1} + y_n - \frac{h^2}{12}\left\{f\left(y^{(o)}_{n+3}\right) + 9f\left(y_{n+2}\right) - 9f\left(y_{n+1}\right) - f\left(y_n\right)\right\}$$

The error estimate is obtained from

$$\text{Error} = \frac{y^{(s+1)}_{y+3} - y^{(s)}_{n+3}}{y^{(s)}_{n+3} - y^{(s-1)}_{n+1}} \tag{4.4}$$

The iteration is terminated whenever

$$\text{Error} < \text{Tolerance} \tag{4.5}$$

5. Numerical Experiments

In order to assess the applicability and accuracy of the method, the formula was computerized in Fortran Programming Language and implemented on a microcomputer with some sample initial value problems adopting double precision arithmetic. Its performance was checked by comparing the approximate solution with the exact solution of the equations using two second-order initial value problems. Its accuracy is determined via the size of the discretization error estimates e_n obtained by subtracting the approximate solution from the corresponding exact solution of the problems.

The two sample problems and their computed results are as shown below.

Problem 1:

The first problem considered is the second-order problem

$$y'' - x(y')^2 = 0,\ y(o) = 1,\ y'(o) = \tfrac{1}{2}$$

whose exact solution is

$$y(x) = 1 + \tfrac{1}{2} l_n\left(\frac{2+k}{2-x}\right)$$

Problem 2:

The other second – order initial value problem solved is:

$$y'' + y = 0,\ y(o) = 0,\ y'(o) = 1$$

with exact solution

$$y(x) = \sin x$$

These problems are solved with fixed stepsize h = $\tfrac{1}{40}$. The results are as shown in tables 1 and 2.

Table 1: Numerical solution of Problems 1, with stepsize h = $^1\!/_{40}$.

N	X_n	$y(x_n)$	y_n	e_n
0	0.000000000	1.000000000	1.000000000	0.000000000
1	0.025000000	1.012500644	1.012500644	0.000000000
2	0.050000000	1.025005221	1.025005221	0.000000000
3	0.075000003	1.037517548	1.037513733	0.000003815
4	0.100000001	1.050041676	1.050026178	0.000015447
5	0.125000000	1.062581539	1.062542558	0.000038981
6	0.15000000	1.075141072	1.075062871	0.000078201
7	0.17500000	1.087724328	1.087587118	0.000137210
8	0.20000000	1.100335360	1.100115299	0.000220060
9	0.2250000	1.112978220	1.112647414	0.000330806
10	0.2500000	1.125657201	1.125183463	0.000873089
11	0.2750000	1.138376474	1.137723446	0.000653028
12	0.3000000	1.151140451	1.150267303	0.000873089
13	0.3250000	1.163953424	1.162815213	0.001138210
14	0.3500000	1.176820040	1.1752366998	0.001453042
15	0.375000060	1.189744830	1.187922716	0.001822114
16	0.400000000	1.202732563	1.200482368	0.002250195
17	0.42500000	1.215788126	1.213045955	0.002742171
18	0.45000000	1.228916645	1.225613475	0.003303170
19	0.47500000	1.242123008	1.238184929	0.003938079
20	0.50000000	1.255412817	1.250760317	0.00465250

Table 2: Numerical solution of Problem 2 with stepsize h = $\frac{1}{40}$

	X_n	$y(x_n)$	y_n	e_n
0	0.000000000	0.000000000	0.000000000	0.000000000
1	0.025000000	0.024997396	0.024997396	0.000000000
2	0.050000000	0.049979169	0.649979169	0.000000000
3	0.075000000	0.074929707	0.074945316	0.000015609
4	0.100000000	0.099833421	0.099895835	0.000062414
5	0.125000000	0.124674730	0.124830723	0.000155993
6	0.1500000000	0.149438143	0.149749979	0.000311837
7	0.175000000	0.174108148	0.174653605	0.000545457
8	0.200000000	0.198669344	0.199541599	0.000872254
9	0.225000000	0.223106384	0.224413961	0.001307577
10	0.250000000	0.247403994	0.249279692	0.001866698
11	0.275000000	0.271546960	0.274111807	0.001564847
12	0.300000000	0.295520246	0.298937321	0.003417075
13	0.325000000	0.319308817	0.323747218	0.004438400
14	0.350000000	0.342897662	0.348541498	0.005643636
15	0.375000000	0.366272599	0.373320162	0.007047564
16	0.400000000	0.389418393	0.398083210	0.008664817
17	0.425000000	0.412320852	0.422830641	0.010509789
18	0.450000000	0.434965611	0.442562456	0.012596846
19	0.475000000	0.457338517	0.472278655	0.014940143
20	0.50000000	0.479425579	0.496979237	0.017553657

6. Conclusion

In this paper, a three-step numerical formula is developed, analysed and used to solve two sample second-order initial value problems of ordinary differential equations. Theoretical analysis show that the method is consistent, zero-stable and convergent while numerical experiment shows that the scheme is quite accurate. The method will therefore be suitable for direct numerical solution of second-order differential equations arising in physics, chemistry, biology and economics.

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