Journal of the Nigerian Association of Mathematical Physics
Volume **16 (May, 2010), pp 221 – 228**

A Further Result on Oscillations in a Non-Linear Boundary Value Problem of a Fourth Order Ordinary Differential Equation

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Abstract

The Eigenvalue approach and the comparison between a linear and nonlinear fourth order differential equation yielded the basis for a theorem on existence of periodic solutions for a nonlinear boundary value problem. By involving β, where $\beta = \min\left\{\frac{1}{2}a_2 - M^2,\ (M+1)^2 - \frac{1}{2}a_2\right\}$ ***in the hypothesis a further theorem is proved on existence of periodic solutions for the nonlinear boundary value problem. Our method of investigation is the Leray-Schauder fixed point technique and the use of integrated equation as the mode for estimating the a priori bounds.***

Keywords: Boundary value problems (BVP), Leray-Schauder fixed point technique, a priori bounds, integrated equation, parameter dependent equation.

1.0 Introduction

Consider the nonlinear fourth order differential equation:

$$x^{(4)} + \varphi(\dddot{x})\dddot{x} + a_2\ddot{x} + \theta(\dot{x}) + f(x) = P\left(t, x, \dot{x}, \ddot{x}, \dddot{x}\right) \tag{1.1}$$

with boundary conditions

$$D^{(r)}x(0) = D^{(r)}x(2\pi), \quad r = 0,\ 1,\ 2,\ 3 \qquad D = \frac{d}{dt} \tag{1.2}$$

where $\varphi,\ f,\ \theta,\ P$ are continuous functions depending on the argument, a_2 is a constant.

It has been clearly shown that one would expect solutions to the 2π periodic BVP for the equations.

$$x^{(4)} + a_1\dddot{x} + a_2\ddot{x} + a_3\dot{x} + a_4x = P\left(t, x, \dot{x}, \ddot{x}, \dddot{x}\right) \tag{1.3}$$

for arbitrary a_1 and a_3, if a_2 and a_4 satisfy

$$\chi(m) \neq 0 \text{ for } m = 1,\ 2,\ \ldots \tag{1.4}$$

[See [1], [5], [6]]. The Equation (1.4) is an improvement on result of [3], which required that $\chi(\lambda) \neq 0$ for all real λ.

Observe from (1.4) that

$$\chi(m) > 0 \qquad \forall m \tag{1.5}$$

if

$$a_2 \leq 0 \text{ and } a_4 > 0 \tag{1.6}$$

Then $a_2 > 0$, the least restriction on a_4 which secures (1.5) can be obtained by completing squares and rewriting $\chi(m)$ on the form

$$\chi(m) = \left(m^2 - \tfrac{1}{2}a_2\right)^2 + a_4 - \tfrac{1}{4}a_2{}^2 \tag{1.7}$$

which shows that (1.5) holds if

$$a_4 > \tfrac{1}{4}a_2{}^2 \tag{1.8}$$

This can be further relaxed when $a_2 > 0$ but $\tfrac{1}{2}a_2 \neq$ integer^2 that is $\tfrac{1}{2}a_2 \neq m^2$ $(m = 1,\ 2,\ ...)$ implying that $M^2 < \tfrac{1}{2}a_2 < (M+1)^2$

$$\tag{1.9}$$

Suppose that (1.9) holds and define a constant $\beta = \beta(a_2) > 0$ by

$$\beta = \min\left\{\tfrac{1}{2}a_2 - M^2,\ (M+1)^2 - \tfrac{1}{2}a_2\right\} \tag{1.10}$$

Then by (1.7)

$$\chi(m) \geq a_4 - \tfrac{1}{4}a_2{}^2 + \beta^2 \qquad \forall m \tag{1.11}$$

so that (1.5) holds provided that

$$a_4 > \tfrac{1}{4}a_2{}^2 - \beta^2 \tag{1.12}$$

which is a distinct improvement on (1.8). [1].

In this paper, a generalization of conditions (1.6), (1.8) and (1.12) on Equation (1.3) have been used in establishing existence of periodic solutions for Equations (1.1) – (1.2) in which $a_1,\ a_2,\ a_3,\ a_4$ are all not necessarily constants. In particular with respect to (1.12), we have the following:

THEOREM 1

Suppose that

(i) $\varphi,\ \theta,\ f,\ P$ are continuous function depending on the arguments shown with a_2 a constant greater than zero

(ii) The constant a_2 satisfies $M^2 < \tfrac{1}{2}a_2 < (M+1)^2$ for some $M > 0$ and there exists a constant $\delta > 0$ such that

$$\inf_{\|x\| \geq 1} \frac{f'(x)}{x} \geq \delta_1 > \tfrac{1}{4}a_2{}^2 - \varepsilon\beta^2 \tag{1.13}$$

where β is defined by (1.10) and ε is in the interval 0≤ε≤1.

(iii) The function P is bounded and 2π periodic in t.

Then Equations (1.1) – (1.2) have at least one 2π periodic solution for arbitrary φ and θ

2.0 GENERAL COMMENTS ON SOME NOTATIONS

Throughout the proof, which follows the capital, $C,\ C_1,\ C_2,\ C_3\ \ldots$ represent positive constants, whose magnitudes depend at most on $\varphi,\ \theta,\ f,\ P$ and the constants a_2. The $C_1,\ C_2,\ C_3,\ \ldots$ with suffices attached

retain their identities throughout the proof of the theorem 2 but the C_j without suffixes are not necessarily the same

in each place of occurrence. The symbols $|.|_\infty,\ |.|_1,\ and\ |.|_2$ in respect of the mapping: $[0:2\pi] \to \mathbf{R}$ shall have their usual meanings. Thus given the function

$\theta:[0,2\pi] \to \mathbf{R}$ then

$$|\theta|_\infty = \max_{0\le t\le 2\pi}|\theta(t)|,\ \ |\theta|_1 = \int_0^{2\pi}|\theta(t)|\,dt,\ |\theta|_2 = \left(\int_0^{2\pi}\theta^2(t)\,dt\right)^{\frac{1}{2}}$$

3.0 PROOF OF THEOREM 1

The proof of theorem 1 is by the Leray-Schauder fixed point technique and we shall consider the parameter λ dependent equation, $(0 \le \lambda \le 1)$

$$x^{(4)} + \varphi(\dddot{x}) + a_2\ddot{x} + \theta(\dot{x}) + f_\lambda(x) = \lambda P \tag{3.1}$$

where

$$f_\lambda(x) = (1-\lambda)\delta_1 + \lambda f(x)$$

By setting

$$\dot{x} = y,\ \dot{y} = z,\ \dot{z} = u,\ \dot{u} = -\lambda\varphi u - a_2 z - \lambda\varphi(y) - f_\lambda(x) + \lambda p \tag{3.2}$$

the equation (3.1) can be written compactly in matrix form

$$\dot{X} = AX + \lambda F(X,t) \tag{3.3}$$

where

$$X = \begin{bmatrix} x \\ y \\ z \\ u \end{bmatrix},\quad A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\delta_1 & 0 & -a_2 & 0 \end{bmatrix},\quad F = \begin{bmatrix} 0 \\ 0 \\ 0 \\ Q \end{bmatrix} \tag{3.4}$$

with $Q = P(t) - \varphi u + a_2 z - \theta(y) - f(x) + \delta_1 x.$

Note that equation (3.1) reduces to a linear equation.

$$x^{(4)} + a_2 r^2 + \delta_1 x = 0$$
(3.5)

when $\lambda = 0$ and to (1.1) when $\lambda = 1$. The eigenvalues of the matrix A defined by (3.4) are the roots of the auxiliary equation

$$r^4 + a_2 r^2 + \delta_1 = 0$$
(3.6)

The equation (3.6) has no roots of the form $r = i\beta$ (β an integer) if

$$\delta_1 > \frac{1}{4} a_2^2$$
(3.7)

Therefore the matrix $\left(e^{-2\pi A} - I\right)$ (I being the identity 4x4 matrix) is invertible. Thus $X = X(t)$ is a 2π periodic solution of (3.3) if and only if

$$X = \lambda TX, \quad 0 \le \lambda \le 1$$
(3.8)

where the transformation T is defined by

$$(TX)(t) = \int_0^{2\pi} \left(e^{-2\pi A} - 1\right)^{-1} e^{A(t-s)} F\left(X(s), s\right) ds \quad [4]$$
(3.9)

Let *S* be the space of all real continuous 4-vector function $\bar{X}(t) = \left(x(t), y(t), z(t), u(t)\right)$ which are of periodic 2π and with norm

$$\left\|\bar{X}\right\|_s = \sup_{0 \le t \le 2\pi} \left\{ \left|x(t)\right| + \left|y(t)\right| + \left|z(t)\right| + \left|u(t)\right| \right\}$$
(3.10)

If the operator T defined by (3.9) is a compact mapping of *S* into itself, then it suffices for the proof of theorem 1 to establish a priori bounds C_1, C_2, C_3, C_4 independent of λ such that

$$|x|_\infty \le C_1, \ |\dot{x}|_\infty \le C_2, \ |\ddot{x}|_\infty \le C_3, \ |\dddot{x}|_\infty \le C_4$$
(3.11)

4.0 VERIFICATION OF (24)

We shall require in addition the use of result

$$\int_0^{2\pi} (u + \tfrac{1}{2} a_2 y)^2 dt \equiv \int_0^{2\pi} (\dddot{x} + \tfrac{1}{2} a_2 \dot{x})^2 dt \ge \beta^2 \int_0^{2\pi} \dot{x}^2 dt$$
(4.1)

Let $x(t)$ be a possible 2π periodic solution of (3.1). Then the main tool to be used here in this verification is the function $V(x, y, z, u)$ defined by

$$V = \lambda \int_0^z s\varphi(s) ds + a_2 yz + uz + y f_\lambda(x) + \lambda \int_0^y \theta(s) ds$$
(4.2)

The time derivative $\dot{V}$ of (4.2) along the solution paths of (3.2) is

$\dot{V} = u^2 + a_2 yu + y^2 f_\lambda'(x) + \lambda zP,$ on completing the squares, we have

$$= \left(u + \tfrac{1}{2}a_2 y\right)^2 + y^2\left(f_\lambda'(x) - \tfrac{1}{4}a_2^{\,2}\right) + \lambda zP$$

(4.3)

In dealing with the term like $y^2 f_\lambda'(x)$ in which $f'(x)$ is positive only when $|x|$ is positive. Consider the function W defined by

$$W = yH(x)$$

(4.4)

where

$$H(x) = \begin{cases} \sin(\frac{\pi x}{4}), & |x| \le 2 \\ sqn\ x, & |x| > 2 \end{cases}$$

Along the solution paths of the Equation (3.2)

$$\frac{d}{dt}\left(yH(x)\right) = y^2 H'(x) + zH(x)$$

(4.5)

By considering the function

$$U = V + \lambda C_0 yH(x)$$

(4.6)

and along the solution paths of (3.2)

$$\frac{d}{dt}(U) = \dot{U} = \left(u + \tfrac{1}{2}a_2 y\right)^2 + y^2\left(f_\lambda'(x) - \tfrac{1}{4}a_2^{\,2}\right) + \lambda zP + \lambda C_0 y^2 H'(x) + \lambda C_0 zH(x)$$

(4.7)

since $|H| \le 2\ \forall x$ and $H'(x) \ge 0\ \forall x$

but $H'(x) \ge \frac{\pi}{4\sqrt{2}}$ when $|x| \le 1$ it follows from (4.6) and (4.7) that C_0 is fixed and large enough, we shall have that for every possible 2π periodic solution of that

$$\int_0^{2\pi}\left(u + \tfrac{1}{2}a_2 y\right)^2 dt + y^2\int_0^{2\pi} f_\lambda'(x)dt + H'(x) - \tfrac{1}{4}a_2^{\,2} + y^2\int_0^{2\pi}\left(\lambda C_0 H'(x) - \tfrac{1}{4}a_2^{\,2}\right)dt = \int_0^{2\pi}\left\{\lambda P + \lambda C_0 H(x)\right\}|z|\,dt$$

$$\int_0^{2\pi}\left(u + \tfrac{1}{2}a_2 y\right)^2 dt + \int_0^{2\pi}\left(\tfrac{1}{4}a_2^{\,2} + \varepsilon\beta^2\right)y^2 dt \le C_2\int_0^{2\pi}|z|dt$$

(4.8)

splitting $\int_0^{2\pi}\left(u + \tfrac{1}{2}a_2 y\right)^2 dt$ as follows

$$\left(1-\varepsilon\right)\int_0^{2\pi}\left(u+\alpha y\right)^2 dt = \left\{\left(1-\varepsilon\right)+\varepsilon\right\}\int_0^{2\pi}\left(u+\alpha y\right)^2 dt$$

$$\alpha = \tfrac{1}{2} a_2$$

By (4.1) and (4.8), we have

$$\left(1-\varepsilon\right)\int_0^{2\pi}\left(u+\alpha y\right)^2 dt \le C_2 \int_0^{2}\left|z\right| dt$$

with $\left(1-\varepsilon\right) > 0$ and that $y = \dot{x}$, $z = \ddot{x}$, $u = \dddot{x}$ the equation implies

$$\left|\dddot{x} + \alpha \dot{x}\right|_2^2 \le C_2 \left|\ddot{x}\right|_1$$

Note that

$$\left|\ddot{x}\right|_1 \le \left(2\pi\right)^{\frac{1}{2}} \left|\ddot{x}\right|_2$$

(4.9)

by Schwartz's inequality

$$\le \left(2\pi\right)^{\frac{1}{2}} C_3 \left|\dddot{x} + \alpha \dot{x}\right|_1$$

From [2]

$$\le C_5 \left|\dddot{x} + \alpha \dot{x}\right|_1 \le C_6 \left|\dddot{x} + \alpha \dot{x}\right|_2$$

by Schwartz's inequality and where $C_6 = \left(2\pi\right)^{\frac{1}{2}} C_5$

This implies that

$$\left|\dddot{x} + \alpha \dot{x}\right|_2 \le C_6$$

(4.10)

From (4.9) and (4.10)

$$\left|\ddot{x}\right|_\infty \le C_8$$

(4.11)

Since $x(0) = x(2\pi)$ implies that there exists $\tau \in \left[0,\ 2\pi\right]$ such that $\dot{x}(\tau) = 0$ then the identity

$$\dot{x}(t) = \dot{x}(\tau) + \int_0^{2\pi} \ddot{x} ds$$ holds.

So that

$$\max_{0 \le t \le 2\pi} \left|\dot{x}(t)\right| \le \int_0^{2\pi} \left|\ddot{x}\right| ds$$

That is $$\le \left(2\pi\right)^{\frac{1}{2}} \left|\ddot{x}\right|_2$$

$$\max_{0 \le t \le 2\pi} \left|\dot{x}(t)\right| \le C_9$$

Hence

$$\left|\dot{x}\right|_\infty \le C_9$$

(4.12)

Now integrate equation (14) directly from $t = 0$ to $t = 2\pi$

$$\int_0^{2\pi} x^{(4)}dt + \int_0^{2\pi} \lambda\varphi(\ddot{x})\dddot{x}dt + \int_0^{2\pi} a_2\ddot{x}dt + \int_0^{2\pi} \lambda\theta(\dot{x})dt + \int_0^{2\pi} f_\lambda(x)dt = \int_0^{2\pi} \lambda P dt$$

Using equations (4.11), (4.12) and (1.2)

$$\int_0^{2\pi} f_\lambda(x)dt = \int_0^{2\pi} \lambda P dt - \int_0^{2\pi} \lambda\theta(\dot{x})dt$$

(4.13)

The boundedness of *P* and the fact that $0 \le \lambda \le 1$ together with (4.12) imply that the right hand side of (4.13) is finite.

That is

$$\left|\int_0^{2\pi} \lambda P dt\right| + \left|\int_0^{2\pi} \lambda\theta(\dot{x})dt\right| \le C_{10}$$

(4.14)

Thus

$$\left|\int_0^{2\pi} f_\lambda(x)dt\right| \le C_{10}$$

or

$$\left|(1-\lambda)\,\delta_1 x + \lambda f(x)\right| \le C_{10}$$

(4.15)

for C_{10} very large implies that $|x(\tau)| \le C_{11}$ for some $\tau \in [0,\ 2\pi]$

(4.16)

Now the identity *t*

$x(t) = x(\tau) + \int_e^t \dot{x}dt$ holds.

Thus

$$\max_{0\le t\le 2\pi}|x(t)| \le |x(\tau)| + \int_0^{2\pi}|\dot{x}|dt$$

$$\le C_{11} + (2\pi)^{\frac{1}{2}}|\dot{x}|_2$$

by Schwartz's inequality. From (4.12)

$$\max_{0\le t\le 2\pi}|x(t)| \le C_{11} + C_2 = C_3$$

and

$$|x|_\infty \le C_{13}$$

(4.17)

To obtain the fourth inequality in (3.11), multiply (3.1) by $x^{(4)}$ and integrate with respect to t from $t = 0$ to $t = 2\pi$.

$$\int_0^{2\pi} x^{(4)2}dt + \int_0^{2\pi} \lambda\varphi(\dddot{x})x^{(4)}dt + \int_0^{2\pi} a_2\ddot{x}x^{(4)}dt + \int_0^{2\pi} \lambda\theta(\dot{x})x^{(4)}dt + \int_0^{2\pi} f_\lambda(x)x^{(4)}dt = \int_0^{2\pi} \lambda P x^{(4)}dt$$

we use equation (4.11), (4.12), (4.17) and the boundedness of P and since φ, θ and f are continuous functions, there are constants C_{14}, C_{15} such that

$$\left|x^{(4)}\right|_2^2 \le C_{14}\left|x^{(3)}\right|_2\left|x^{(4)}\right|_2 + C_{15}\left|x^{(4)}\right|_2$$

$$\le C_{16}\left|x^{(4)}\right|_2$$

(4.18)

where $C_{16} = C_{15} + C_{14}C_7$

so that

$$\left|x^{(4)}\right|_2 \le C_{16}$$

from which because of (1.2) with r = 3 then

$$\left|x^{(3)}\right|_\infty \le (2\pi)^{\frac{1}{2}}$$

The estimates (4.11), (4.12), and (4.17) verify the inequality (3.11) and hence the proof theorem 1

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