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On the stability criteria for prey-predator generalized model

E.A Bakare[1] and R.O Ayeni[2]
[1]Department of Computer and Information Science,
Lead City University,Ibadan,Nigeria
emmabak2000@yahoo.com
[2]Department of Pure and Applied Mathematics,LAUTECH,
Ogbomosho,Nigeria

[1]Corresponding author: email emmabak2000@yahoo.com: Tel. +2348034867587

Abstract

We formulate a generalized model for prey-predator interaction. We discuss the existence and uniqueness of solution of the model. Of particular interest are the criteria for the stability of the critical points.

Keywords: generalized model, uniqueness, existence, stability, and prey-predator.

1.0 Introduction

When species interact, the population dynamics of each species is affected. In general there is a whole web of interacting species, called a trophic web, which makes for structurally complex communities. We consider here systems involving two or more species, concentrating particularly on two-species systems. There are three main types of interaction.

(i) If the growth rate of one population is decreased and the other increased the populations are in a predator-prey situation.
(ii) If the growth rate of each population is decreased, then it is competition.
(iii) If each population's growth rate is enhanced, then it is called mutualism or symbiosis [8].

Some mathematical models have been developed in this area.
In 1926, [13] first proposed a simple model for the predation of one species by another to explain the oscillatory levels of certain fish catches in the Adriatic. This model was based on four assumptions.
First, the prey grows unboundedly in a Malthusian way in the absence of any predation.Secondly; the effect of the predation is to reduce the prey's per capita growth rate by a term proportional to the prey and the predator populations.
Thirdly, in the absence of any prey for sustenance the predator's death rate results in exponential decay.Fourthly, the prey's contribution to the predator's growth is proportional to the available prey as well as the size of the predator population. The model, is $\frac{dN}{dt} = N(a-bp)$ and $\frac{dP}{dt} = P(cN-d)$
When N is the prey population and P is the predator population. This model also called Lokta-Volterra model was analyzed.
Murray [8] modified the Lokta-Volterra Model by changing of the assumptions made by Volterra.The model he obtained is: $\frac{dN_1}{dt} = r_1 N_1\left[1-\frac{N_1}{K_1}-b_{12}\frac{N_2}{K_2}\right]$, $\frac{dN_2}{dt} = r_2 N_2\left[1-\frac{N_2}{K_2}-b_{21}\frac{N_1}{K_2}\right]$ where $r_1, k_1,$ r_2, k_2, b_{12}, b_{21} are all positive constants. This model was analyzed and the conditions for stability established. [2] presented some results on the dynamic complexity of coupled predator-prey systems. [3,4] studied in detail a modified Lotka-Volterra system with logistic growth of the prey and with both predator and prey dispersing by diffusion.

"Predator-Prey models are arguably the building blocks of the bio and ecosystems as biomasses are grown out of their resource masses. Species compete, evolve and disperse simply for the purpose of seeking resources to sustain their struggle for their very existence. Depending on their specific settings of applications, they can take the forms of resource- consumer, plant-herbivore, parasite-host, tumor cells (virus)-immune system, susceptible-infectious interactions, etc.They deal with the general loss-win interactions and hence may have applications outside of ecosystems. When seemingly competitive interactions are carefully examined, they are often in fact some forms of predator-prey interaction in disguise" [5].

Another approach to modeling the interaction between prey and predators was developed to account as well for organisms (such as bacteria) taking up nutrients and this is called Jacob-Mond Model. This model was discovered independently in the several diverse applications. It is akin to the Haldane-Briggs Model and Michaelis-Menten Model in Biochemistry the Jacob-Mond Model in microbial ecology and the Beverton-Holt model in fisheries. It serves as one of the important building blocks in studies of complex biochemical reactions and in ecology [12].In this work, we consider the classical predator-prey problem. We study an ecological situation involving two similar species competing for a limited food supply for example, two species of fish in a pond that do not prey on each other but do compete for the available food. Let R_1 and R_2 be the populations of the two species at time t.

2.0 Analysis Of The Model

Consider the general prey-predator model for an n-species system

$$\frac{dR_i}{dt} = R_i\left[b_i + \sum_{i=1}^{n} a_{ij}R_i\right] \tag{2.1}$$

i=1, 2 …n

The equation represent multispecies prey-predator cases where R_i's represents the population of different species at time t where $a_i>0$, $b_i>0$ are constants and represents a given finite source of food.

One species

In this case equation (2.1) is reduced to one species competing for a given finite source of food.

$$\frac{dR}{dt} = R(b + aR) \tag{2.2}$$

Where a>0, b>0 are constants and R (0)>0.This equation has an exact solution.

$$R(t) = \frac{be^{bt}}{\frac{b + aR(0)}{R(0)} - ae^{bt}} \qquad \text{for } b \neq 0 \tag{2.3}$$

$$R(t) = \frac{R(0)}{1 - aR(0)t} \qquad \text{for } b = 0 \tag{2.4}$$

Two species (Of the third order) with death rate (α, β) competing for a common ecological niche. The prey-predator model for this case takes the following form

$$\frac{dR_1}{dt} = R_1b_1 + a_{11}R_1^3 + a_{12}R_1R_2 - \alpha R_1^n$$

$$\frac{dR_2}{dt} = R_2b_2 + a_{22}R_2^3 + a_{21}R_2R_1 - \beta R_2^m$$

If (α, β) = (0, 0)

Then

$$\frac{dR_1}{dt} = R_1b_1 + a_{11}R_1^3 + a_{12}R_1R_2$$

$$\frac{dR_2}{dt} = R_2 b_2 + a_{22} R_2^3 + a_{21} R_2 R_1$$

Finding the critical solutions

We set $\frac{dR_1}{dt} = 0$ and $\frac{dR_2}{dt} = 0$

Therefore $R_1 b_1 + a_{11} R_1^3 + a_{12} R_1 R_2 = 0$ (2.5)

$$R_2 b_2 + a_{22} R_2^3 + a_{21} R_2 R_1 = 0 \quad (2.6)$$

From (2.5) $R_1(b_1 + a_{11} R_1^2 + a_{12} R_2) = 0$

$R_1 = 0$ and $b_1 + a_{11} R_1^2 + a_{12} R_2 = 0$

$b_1 + a_{11} R_1^2 + a_{12} R_2 = 0$

$$R_2 = \frac{-b_1 - a_{11} R_1^2}{a_{12}}$$

Since $R_1 = 0$, $R_2 = {-b_1}/{a_{12}}$

From (2.6) $R_2(b_2 + a_{22} R_2^2 + a_{21} R_1) = 0$

$R_2 \neq 0$ and $b_2 + a_{22} R_2^2 + a_{21} R_1 = 0$

Since $R_1 = 0$ and $R_2(b_2 + a_{22} R_2^2) = 0$

$R_2 = 0$ implies that $R_2^2 = {-b_2}/{a_{22}}$

Since $R_2 = 0$, and $R_1 = {-b_2}/{a_{12}}$, $R_2^2 = \frac{b_1^2}{(a_{12})^2} = \frac{-b_2}{a_{22}}$

Therefore $R_2 = 0$ and $R_1 = {-b_2}/{a_{21}}$

Hence the critical solutions are (0, 0), $(0, {-b_1}/{a_{12}})$ and $({-b_2}/{a_{21}}, 0)$

(1) **Critical solution** (0, 0) $\frac{dR_1}{dt} = R_1 b_1 + a_{11} R_1^3 + a_{12} R_1 R_2$

$$\frac{dR_2}{dt} = R_2 b_2 + a_{22} R_2^3 + a_{21} R_2 R_1$$

We write the system of equation in this form

$$\begin{pmatrix} \frac{dR_1}{dt} \\ \frac{dR_2}{dt} \end{pmatrix} = \begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \begin{pmatrix} a_{12} R_1 R_2 + a_{11} R_1^3 \\ a_{21} R_1 R_2 + a_{22} R_2^3 \end{pmatrix}$$

Let A= $\begin{pmatrix} b_1 & 0 \\ 0 & b_2 \end{pmatrix}$

If $|A - \lambda I| = 0$

$$\begin{vmatrix} b_1-\lambda & 0 \\ 0 & b_2-\lambda \end{vmatrix} = 0$$

$$(b_1-\lambda)(b_2-\lambda)=0$$

$$\lambda_1 = b_1 \text{ and } \lambda_2 = b_2$$

Hence the critical solution (0, 0) is

Case 1; If $b_1 < 0$ and $b_2 < 0$ then (0, 0) is asymptotically stable

Case 2; If b_1 or $b_2 > 0$ then (0, 0) is unstable.

2) **Critical solution** $(0, -b_1/a_{12})$

Let $R_1 = 0$ and $R_2 = -b_1/a_{12}$

We set $R_2 = -b_1/a_{12}$

$U = R_1$ and $V = R_2 + b_1/a_{12} \Rightarrow R_2 = V - b_1/a_{12}$

$$\therefore \frac{du}{dt} = \frac{dR_1}{dt} = ub_1 + a_{11}u^3 + a_{12}u\,(v - b_1/a_{12})$$

$$= ub_1 + a_{11}u^3 + a_{12}uv - ub_1$$

$$= ub_1 - ub_1 + a_{11}u^3 + a_{12}uv$$

$$= a_{11}u^3 + a_{12}uv$$

$$\therefore \frac{du}{dt} = \frac{dR_1}{dt} = a_{11}u^3 + a_{12}uv$$

$$\frac{dv}{dt} = \frac{dR_2}{dt} = (v - b_1/a_{12})b_2 + a_{21}(u)(v - b_1/a_{12}) + a_{22}\,(v - b_1/a_{12})^3$$

$$= b_2 v - \frac{b_1 b_2}{a_{12}} + a_{21}uv - a_{21}\frac{b_1 u}{a_{12}} + a_{22}\,(v - b_1/a_{12})^3$$

$$\frac{dv}{dt} = \frac{dR_2}{dt} = b_2 v - a_{21}\frac{b_1 u}{a_{12}} - \frac{b_1 b_2}{a_{12}} + a_{21}uv + a_{22}\,(v - b_1/a_{12})^3$$

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -a_{21}b_1/a_{12} & b_2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a_{11}u^3 + a_{12}uv \\ a_{21}uv + a_{22}(v - \frac{b_1}{a_{12}}) - \frac{b_1 b_2}{a_{12}} \end{pmatrix}$$

Let $A = \begin{pmatrix} 0 & 0 \\ -a_{21}b_1/a_{12} & b_2 \end{pmatrix}$

If $\left|A-\lambda I\right|=0$ ie $\begin{vmatrix} 0-\lambda & 0 \\ -a_{21}b_1/a_{12} & b_2-\lambda \end{vmatrix}=0$

$$\begin{vmatrix} 0-\lambda & 0 \\ -a_{21}b_1/a_{12} & b_2-\lambda \end{vmatrix}=0$$

$$-\lambda(b_2-\lambda)=0$$

$$\lambda_1=0 \text{ or } \lambda_2=b_2$$

ie $\lambda_1=0$ and $\lambda_2=b_2$

Hence the critical solution $(0,\frac{-b_1}{a_{12}})$ is not asymptotically stable and it is not unstable since

$$\lambda_1=0 \text{ and } \lambda_2=b_2.$$

3) **Critical Solution** $(\frac{-b_2}{a_{21}},0)$

Let $R_1=\frac{-b_2}{a_{21}}$, and $R_2=0$

$\therefore$ We set $u=R_1+\frac{b_2}{a_{21}}$ and $v=R_2$

$R_1=u-\frac{b_2}{a_{21}}$ and $v=R_2$

$$\therefore \frac{du}{dt}=\frac{dR_1}{dt}=\left(u-\frac{b_2}{a_{21}}\right)b_1+a_{11}\left(u-\frac{b_2}{a_{21}}\right)^3+a_{12}\left(u-\frac{b_2}{a_{21}}\right)(v)$$

$$=b_1u-\frac{b_1b_2}{a_{21}}+a_{12}uv-\frac{a_{12}b_2v}{a_{21}}+a_{11}\left(u-\frac{b_2}{a_{21}}\right)^3$$

$$=b_1u-\frac{a_{12}b_2v}{a_{21}}+a_{12}uv-\frac{b_1b_2}{a_{21}}+a_{11}\left(u-\frac{b_2}{a_{21}}\right)^3$$

$$\frac{dv}{dt}=\frac{dR_2}{dt}=vb_2+a_{21}(u-\frac{b_2}{a_{21}})(v)+a_{22}v^3$$

$$=vb_2+a_{21}uv-vb_2+a_{22}v^3$$

$$=vb_2-vb_2+a_{21}uv+a_{22}v^3$$

$$\frac{dv}{dt}=a_{21}uv+a_{22}v^3$$

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} b_1 & -a_{12}b_2/a_{21} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a_{12}uv + a_{11}(u - \frac{b_2}{a_{21}})^3 - \frac{b_2 b_1}{a_{21}} \\ a_{12}uv + a_{22}v^3 \end{pmatrix}$$

Let A = $\begin{pmatrix} b_1 & -a_{12}b_2/a_{21} \\ 0 & 0 \end{pmatrix}$

If $|A - \lambda I| = 0$

ie $\begin{vmatrix} b_1 - \lambda & -a_{12}b_2/a_{21} \\ 0 & 0 - \lambda \end{vmatrix} = 0$

$$(b_1 - \lambda)(-\lambda) = 0$$

$$-\lambda b_1 + \lambda^2 = 0$$

$$\lambda^2 - b_1\lambda = 0$$

$$\lambda(\lambda - b_1) = 0$$

$\lambda_1 = 0$ or $\lambda_2 = b_1$

ie $\lambda_1 = 0$ and $\lambda_2 = b_1$

Hence the critical solution $(\frac{-b_2}{a_{21}}, 0)$ is not asymptotically stable and it is not unstable since $\lambda_1 = 0$ and $\lambda_2 = b_1$.

IF $(\alpha, \beta) \neq 0$

Then

$$\frac{dR_1}{dt} = R_1 b_1 + a_{11}R_1^3 + a_{12}R_1 R_2 - \alpha R_1^n$$

$$\frac{dR_2}{dt} = R_2 b_2 + a_{22}R_2^3 + a_{21}R_2 R_1 - \beta R_2^m$$

Where n=m=1

$$\therefore \frac{dR_1}{dt} = R_1 b_1 + a_{11}R_1^3 + a_{12}R_1 R_2 - \alpha R_1$$

$$\frac{dR_2}{dt} = R_2 b_2 + a_{22}R_2^3 + a_{21}R_2 R_1 - \beta R_2$$

Finding the critical solutions

We set $\frac{dR_1}{dt} = 0$ and $\frac{dR_2}{dt} = 0$

$$R_1 b_1 + a_{11}R_1^3 + a_{12}R_1 R_2 - \alpha R_1 = 0 \quad (2.7)$$

$$R_2 b_2 + a_{22}R_2^3 + a_{21}R_2 R_1 - \beta R_2 = 0 \quad (2.8)$$

From (2.7)

$$R_1(b_1 + a_{11}R_1^2 + a_{12}R_2 - \alpha) = 0$$

$R_1 = 0$ and $b_1 + a_{11}R_1^2 + a_{12}R_2 - \alpha = 0$

$R_1 = 0$ and $R_2 = 0$

$$R_2 = \frac{-b_1 - a_{11}R_1^2 + \alpha}{a_{12}}$$

Since $R_1 = 0$ and $R_2 = \dfrac{-b_1 + \alpha}{a_{12}}$

$$\therefore\ R_1 = 0 \text{ and } R_2 = \frac{-b_1 + \alpha}{a_{12}}$$

From (2.7) $R_2(b_2 + a_{22}R_2^2 + a_{21}R_1 - \beta) = 0$

$R_2 \neq 0$ and $b_2 + a_{22}R_2^2 + a_{21}R_1 - \beta = 0$

since $R_1 = 0$

$R_2(b_2 + a_{22}R_2^2 - \beta) = 0$

$R_2 \neq 0$

$$\Rightarrow R_2^2 = \frac{-b_2 + \beta}{a_{22}}$$

Since $R_2 = 0$, $R_1 = \dfrac{-b_2 + \beta}{a_{21}}$, $R_2^2 = \dfrac{(-b_1 + \alpha)^2}{a_{12}^2} = \dfrac{-b_2 + \beta}{a_{22}}$

$$\therefore\ R_2 = 0 \quad \text{and} \quad R_1 = \frac{-b_2 + \beta}{a_{21}}$$

Hence the critical solutions are (0, 0), $(0, \dfrac{-b_1 + \alpha}{a_{12}})$ and $(\dfrac{-b_2 + \beta}{a_{21}}, 0)$

1) **Critical solution** (0,0)

$$\frac{dR_1}{dt} = (b_1 - \alpha) + a_{11}R_1^3 + a_{12}R_1R_2$$

$$\frac{dR_2}{dt} = (b_2 - \beta) + a_{22}R_2^3 + a_{21}R_2R_1$$

We write the system of equation in this form

$$\begin{pmatrix} \dfrac{dR_1}{dt} \\ \dfrac{dR_2}{dt} \end{pmatrix} = \begin{pmatrix} b_1 - \alpha & 0 \\ 0 & b_2 - \beta \end{pmatrix} \begin{pmatrix} R_1 \\ R_2 \end{pmatrix} + \begin{pmatrix} a_{11}R_1^3 + a_{12}R_1R_2 \\ a_{21}R_2R_1 + a_{22}R_2^3 \end{pmatrix}$$

Let A = $\begin{pmatrix} b_1 - \alpha & 0 \\ 0 & b_2 - \beta \end{pmatrix}$

If $|A - \lambda I| = 0$

$$\begin{vmatrix} b_1-\alpha-\lambda & 0 \\ 0 & b_2-\beta-\lambda \end{vmatrix} = 0$$

$$(b_1 - \alpha) - \lambda)\,(b_2 - \beta) - \lambda) = 0$$

$$\lambda_1 = b_1 - \alpha \text{ and } \lambda_2 = b_2 - \beta$$

Hence the critical solution (0,0) is

Case 1;If $b_1 - \alpha < 0$ and $b_2 - \beta < 0$ then (0,0) is asymptotically stable

Case 2;If $b_1 - \alpha$ or $b_2 - \beta$ >o then (0,0) is unstable.

2) **Critical solution** $(0, \dfrac{-b_1+\alpha}{a_{12}})$

Let $R_1 = 0$ and $R_2 = \dfrac{-b_1+\alpha}{a_{12}}$

We set

U = R_1 and V = $R_2 + \dfrac{b_1-\alpha}{a_{12}} \Rightarrow R_2 = v - (\dfrac{b_1-\alpha}{a_{12}})$

$$\frac{du}{dt} = \frac{dR_1}{dt} = ub_1 + a_{11}u^3 + a_{12}u(v - (\frac{b_1-\alpha}{a_{12}})) - \alpha u$$

$$= ub_1 + a_{11}u^3 + a_{12}uv - ub_1 - \alpha a_{12}u - \alpha u$$

$$= ub_1 - ub_1 + a_{11}u^3 + a_{12}uv - \alpha a_{12}u - \alpha u$$

$$= -\alpha a_{12}u + a_{11}u^3 + a_{12}uv - \alpha u$$

$$\therefore \frac{du}{dt} = \frac{dR_1}{dt} = -\alpha a_{12}u + a_{11}u^3 + a_{12}uv - \alpha u$$

$$\frac{du}{dt} = \frac{dR_1}{dt} = (-\alpha a_{12} - \alpha)u + a_{11}u^3 + a_{12}uv$$

$$\frac{dv}{dt} = \frac{dR_2}{dt} = (v - (\frac{b_1-\alpha}{a_{12}})) + a_{21}(u)(v - (\frac{b_1-\alpha}{a_{12}})) + a_{22}(v - (\frac{b_1-\alpha}{a_{12}}))^3 - \beta v$$

$$\frac{dv}{dt} = \frac{dR_2}{dt} = b_2v - \frac{b_1b_2 + \alpha b_2}{a_{12}} + a_{21}uv - \frac{a_{21}ub_1 + a_{21}u\alpha}{a_{12}} + a_{22}(v - (\frac{b_1-\alpha}{a_{12}}))^3 - \beta v$$

$$= b_2v - \beta v - (\frac{a_{21}b_1u}{a_{12}} - \frac{a_{21}\alpha}{a_{12}})u + a_{21}uv - \frac{b_1b_2 + \alpha b_2}{a_{12}} + a_{22}(v - (\frac{b_1-\alpha}{a_{12}}))^3$$

$$= (b_2 - \beta)v - (\frac{a_{21}b_1u}{a_{12}} - \frac{a_{21}\alpha}{a_{12}})u + a_{21}uv - \frac{b_1b_2 + \alpha b_2}{a_{12}} + a_{22}(v - (\frac{b_1-\alpha}{a_{12}}))^3$$

We write the system of equation in this form

$$\begin{pmatrix} \dfrac{du}{dt} \\ \dfrac{dv}{dt} \end{pmatrix} = \begin{pmatrix} -\alpha a_{12} - \alpha & 0 \\ \dfrac{a_{21}\alpha - a_{21}b_1}{a_{12}} & b_2 - \beta \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a_{11}u^3 + a_{12}uv \\ a_{21}uv - \dfrac{b_1b_2 + \alpha b_2}{a_{12}} + a_{22}(v - (\dfrac{b_1-\alpha}{a_{12}})^3 \end{pmatrix}$$

Let $\quad$ **A** = $\begin{pmatrix} -\alpha a_{12}-\alpha & 0 \\ \frac{a_{21}\alpha - a_{21}b_1}{a_{12}} & b_2-\beta \end{pmatrix}$

If $\quad |A-\lambda I| = 0$

ie $\quad \begin{vmatrix} -\alpha a_{12}-\alpha-\lambda & 0 \\ \frac{a_{21}\alpha - a_{21}b_1}{a_{12}} & b_2-\beta-\lambda \end{vmatrix}$

$$\{(-a_{12}-\alpha)-\lambda)((b_2-\beta)-\lambda)\} = 0$$

ie $\quad \lambda_1 = -\alpha a_{12} - \alpha$ or $\lambda_2 = b_2 - \beta$

Hence the critical solution (0, $\frac{-b_1+\alpha}{a_{12}}$) is

Case 1 : If $(-\alpha a_{12}-\alpha) < 0 and b_2 - \beta < 0$ then the critical solution (0, $\frac{-b_1+\alpha}{a_{12}}$) is asymptotically stable

Case 2: If $(-\alpha a_{12}-\alpha) > 0 or b_2 - \beta > 0$ then the critical solution (0, $\frac{-b_1+\alpha}{a_{12}}$) is unstable

3) The Critical solution ($\frac{-b_2+\beta}{a_{21}}$,0)

Let $R_1 = \frac{-b_2+\beta}{a_{21}}$ and R_2 =0

$\therefore$ We set u = $R_1 + \frac{b_2-\beta}{a_{21}}$ and v = R_2

R $_1$ = $u - \frac{b_2+\beta}{a_{21}}$ and v = R_2

$$\frac{du}{dt} = \frac{dR_1}{dt} = (u - \frac{b_2+\beta}{a_{21}})b_1 + a_{11}(u - \frac{b_2+\beta}{a_{21}})^3 + a_{12}(u - \frac{b_2+\beta}{a_{21}})(v) - \alpha(\frac{-b_2+\beta}{a_{21}})$$

$$= b_1 u - \frac{b_1b_2+\beta b_1}{a_{21}} + a_{11}(u - \frac{b_2+\beta}{a_{21}})^3 + a_{12}uv - \frac{a_{12}b_2v + a_{12}\beta v}{a_{21}} + \frac{\alpha b_2 - \alpha\beta}{a_{12}}$$

$$= b_1 u - \frac{a_{12}b_2v}{a_{21}} + \frac{a_{12}\beta v}{a_{21}} + a_{11}(u - \frac{b_2+\beta}{a_{21}})^3 - \frac{b_1b_2+\beta b_1}{a_{21}} + a_{12}uv$$

$$\frac{dv}{dt} = \frac{dR_2}{dt} = vb_2 + a_{21}(u - \frac{b_2+\beta}{a_{21}})(v) + a_{22}(v^3) - \beta v$$

$$= vb_2 - \beta v + a_{21}uv - b_2v + \beta v + a_{22}v^3$$

$$= vb_2 - vb_2 - \beta v + \beta v + a_{12}uv + a_{22}v^3$$

$$\frac{dv}{dt} = \frac{dR_2}{dt} = a_{21}uv + a_{22}v^3$$

$$\begin{pmatrix} \frac{du}{dt} \\ \frac{dv}{dt} \end{pmatrix} = \begin{pmatrix} b_1 & \frac{a_{12}\beta + a_{12}b_2}{a_{21}} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{pmatrix} a_{11}\left(u - \frac{b_2 + \beta}{a_{21}}\right)^3 - \frac{b_1 b_2 + \beta b_1}{a_{21}} + a_{12}uv \\ a_{21}uv + a_{22}v^3 \end{pmatrix}$$

Let $\quad A = \begin{pmatrix} b_1 & \frac{a_{12}\beta + a_{12}b_2}{a_{21}} \\ 0 & 0 \end{pmatrix}$

If $\lfloor A - \lambda I \rfloor = 0$

Then $\begin{vmatrix} b_1 - \lambda & \frac{a_{12}\beta + a_{12}b_2}{a_{21}} \\ 0 & 0 - \lambda \end{vmatrix} = 0$

$$(b_1 - \lambda)(0 - \lambda) = 0$$
$$\lambda(b_1 - \lambda) = 0$$
$$-\lambda b_1 + \lambda^2 = 0$$
$$-\lambda^2 - \lambda b_1 = 0$$
$$\lambda(\lambda - b_1) = 0$$
$$\lambda_1 = 0 or \lambda_2 = b_1$$

ie $\lambda_1 = 0$ and $\lambda_2 = b_1$

Hence the critical solution $(\frac{-b_2 + \beta}{a_{21}}, 0)$ is not asymptotically stable and it is not unstable since $\lambda_1 = 0$ and $\lambda_2 = b_1$.

Existence and uniqueness of solutions criteria

Theorem 2.1: Let $f_1(R_{1,1,} R_{1,2,} t)$ and $(\frac{\partial f_1(R_{1,1,} R_{1,2,} t)}{\partial R_{1,1}})$ be continuous on D. Then $f_1(R_{1,1,} R_{1,2,} t)$ is Liptschitz continuous in $R_{1,1,} R_{1,2}$ over D.

Proof: Let $(R_{1,1,} R_{1,2,} t)$ and $(R_{2,1,} R_{2,2,} t)$ be continuous points in D.For fixed t, $\frac{\partial f_1(R_{1,1,} R_{1,2,} t)}{\partial R_{1,1}}$ is a function of

$(R_{1,1,} R_{1,2})$ and so we apply the mean value theorem of differential calculus to obtain

$$\left| f_1(R_{1,1,} R_{1,2,} t) - f_1(R_{2,1,} R_{2,2,} t) \right| \le \frac{\partial f_1}{\partial R_{1,1}} \left| R_{1,1} - R_{2,1} \right| + \left| R_{2,1} - R_{2,2} \right|$$

Where $\quad K = \frac{\partial f_1(R_{1,1,} R_{1,2,} t)}{\partial R_{1,1}}$

$$f_1(R_{1,1,}R_{1,2,}t) = R_{1,1}b_1 + a_{11}R_{1,1}^3 + a_{12}R_{1,1}R_{1,2}$$

$$f_1(R_{2,1,}R_{2,2,}t) = R_{2,1}b_2 + a_{12}R_{2,1}R_{2,2} + a_{11}R_{2,2}^3$$

$$\frac{\partial f_1(R_{1,1,}R_{1,2,}t)}{\partial R_{1,1}} = b_1 + 3a_{11}R_{1,1}^2 + a_{12}R_{1,2} \qquad \text{where } R_1(0) = R_{1,0}$$

$$\left|(R_{1,1}b_1 + a_{11}R_{1,1}^3 + a_{12}R_{1,1}R_{1,2}) - (R_{2,1} + a_{12}R_{2,1}R_{2,2} + a_{11}R_{2,2}^2)\right| \le \frac{\partial f_1}{\partial R_{1,1}}\left|R_{1,1} - R_{2,1}\right| + \left|R_{2,1} - R_{2,2}\right|$$

Hence $f_1(R_{1,1,}R_{1,2,}t)$ is Lipschitz continuous and the criteria for the existence and uniqueness of solutions is satisfied**.**

3.0 SUMMARY AND CONCLUSION

In this paper, we try to modify the work of Khaled Batiha [6], 2007 on the Numerical Solutions of the Multispecies Predator-Prey Model by Variational iteration method. He tried to solve the Multispecies Predator-Prey Model numerically using Variational iteration Method(VIM) and the Adomian Decomposition Method(ADM).He was able to show that the VIM has the advantage of being more concise for numerical and analytical purposes.We have tried to solve the Predator-Prey generalized model with two species of the third order with death rate analytically. We modified the work of Khaled Batiha[6], 2007 by making our model of the same two species but third order and with death rate (α, β).We derived the critical points and we try to play around the criteria for stability of the critical points. We try to solve for the uniqueness and existence of the model. We observe that as the given finite source of food b_i 's is getting reduced the system is likely to become Asymptotically Stable and become Unstable as the given finite source of food b_i 's is getting increased. We have made the assumption that the populations of R_1 and the population of R_2 are two species of animals that do not prey on each other but do compete for the available food. This assumption is biologically restrictive. It may hold for some Carnivore-Carnivore or Omnivores-Omnivores or even among aquatic organism.

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