Journal of the Nigerian Association of Mathematical Physics
Volume 16 (May, 2010), pp 253 – 264

On A Two-Stage Supply Chain Model In The Manufacturing Industry

[1]Bakare Emmanuel Afolabi, [2]Ogban Gabriel Iyam and [3]Chima R. Nwozo

[1]Department of Computer and Information Science,
Lead City University,Ibadan,Nigeria
[2]Department of Mathematics and Statistics
Cross River University of Technology, Calabar,Nigeria
[3]Department of MathematicsUniversity of Ibadan,
Ibadan, Nigeria
[1]Corresponding author: email emmabak2000@yahoo.com : Tel. +2348034867587

Abstract

We model a two-stage supply chain where the upstream stage (stage 2) always meet demand from the downstream stage (stage 1).Demand is stochastic hence shortages will occasionally occur at stage 2.Stage 2 must fill these shortages by expediting using overtime production and/or backordering. We derive optimal inventory control policies under Decentralized, Coordinated and Centralized control. The centralized control model is applied to a manufacturing industry(Associated Match Industry,Ibadan).

1.0 Introduction

In this research work, we study a two-stage supply chain where the upstream facility always meets demand from the downstream facility. We assume that demand is stochastic, and hence, shortages will occasionally occur at the upstream facility. In order to fill these shortages, the upstream facility must employ either overtime production or backordering. We study this supply chain under the centralized control where a single controller makes all the decisions for both facilities.

In traditional supply chain situations, downstream facilities make decisions about their order quantities without regard to the actual inventory available upstream. If the Upstream facilities do not have enough inventory on hand to fill the orders, it is often assumed that the downstream facility will take what it can get and backorder the rest. In order to avoid these shortages, the upstream facilities have traditionally set their inventory level high enough so that the likelihood of not meeting downstream demand is low. However, the shift towards lean inventory has caused a reduction in inventories, possibly increasing the likelihood of these shortages. Moreover, the cost of premium freight (already high, though hard to estimate) is certainly not decreasing in today's competitive markets. Therefore, many facilities use various forms of expediting to meet supply requests when shortages occur.

We have modeled our problem after the actual inventory control problems faced by a large match industry in Ibadan, which we refer to as "Associated Match Industry". Associated Match Industry produces matches. The associated Match Industry has three branches (NMC, Safa and Chambon) which work together to produce matches. Backordering is considered an option in both facilities.

In our model, we have attempted to capture the essence of the interaction between NMC and Safa. We feel that our model may apply to the Computer and Electronics Industries as well, where many manufacturers have reduced or even eliminated their requirements for warehousing and receive parts in just-in-time fashion.

Finally we feel that our results yield new insight into a common assumption in the inventory literature. In most single location inventory models, it is assumed that supply requests upstream are always met without

considering how and at what cost. Our results show that supply requests can always be met upstream using some form of expediting, but that may be much less expensive for the system if the downstream facility is sensitive to the inventory situation upstream and adjusts supply requests accordingly.

We have determined the optimal policies for the supply chain under centralized control when both options for filling shortages are available. However, for clarity of exposition, we will consider only one of the two mathematically equivalent options, over time production, throughout most of this paper. The results are the same if we consider only backordering. We include the slightly more complicated optimal policies and proofs.

We consider the situation where the two stages are actually part of a single firm and are controlled by a single manager. That manager has complete information from both stages, makes all the decisions and attempts to minimize the total system costs. This manager knows the inventory levels at both stages and makes the different decisions concerning production at stage 1, regular production at stage 2 and overtime production at stage 2.We develop the relaxed version of our problem. We also derive the optimal inventory control policy for stage 1 under the relaxed conditions. We show that stage 1 will occasionally "under-order" to avoid the expedition at stage 2 but will force expediting at stage 2 if the system inventory is very low. We show that the optimal inventory control policy for the entire system is base-stock under the relaxed conditions. We prove that the optimal relaxed policies are actually the optimal policies for the original, fully constrained problem and we list the optimal policies for both stages and the system. We include backordering as an expediting option. Finally, we conclude and discuss managerial insights for the centralized model.

The literature on supply chain with expediting or set-up costs upstream is limited. [3] consider supply chains with stochastic demand, constant lead times, and with set-up costs under centralized control. [7] consider an m + 1 stage supply chain where decisions must be made at each echelon about how many products to ship by regular means (which takes one period) and how many to expedite (which occurs simultaneously). Each expedited unit incurs a per unit cost higher than the per unit cost of regular shipping. They show that a "top-down" base-stock" policy is optimal where the upstream managers ignore downstream decisions.[9] consider a two-stage system with one warehouse and several retailers. They assume modified one-for-one (s-1, S) policies for both regular and expedited orders and develop a procedure to find optimal policy parameters.

Although our paper is related to those just mentioned, our paper is related to papers where outsourcing is used to meet shortages, but our results still differ. One reason is that we include a fixed cost when shortages occur, whereas most outsourcing assumptions only include a per unit cost. For example, in [4] the cost of outsourcing is the same as it is from within the supply chain. In [8], the author transform their outsourcing units into backorder penalties, effectively allowing negative inventory levels upstream, even though supply requests are always met.

Finally, in our proof of the centralized model, we use the results discussed by the following authors: [10], [12], [2] and [11], [5] and [6]. In Zhang's paper, he shows that (s,S) policies are optimal, given that the expected one period cost function in quasi convex, which we show using a result from [10]. We use this result to prove the optimal policy for the system inventory. The text by Bertsekas has several useful propositions; specifically, one proposition states that if the optimal function satisfies Bellman's equation under assumptions we show to be true in our model. Finally we assume that our demand distribution is log concave.

The rest of this paper is organised as follows:

In section 2, we define our model, develop cost functions, substitute system variables for supplier variables and relax two constraints. Under these relaxed conditions, we determine the optimal inventory control policy for stage 1, we show that stage 2 occasionally "under-order" to avoid expediting at stage 2 but will forced expediting at stage 2 if the system inventory is very low in section 3. In section 4 again under the relaxed conditions, we show that the optimal inventory control policy for the entire system is base-stock. We prove that the optimal relaxed policies are actually the optimal policies for the original, fully constrained problem and we list the optimal policies for both stages and the system. We include Backordering as an expediting option. Finally in section5, we conduct a numerical computation and discuss Management insights. Section 6 concludes the paper.

1.1 Basic Concepts And Definitions

A supply network or chain for a product (services or total solution to consumer needs) consists of several companies that are involved in the manufacturing and delivery of the production from raw material to its end consumer.

Holding Cost/Carrying Cost: This is the cost incurred in the process of keeping an item of inventory in the store.

Inventory: An inventory is any resource that has value or satisfaction of future needs e.g. raw materials, finished goods, loosed tools and consumables etc. It is a detailed list of all items in stock.

Inventory Control: An inventory control is concerned with decisions such as when to order or produce, how much to order or produce, what types of control system should be in place to minimize total inventory cost.

Set-up Cost(Ordering Cost): This is the cost incurred in making stock available in the present location. If the goods are purchased from outside suppliers, those costs are called Ordering Costs. If the items are produced in the company, they are referred to as Set-up Costs i.e. the costs are incurred from the time the request is made for the items from the stores to the time the request are returned into the stores.

Policy: A policy is a rule that determines a decision given.

Backordering Cost: This is incurred when a customer's demand cannot be fulfilled because inventory is completely depleted. For a firm incurring a temporary shortage in supplying its supplying its customer's goodwill due to delay.

(s,S) Policy: This is a 2-parameter decision rule where s=inventory level at which an order is placed and S= inventory level to which to order.

1.2 DEFINITION OF VARIABLES

$\mathbf{D_t}$ = the exogenous demand during period t

$\mathbf{x_{1,t}}$ = the stage 1 inventory position at the start of period t

$\mathbf{z_{1,t}}$ = the stage 1 production quantity during period t

$\mathbf{y_{1,t}}$ = the stage 1 inventory level after production during period t

$\mathbf{x_{2,t}}$ = the stage 2 inventory position at the start of period t.

$\mathbf{z_{2,t}}$ = the stage 2 regular production quantity during period t

$\mathbf{y_{2,t}}$ = the stage 2 inventory level after regular production during period t

$\mathbf{x_{2,t}}$ = the stage 2 inventory position at the start of overtime after receiving demand from stage 1 during period t

$\mathbf{z_{2,t}}$ = the stage 2 overtime production quantity during period t

$\mathbf{y_{2,t}}$ = the stage 2 inventory level after overtime production during period t

During the production processes at each stage, various costs are incurred. At stage 1, linear costs are assessed for production (c_1), holding (h_1) and backordering (b_1). At stage 2, linear costs are assessed for production (C_2), holding (h_2), and backordering (b_2). Overtime production incurs linear (C_0) plus fixed (K_0) costs. These costs are assumed to be discounted every period by a factor of α with $0< \alpha < 1$. Throughout this paper, we utilize the following notation:

$$X^+ = \begin{cases} x \text{ if } x > 0 \\ 0 \text{ otherwise} \end{cases} \qquad X^- = \begin{cases} x \text{ if } x < 0 \\ 0 \text{ otherwise} \end{cases} \qquad \delta(x) = \begin{cases} x \text{ if } x > 0 \\ 0 \text{ otherwise} \end{cases}$$

1.3 MODEL ASSUMPTIONS:

(A_1) $0 < \alpha < 1$

(A_2) Demand is discrete, stationary, non-negative and log-concave
(A_3) For all t, $0 < u < \infty$
(A_4) $c_2 < c_0$
(A_5) $b_1 > (1- \alpha) C_1$

(A_6) $b_1 > (1- \alpha) C_1 + (C_0/\alpha - C_2)$
(A_7) $h_2 < \alpha(h_1 +(1- \alpha) C_1)$

A function F (w) is said to be log concave in x if log (F(x) is concave in X i.e. the assumption means that we require our demand distribution to have a smooth shape with at most one mode). For the discrete distribution that we counter, we use two nice properties of log concave functions: the fraction $\frac{F(x+1)-F(x)}{F(x)}$ is non-increasing in x and the convolution of a quasi convex function with a log concave demand distribution remains quasi convex.

2.0 THE SUPPLY CHAIN MODELS

We model a two-stage supply chain where the upstream stage (stage 2) always meet demand from the downstream stage (stage 1).Demand is stochastic hence shortages will occasionally occur at stage 2.Stage 2 must fill these shortages by expediting using overtime production and/or backordering.We derive optimal inventory control policies under Decentralized, Coordinated and Centralized control.The centralized control model is applied to a manufacturing industry (Associated Match Industry, Ibadan).

2.1 DECENTRALIZED CONTROL

Under decentralized control, stage 1 ignores stage 2 and follows a simple base-sock policy. Stage 2 also follows a simple base-stock policy if there is no set-up cost for regular production. When we include this set-up cost at stage 2 two decisions must be made: how much to produce during regular production and how much to produce during overtime production. We show that the optimal regular production policy is an (s, S) policy and that the optimal overtime production policy depends on the cost parameters.

2.2 COORDINATED CONTROL

To coordinate the two stages, we develop two contracts. Both contracts depend on a two-tiered wholesale cost and a linear transfer payment. Contract A achieves system optimality but requires the two stages to share cost information. Without sharing cost information contract B achieves near-optimality for the system (optimality for the average cost case).Under both contracts an appropriate transfer payment may be negotiated that benefits both stages.

2.3 CENTRALIZED CONTROL

Under centralized control which is where we focused, the two stages work together to minimize system costs. By substituting system variables for stage 2 variables and relaxing some constraints. We show that the optimal policy at stage 2 has two order-up-to levels and depends on the available system inventory. We also show that the optimal policy for stage 2 is to ensure the system base-stock is achieved.

2.4 OPTIMAL POLICIES

We are interested in finding the optimal policy π out of all possible admissible policies Π and hence the optimal total discounted cost over the infinite horizon $f(x_0)$.This function will exist according to proposition 1.1 of page 3 of Bertsekas if g (period "k" variables) ≥ 0. We seek to solve the optimal cost function $f^* (x_0) = \min f_{\Pi} (x_0)$ where $\pi E \ \Pi$ in order to determine the optimal inventory control policies.

2.5 THE RELAXED PROBLEM

In this problem, the centralized manager makes three decisions at once about production at the two stages. So for the one period experienced by the entire system are

$g_{cen}(x_1,y_1,x_2,y_2,D) \equiv K_0\delta((y_1-x_1)-x_2) + c_0((y_1-x_1)-x_2)^+ + h_2(x_2-(y_1$

$x_1))^+ + \alpha(c_1(y_1-x_1) + c_2(y_2-(x_2-(y_1-x_1))^+) + h_1(y_1-D)^+ + b_1(y_1-D)^-)$

With $y_{1\leq}x_1$ and $y_2 \geq (x_2-(y_1-x_1))^+$.

The first two terms are overtime production costs, the third term is the holding and backordering costs at stage 2. The further term is the production cost at stage 1, the fifth term is the production cost at stage 2, and the last two terms are holding and backordering costs at stage 1. Note that we assume there is a fixed cost for regular production at either stage for the remainder of the paper. Clearly $g_{cen}(.) \geq 0$ and hence the Optimal cost function $f_{cen}(x_1,x_2)$ satisfies

$$f^*_{cen}(x_1,x_2) = \min_{\substack{y_1 \geq x_1 \\ y_2 \geq (x_2-(y_1-x_1))^+}} E_D(g_{cen}(x_1,y_1,x_2,y_2,D) + \alpha f^*_{cen}((y_1-D), y_2))$$

The argument that minimizes this equation is the optimal inventory control policy which we seek. Moving the $-\alpha c_1x_1$ back to the preview period as $-\alpha c_2(y_1-D)$ and rearranging terms. We get:

$$g_{cen,m}(x_1,y_1,x_2,y_2,D) \equiv \alpha(1-\alpha)c_1y_1+\alpha_2c_1D+k_0\delta((y_1-x_1)-x_2) + c_0((y_1-x_1)-x_2)^+ + (h_2-\alpha c_2)(x_2-(y_1-x_1))^+ + \alpha(c_2y_2+\alpha(h_1(y_1-D)^+ + b_1(y_1-D)^-) \quad (2.1)$$

Under the same restriction. Note that $f^*cen(x_1,x_2) = -\alpha c_1x_2 + f^*cen,m(x_1,x_2)$. We originally tried to solve this problem in terms of stage 1 and stage 2 variables but found that the solution lent itself more easily to stage 1 and system variables. Define the system inventory position as $x_s=x_1+x_2$ and the system inventory information (x_1,x_2) and (x_1,x_s). We substitute these variables and rewrite g_{cen}, m(.) as

$$g_{cen},m(x_1,x_s,y_1,y_s,D) = \alpha(1-\alpha)c_1y_1 + \alpha^2c_1D + \alpha(h_1(y_1-D) + b_1(y_1-D)^-)^+ + k_0\delta(y_1-x_s) + (c_0-\alpha c_2)(y_1-x_s)^+ + h_2(x_s-y_1)^+ + \alpha c_1(y_1-x_s) \quad (2.2)$$

with $y_1 \geq x_1$ and $y_s \geq y_1+(x_s-y_1)+$. Note that the seconds restriction is equivalent to $y_s \geq \max(y_1,x_s)$.Also, we can rewrite $g_{cen,\ m}(.)$ as

$$g_{cen},m(x_1,y_1,x_s,y_s,D) = L_1(y_1,D) + L_2(y_1,x_s) + \alpha c_2(y_1-x_s)$$

Where $L_1(y_1, D)$ represents the terms on equation (2.1) and $L_2(y_1,x_s)$ represents the terms on equation (2.2).

We can now rewrite the fully constrained optimal cost function which we would like to solve namely

$$f_{cen,m}(x_1,x_s) = \min_{\substack{y_1 \geq x_1 \\ y_s \geq \max(x_s,y_1)}} E_D[g_{cen,m}(x_1,y_1,x_s,y_s,D) + f^*_{cen},m(y_1-D)(y_s-D)]$$

To solve this equation, we relax some of the constraints; later we show that these constraints are always met by the optimal solution to the relaxed problem and thus solve the original fully constrained problem.

First we drop the constraints that $y_1>x_s$.For later reference we label the relaxed assumptions:

(R_1) $y_1 \geq x_1$ and

(R_2) $y_s \geq y_1$, when $y_1>x_s$

After relaxing the constraints, our relaxed cost per period has the same costs as $g_{cen,\ m}(.)$ but with only one constraint;

$$g_{cen,s}(y_1,x_s,y_s,D) \equiv L_1(y_1,D) + L_2(y_1,x_s) + \alpha c_2(y_s-x_s)$$

With $y_s \geq x_s$.

THEOREM 2.1 $g_{cen,\ r}(y_1,x_s,y_s,D) \geq 0$

Proof: Every term of $g_{cen,\ r}(y_1,x_s,y_s,D)$ is non-negative except possibly the first term if $y_1 < 0$

$g_{cen,\ r}(y_1,x_s,y_s,D) \geq \alpha(1-\alpha)c_1,y_1 - \alpha b_1y_1 + \alpha b_1E_D(D)$

$$= \alpha ((1-\alpha)c_1-b_1)y_1+\alpha b_1\mu \geq 0$$

Because $g_{cen,r}(y_1,x_s,y_s,D)\geq 0$, proposition 1.1 of Bertsekas holds and the relaxed optimal cost function f^*_{cen},r satisfies

$$f^*_{cen,r}(x_s) = \min_{y_1,ys\geq xs} E_D(g_{cen,r}(y_1,x_s,y_s,D) + \alpha f^*_{cen,r}(y_s-D) \quad 2.3$$

$$=\min_{y_1,y_s\geq x_s} (E_D (L_1 (y_1, D)) + L_1 (y_1,x_s) + \alpha c_2(y_s-x_s) + \alpha E_D (f^*_{cen},r(y_s-D)$$

Here it is important to notice that under the relaxed condition y1 has o effect on either y_s or the costs to go, $\alpha E_D (f^*_{cen,r}(y_s-D))$. Thus

$$f^*_{cen,r} (x_s) = \min_{y_s\geq x_s}(E_D(L_1(y_1, D) + L_2(y_1,x_s)) + \alpha c_2(y_s-x_s) + \alpha E_D(f^*_{cen,r} (y_s-D))$$

$$=\min (m (x_s) + \alpha c_2(y_s-x_s) + \alpha E_D(f^*_{cen,r}(y_s-D))) \quad 2.4$$

Where $m (x_s) \equiv \min \{E_D(L_1(y_1, D)) + L_2(y_1,x_s)\}$. Finding the optimal inventory policy for stage 1 has become a myopic problem which we solve in the next section. For later reference, we also define the related function $m' (x_s) \equiv m(x_s) - \alpha 2c_1\mu$ where $m'(x_s)$ represents all of the system costs that are not due to production. For convenience, we study $m(x_s)$ below but test our final result in terms of $m'(x_s)$.

3.0 STAGE 1 RELAXED OPTIMAL POLICIES:

In this section, we determine the optimal inventory policy for stage 1 for the relaxed problem. We study the function $m (x_s)$ and show that the stage 1 policy depends only on the system inventory level x_s. Define $N_H(y_1)$ and $N_L(y_1)$ as:

$$N_H(y_1)=(\alpha(1-\alpha)c_1-h_2)y_1+\alpha E_D (\alpha c_1D+h_1(y_1-D)^++b_1(y_1-D)^-)$$

And

$$N_L(y_1)\equiv(\alpha(1-\alpha)c_1-c_0-\alpha c_2) y+\alpha E_D(\alpha c_1D+h_1(y_1-D)^++b_1(y_1-D)^-)$$

The function $N_H(y_1)$ corresponds to the stage 1 costs when $y_1\leq x_s$ and the function $N_H(y_1)$ corresponds to the stage 1 costs when $y_1>x_s$. We now have that

$$m(x_s) = \min_{y_1} (E_D (L_1(y_1D)) + L_2(y_{1,}x_s)$$

$$= \min_{y_1} \begin{cases} E_D (L_1(y_{1,}D))+ h_2 (x_s-y_1) \\ E_D (L_1(y_1, D) +k_0+(c_o-\alpha c_2)(y_1-x_s) \end{cases}$$

$$= \min_{y_1} \begin{cases} h_2x_s + \min_{y_1\leq x_s}(N_H(y_1) \\ K_0-(c_0-\alpha c_2)x_s + \min_{y_1>x_s} \{N_L(y_L)\} \end{cases}$$

Before continuing our study of $m (x_s)$, we derive properties for $N_1(y_1))$

1. $N_L(y_1)$ and $N_H(y_1)$ are convex in y_1
2. $0\leq y_L\leq y_H\leq \infty$

Returning to our study of $m (x_s)$ and defining $N(x_s) \equiv E_D(L_1(x_s,D))$, we have that

$$m(x_s) = \min \begin{cases} h_2x_s+N_H(y_H) & \text{if } x_s\geq y_H \\ N(x_s) & \text{if } t_L \leq x_s < y_H \\ K_0-(c_0-\alpha c_2)x_s+N_L(y_L) & \text{if } x_s<t_L \end{cases}$$

Define the as the smallest w such that $N(w) \leq K_0 – (C_0-\alpha c_2) w+N_1 (y_1$. We get that

$$m(x_s) = \begin{cases} y_H & \text{if } x_s\geq yH \end{cases}$$

x_s if $t_L \le x_s < y_H$

y_L if $x_s<t_L$

Proof: By the definition of $m(x_s)$

4.0 SYSTEM RELAXED OPTIMAL POLICY

In this section, we determine the relaxed optimal policy for the system. Given m (x_s), we now have the optimal relaxed cost functions in terms of system variables only, we have

$f^*_{cen},r(x_s) = \min(m(x_s) + \alpha c_2(y_s-x_s) + \alpha E_D(f^*_{cen,r}(y_s-D))$

We move the m (x_s) and $-\alpha c_2 x_s$ terms back to the previous period as $\alpha m(y_s\text{-}D)$ and $\alpha_2 c_2$ (y_s-D) respectively and get

$f^*_{cen,s}(x_s) = \min \;\; \alpha(1-\alpha)\, c_2 y_s + \alpha E_D(m(y_s-D)$

$$= \min_{y_s \ge x_s} \left\{ G_{cen,\,s}(y_s) + \alpha_2 c_2 \mu + \alpha E_D(f^*_{cen,\,s}(y_s-D)) \right\}$$

Where $G_{cen,\,s}(y_s) = \alpha(1-\alpha)c_2 y_s + \alpha E_D(m(y_s-D))$

We need to justify two steps next. First we can move the two terms back a period and $F^*_{cen,s}(.)$ will have the same optimal policy as f^*_{cen} (.)

We have that

$f^*_{cen,r}(x_s) = m(x_s) - \alpha c_2 x_s + f^*_{cen,s}(x_s)$

THEOREM 4.1. For the relaxed problem, the optimal inventory control policy for the system inventory is a base-stock policy.

Proof: Consider $g_{cen,s}(y_s)$:

$G_{cen},s(ys) = \alpha((1-\alpha)\, c_2 y_s + m(y_s-D)$

$= \alpha((1-\alpha)\, (c_2 y_s - c_2 D + c_2 D) + m(y_s-D))$

$= \alpha((1-\alpha)\, c_2 D + g^+(y_s-D)) \ge 0$

Where the inequality holds because $g+(.)\ge 0$. Note from Lemma 3 that g+(.)is a quasiconvex function with a minimum point. Now consider $G_{cen,s}(y_s)$:

$G_{cen,s}(y_s) = \alpha E_D\, ((1-\alpha)c_2 y_s + m(y_s-D))$

$= \alpha((1-\alpha)c_2 E_D(D) + E_D(g+(y_s-D)))$

The first term is a constant, and the second term is a convolution of a quasiconvex function ($g+(y_s$-D)) and a log concave probability distribution by Assumption(A2). Thus Gcen,s(.) is a quasiconvex function according to [10] and [1].

4.1 FULLY CONSTRAINED OPTIMAL POLICIES

In this section, we show that the optimal inventory control policy that solves the relaxed problem also solves the original fully constrained problem. From section 2 we have that

$f^*_{cen}(x_1,x_2) = -\alpha c_1 x_1 + f^*_{cen,m}(x_1, x_s)$

From the previous section, we have that

$f^*_{cen,r}(x_s) = m(x_s) - \alpha c_2 x_s + f^*_{cen,s}(x_s)$

The missing piece of the puzzle is to show that

$f^*_{cen,m}(x_1,x_s) = f^*_{cen,r}(x_s)$

By doing so, we verify that the optimal policies listed in section 3 and 4 are truly optimal. We must show that the optimal policies for $f^*_{cen,r}(x_s)$ minimize $f^*_{cen,m}(x_1,x_s)$ and that both (R1) and (R2) are met.

Theorem 4.2:

$f^*_{cen,m}(x_1,x_s) = f^*_{cen,r}(x_s)$

Proof: The optimal policies for $f^*_{cen,r}(x_s)$ minimize the costs for $f^*_{cen,m}(x_1,x_s)$ because both relaxed constraints are met and y_1 does not affect y_s or the costs to go.

For the original, fully constrained problem, we now know the optimal policies for stage 1, stage 2, and for the system.

$$y^*_{cen,1} = \begin{cases} y_H & \text{if } x_s \geq y_H \\ x_s & \text{if } t_L \leq x_s < y_H \\ y_L & \text{if } x_s < t_L \end{cases}$$

$$y^*_{cen,s} = \begin{cases} x_s & \text{if } x_s > S^*_{cen} \\ S^*_{cen} & \text{if } x_s \leq S^*_{cen} \end{cases} \text{ and}$$

$$y^*_{cen,2} = y^*_{cen,s} - y^*_{cen,1}$$

Now, under the assumption that the initial system inventory is less than the system base-stock level, $x_s \leq S^*_{cen}$, we can further calculate $f^*_{cen}(x_1,x_s)$

$f^*_{cen,1}(x_1,x_s)$

$= -\alpha c_1x_1 + f^*_{cen,m}(x_1,x_s)$

$= -\alpha c_1x_1 + m(x_s) - \alpha c_2x_s + \min(\alpha(1-\alpha)c_2y_s + \alpha E_D(m(y_s-D)) + \alpha^2c_2\mu + \alpha E_D(f^*_{cen,s}(y_s-D)))$

$= -\alpha c_1x_1 + m(x_s) - \alpha c_2x_s + \alpha(1-\alpha)c_2s^*_{cen} + \alpha E_D(m(S^*_{cen}-D)) + \alpha^2c_2\mu + \alpha E_D(f^*_{cen,s}(S^*_{cen}-D))$

$= -\alpha c_1x_1 + m(x_s) - \alpha c_2x_s + \alpha((1-\alpha)c_2S^*_{cen} + E_D(m(S^*_{cen}-D)) + \alpha c_2\mu)(1+\alpha)$

$$= -\alpha c_1x_1 + m(x_1) - \alpha c_2x_s + \frac{\alpha}{1-\alpha}((1-\alpha)c_2S^*_{cen} + E_D(m(S^*_{cen}-D)) + \alpha c_2m)$$

$$= -\alpha c_1x_1 + m'(x_s) + \alpha c_2(S^*_{cen}-x_s) + \frac{\alpha}{1-\alpha}(E_D(m(S^*_{cen}-D)) + \alpha(c_1+c_2)\mu$$

4.3 OPTIMAL POLICIES WITH BACKORDERING

In this section, we study the problem where both overtime production and Backordering are viable options. We follow the analysis from the previous five sections but add additional comments and proofs when necessary. In this problem, the overtime production decision is a real decision. If a shortage exists, the manager must choose how much of the shortage to fill with overtime production and how much to fill with backordering. We will later show that it is optimal to pick one of the other expediting options, but never both. We modify assumption (A6)

(A9) $b_1 \geq (1-\alpha c_1) + b_2$

(A10) $k_0 \geq \alpha b_c$

(A11) $c_0 \leq \alpha(b_2+c_2)$

Using the same notation as before, we now develop the various cost equations:

$g_{cen}(x_1,y_1,x_2,y_2,z_2,D) \equiv k_0\delta(z_2 + h_2(x_2-(y_1-x_1)+z_2) +$
$+\alpha b_c\delta((x_2-(y_1-x_1)+z_2)) + \alpha b_2(x_2-(y_1-x_1)+z_{2+}\alpha c_1(y_1 - x_s)$
$+\alpha c_2(y_2-(x_2-(y_1-x_1)+z_2)) + \alpha(h_1(y_1-D)^+ + b_1(y_1-D)^-$

Under the same restriction, substituting system variables, we get

$G_{cen,m}(x_1,y_1,xs,ys,z,D) \equiv$

$\alpha(1-\alpha)c_1y_1 + \alpha(\alpha c_1D + h_1(y_1-D) + +b_1(y_1-D)-)$ (4.1)

$+k_0\delta(z_2) + (c_0-\alpha c_2)z_2 + h_2(x_s-y_1+z_2) +$ (4.2)

$+b_c\delta((xs-y_1+z_2)-)+\alpha cp(xs-y_1+z_2)$ (4.3)
$+\alpha c_2\,(y_s-x_s)$

with $y_1 \geq x_1, z_2 \geq 0$, and $ys \geq y_1+(x_s-y_1,+z_2)+ = \max\{y_1,x_s+z_2\}$. We can rewrite $g_{cen,m}(.)$ as
$g_{cen,m}(x_1,y_1,x_s,y_s,z_2,D) = L_1\,(y_1,D)+L_2\,(y_2,xs,z_2) + \alpha c_2\,(y_s-x_s)$

Where $L_1\,(y_1, D)$ represents the terms on the equation (4.1) and $L_2\,(y_2,x_s,z_2)$ represents the terms on equations(4.2) and (4.3)

Again, we relax some constraints. First we drop the constraint that $y_1 \geq x_1$. Second, we relax the constraint on the system inventory position so that only $y_s \geq x_s$. For later reference, we label the relaxed assumptions as:

(R1) $y_1 \geq x_1$ [R1=Relaxed assumption 1] and [R2= Relaxed assumption 2]

(R2) $y_s \geq \max\,(y_1,x_s+z_s, D) \equiv L_1\,(y_1,D) + \alpha\, c_2(y_s-x_s)$

With $z_2 \geq 0$ and $y_s \geq x_s$. The function $g_{cen,r}(.)$ can be shown to be non-negative by analysis similar to Lemma and we can thus use the same result from Bertsekas for the optimal cost function $f^*_{cen,r}$

$f^*_{cen,r}(x_s)$

$$=\min \quad E_D(g_{cen},r(y_1,x_s,y_s,z_2,D)+ \alpha\, f^*_{cen},r(y_s-D)) \qquad (4.4)$$

$$\underset{y_s \geq x_s, z_2 \geq 0, y_1}{}\; = \min_{y_s > x_s, z_2 \geq 0, y_1} \left\{ \begin{array}{l} E_D(L_1,(y_1,D))+L_2(y_1,x_s,z_2) \\ + \alpha c_2(y_s-x_s)+ \alpha E_D(f^*_{cen},r(y_s-D)) \end{array} \right\}$$

Again, it is important to notice that under the relaxed conditions y_1 and z_2, have no effect on either y_s or the cost to go, $\alpha E_D\,(f^*_{cen,r}(y_s-D)$ Thus,

$$f^*_{cen},r(x_s) = \min \left\{ \begin{array}{l} (E_D L_1(y_1,D))+ L_2(y_1,x_{1,}z_2)\} \\ \quad z_2>0_1 y_1 \\ +\alpha c_2(y_s-x_s) +\alpha E_D(f^*_{cen,r}(y_s-D)) \end{array} \right\}$$

Where $m\,(x_s)=\text{Min}\,\{E_D L_1(y_1, D)\} + L_2(y_1,x_s,z_2)\}$. Finding the optimal inventory policy for stage 1 has become a myopic problem that now depends on y_1 and z_2. Now consider $m(x_s)$ under two cases. When stage 1 does not order more than the system inventory on hand ($y_1 \leq x_s$) and when stage 1 does order more than the system inventory on hand ($y_1 > x_s$). In the first case, we get that

$$L_2(y_1,x_s,z_2) = k_0\delta(z_2) + (c_0-\alpha c_2)\, z_2+h_2\,(x_s-y_1+z_2)$$
$$= k_0\delta(z_2) + (h_2+c_0-\alpha c_2)z_2+h_2\,(x_s-y_1)$$

Which is minimized when $z_2=0$ and thus $L_2\,(y_1,x_s,z_2)=h_2\,(x_s-y)+\alpha c_2 x_s$ when $y_1 \leq x_s$.

In other words, if there is not a shortage don't use overtime production.

In the second case, there are options.

$L_2\,(y_1,x_s,z_2)=$

$\alpha b_c+\alpha b_2\,(y_1-x_s)$	If $z_2=0$
$k_0+\alpha b_c + (c_0-\alpha(b_2+c_2))z_2+\alpha b_2\,(y_1-x_s)$	If $0<z_2<y_1-x_s$
$k_0+ (c_0-\alpha c_2)(y_1-x_s)$	If $z_2=y_1-x_s$
$k_0+ (h_2+c_0-\alpha c_2)z_2+h_2\,(x_s-y_1)$	If $z_2>y_1-x_s$

Before returning to $m\,(x_s)$, define $Nm(y_1)$ as

$N_m(y_1)= \alpha\{(1-\alpha)c_1+b_2\}\, y_1+\alpha E_D(\alpha c_1 D+h_1\,(y_1-D)^+ +b_1\,(y_1-D)^-)$

Define $y_m=\text{argmin}\,\{N_m(y_1)$ is convex and we have $0 \leq y_m \leq y_L \leq y_H < \infty$

[$0 \leq y_m$ by assumption (A6) $y_m \leq y_L$ by assumption (A10), $y_L \leq y_H$ by algebra and $y_H < \infty$ as before]

We have

$$m(xs) = \min_{Z_2 \geq 0, y_1} \{E_D(L_1\,(y_1,D))+ L_1\,(y_1,x_s,z_2)$$

$$(4.5) \quad = \begin{cases} h_2x_s+N_H(y_H) & \text{If } x_s \geq y_H \\ N(x_s) & \text{If } t_M \leq x_s < y_H \\ \alpha b_c-\alpha b_2x_s+N_M(Y_M) & \text{If } t_L \leq x_s < t_M \\ K_0-(c_0-\alpha c_2)x_s+N_L(y_L) & \text{If } x_s < t_L \end{cases}$$

Where t_M is defined as the smallest w such that $N(w) \leq \alpha b_c-\alpha b_2w+N_m(Y_m)$ and t_L is defined as

$$t_L = \frac{K_0+N_L(y_L)- \alpha b_c - N_M(y_M)}{c_0-\alpha(c_b + c_2)}$$

We have now defined $m(x_s)$ explicitly and again determined the relaxed optimal Inventory Control Policy at stage 1.

$$(4.6) \quad y^*_{cen,1} = \begin{cases} y_H & \text{If } x_s \geq y_H \\ x_s & \text{If } t_M \leq x_s < y_H \\ y_m & \text{If } t_1 \leq x_s < tm \\ y_L & \text{If } x_s < t_L \end{cases}$$

From equation (4.5) we have that

$$f^*_{cen}(x_s)=\min_{y_s \geq x_s}\{m(x_s)+ \alpha c_2(y_s-x_s) +\alpha E_D \{F^*_{cen,s}(y_s-D)\}\}$$

As earlier we move the $m(x_s)$ and $-\alpha c_2x_s$ terms back to get

$$f^*_{cen},s(x_s) = \min \{G_{cen},s(y_s) + \alpha c_2\mu+\alpha E_D(f^*_{cen,r}(y_s -D))\}$$

Where $G_{cen,s}(y_s) = \alpha(1-\alpha) c_2y_s+E_D(m(y_s-D))\}$.

5.0 Application

We consider a problem that has a Poisson demand with mean 20 in a match Industry (AMI) Ibadan, where stage 1 which is the downstream stage is called NMC and Stage 2 which is the Upstream stage is referred to as SAFA. The per unit cost at Stage 1 are: G=10, h_1=4, b_1=20

The per unit cost at Stage 2 are

$C_2=4$	$b_2=10$	$y_1=20$
$H_2=1$	$bc=100$	$y_2=1000$
$C_0=5$	$x_1=600$	
$K_0=200$	$x_2=1000$	
$\alpha=0.9$	$Nm(y_1)=15$	

Note that all the values are in ten thousands.

We first calculate the Optimal Inventory Control parameters for stage 2: t_1,y_1 and y_H

$Y_H=52.5, Y_L=88.5, Y_M=202.5$

$N_M(Y_M) = 6108.75$

$N_L(Y_L)=1845.15$

$t_L=546$

$S^*_{1,dec}=38.4$

$S^*_{2,dec}=45$

$f^*_{1,dec} =660$, $f^*_{2,dec}=440$, $S^*_{cen}=30$

The percentage Inventory reduction which is determined by

$= S^*_{1,dec} +S^*_{2,dec}-S^*_{cen}$

$$S^*_{1,dec}+S^*_{2,dec}$$

$$= \frac{38.45+45-30 \times 100}{38.45+45}$$

$$= \frac{53.4 \times 100}{83.4} = 64\% \text{ reduction}$$

this is one factor that contributes to cost savings. Another factor that contributes to cost savings is how often Stage 2 is forced to run overtime production.

6.0 CONCLUSION AND INSIGHTS

We have studied the two-stage supply chain under centralized Control. We have shown that the optimal Inventory Control policies for both stages depend only on the system inventory x_s, and that the optimal policy for the system inventory is a base-stock policy. In the first sections, we assumed that overtime production was the only method of expediting. We later finally described the numerical application of the problem.

The main managerial insight gained from this paper is that to cut costs in this kind of supply chain, Stage 1 must be sensitive to the amount of Inventory available at Stage 2. Stage 1 must be willing to occasionally under-order to save significant overtime production costs (or backordering costs) at Stage 2. By the same token, Stage 2 must be willing to produce extra units when Stage 1 under-orders trusting that stage 1 will want those additional units the next period.

References

1. An M.Y. 1995 "Log concave Probability Distribution: Theory and Statistical Testing" Department of Economics, Duke University, Durham

2. Bertsekas D.1995 Dynamic Programming and Optimal Control (Volume 2). Athena Scientific, Behnont. Massachusetts.

3. Chen F. and Y.Zheng 1994 "Lower Bounds for Multi-Echelon Stochastic Inventory Systems" Management Science 40, 1426-1443.

4. Gavirneni S., Kapuscinski and S.Tayur 1999 "Value of information in Capacitated Supply Chains"Management Science 45,16-24.

5. Huggins, E and T. Oslen 2001. "Inventory Control with Overtime and Premium Freight" Submitted to Operation Research

6. Huggins E.and T. Oslen 2002 "Supply Chain management with.Overtime and Premium Freight" Submitted to Industrial and Operation Engineering in the University of Michigan (A PhD Thesis).

7. Lawson D. and E. Porteus 2000."Multistage Inventory Management with Expediting"Operations Research 48,878-893.

8. Lee H.L.,K.C. So and C.S. Tang 2000."The Value of Information Sharing in a Two-level Supply Chain"Management Science 46.626-643.

9. Moinzadeh K. and F. Aggarwal 1997 " An International Based Multi-echelon Inventory System with Emergency Orders"Operations Research

10. Porteus E. 1990 "Stochastic Inventory Theory"Stochastic Models.D.Heyman and M.Sobel (eds) Elsevier Science Publishers,New York

11. Rosling K. 1998 “Applicable Cost Rate Function for single item Inventory Control”Accepted by Operations Research.

12. Zhang V.1996 “Ordering Policies for an Inventory System with Three Supply Modes”Naval Research Logistics13,691-708.