Journal of the Nigerian Association of Mathematical Physics
Volume 16 (May, 2010), pp 215 – 220

Convergence of Crank-Nicolson-Galerkin discrete scheme for Stochastic hyperbolic equation in the Maximum-norm

Ignatius N. Njoseh

Department of Mathematics and Computer Science
Delta State University
Abraka, Nigeria
Corresponding Author: I. N. N. ; Tel. +2348035786279

Abstract

We studied the maximum-norm error estimate for the Galerkin finite element discretization in time of a stochastic wave equation by the Crank-Nicolson time stepping finite difference method. The error estimate was obtained by using the notions of rational function and resolvent estimates.

1.0 Introduction

This is an extension of [6] where the equation of study is the strongly damped stochastic wave equation

$$\begin{aligned} u_{tt} - \alpha\Delta u_t - \Delta u = dW \quad & in \ \ \Omega\times[0,T] \\ u(\cdot,t) = 0 \ \ on \ \ \partial\Omega & \\ u(0,\cdot) = x_0,\ u_t(0,\cdot) = x_1 \quad & in \ \ \Omega\times[0,T] \end{aligned} \qquad (1.1)$$

where Ω is a bounded domain in $\mathbf{R}^d$; $d \le 3$, with smooth boundary $\partial\Omega$. W is a Wiener process defined on a filtered probability space $(\Omega, F, P, \{F_t\}_{t\ge 0})$. $-\Delta = A$ denotes the Laplacian.

[5] and [6] studied the finite element method of the deterministic form of (1.1) and proved error estimates in the L_2- and maximum-norms. Other authors who have worked on the finite element method of (1.1) but applying the backward Euler method for its fully discrete scheme include [1], [2], [3], and [4] some references therein. In this work we extend the studies of [6] and prove maximum-norm error estimates for the Crank-Nicolson-Galerkin discrete scheme of (1.1).

2.0 Finite Element Analysis

First we formulate (1.1) as a first order system in time by setting $A = -\Delta$,

$$y = \begin{bmatrix} u \\ u_t \end{bmatrix} \ \ and \ \ B = \begin{bmatrix} 0 & -I \\ A & \alpha A \end{bmatrix} \qquad (2.1)$$

where the functions lie in the domain $D(A) = H_0^1 \cap H^2$, and thus the components of those in D(B) vanishes on $\partial\Omega$. We then have

$$\begin{aligned} y_t + By = dW, \qquad & t > 0 \\ y(0) = y_0 = (x_0, x_1)^T & \end{aligned} \qquad (2.2)$$

which has a unique solution

$$y(t) = e^{-tB} y_0 \text{ for any } y_0 \in L_2 \times L_2$$

with $E(t) = e^{-tB}$.

2.1 Galerkin Discretization

Let S_h be a family of finite elements spaces, which consists of continuous piecewise linear finite elements that vanish on the boundary with respect to the triangulation T_h of $\Omega_h \subset \Omega$ with boundary nodes of h on Ω_h on $\partial\Omega$. We also have that $\{S_h\} \subset H_0^1$. Then the semidiscrete problem is to find $u_h(t) \in S_h$, such that,

$$u_{h,tt} + \alpha A_h u_{h,t} + A_h u_h = P_h dW, \quad t > 0, \quad u_h(0) = x_{0h}, \; u_t(0) = x_{1h}$$

(2.3)

and (2.2) becomes

$$y_{h,t} + B_h y_h = P_h dW, \qquad t > 0, \; y_h(0) = y_{0h} = (x_{0h}, x_{1h})^T$$

(2.4)

For the fully discrete scheme, let r(z) be a rational function approximating e^{-z} to order p, i.e., such that

$$r(z) = e^{-z} + O(z^{p+1}), \quad for \;\; z \to 0, \; where \;\; p \geq 1$$

(2.5)

and which is A-stable, so that

$$|r(z)| \leq 1, \quad for \;\; \mathrm{Re}(z) \geq 0$$

(2.6)

We define an approximation $Y_n = (U_n, V_n)^T$ to the solution of (1.1) at time $t_n = nk$; where k is the time step, by

$$Y_n = r(kB_h) Y_{n-1}, \quad for \;\; n \geq 1, \quad with \;\; Y_0 = y_{0h} = (x_{0h}, x_{1h})^T$$

(2.7)

Applying the Crank-Nicolson approximations which corresponds to the rational function

$$r(z) = (1 - \frac{1}{2} z)/(1 + \frac{1}{2} z) \text{ gives}$$

$$(1 + \frac{1}{2} kB_h) Y_n = (1 - \frac{1}{2} kB_h) Y_{n-1}, \;\; for \; n \geq 1, \;\; with \;\; Y_0 = y_{0h} = (x_{0h}, x_{1h})^T$$

This is written as

$$(1 + (\frac{1}{2}\alpha k + \frac{1}{4}k^2) A_h) V_n = (1 - (\frac{1}{2}\alpha k + \frac{1}{4}k^2) A_h) V_{n-1} - kA_h U_{n-1} + \frac{1}{2}\int_{t_{n-1}}^{t_n} P_h dW(s),$$

$$U_n = U_{n-1} + \frac{1}{2} k (V_n + V_{n-1}), \quad for \;\; n \geq 1, \;\; U_0 = x_{0h}, V_0 = x_{1h}$$

when express in terms of the components of $Y_n = (U_n, V_n)^T$.

Eliminating V_n we find that, with $\bar{\partial} U_n = (U_n - U_{n-1})/2k$ and $\hat{U}_n = \frac{1}{2}(U_n + 2U_{n-1} + U_{n-2})$,

$$(\bar{\partial} U_n, \chi) + \alpha A(\tilde{\partial} U_n, \chi) + A(\hat{U}_n, \chi) = P_h \Delta \hat{W}, \qquad \forall \;\; \chi \in V_h, \;\; n \geq 2 \qquad (2.8)$$

$$U_0 = x_{0h}, \;\; \bar{\partial} U_1 = \frac{1}{2} x_{1h} + \frac{1}{2}(1 + \frac{1}{2}(\alpha k + k^2) A_h)^{-1} (x_{1h} - kA_h x_{0h} + \int_{t_{n-1}}^{t_n} P_h dW(s))$$

3.0 Maximum-norm error estimates

3.3.1 Resolvent Estimates

Here we consider A as a densely defined operator in the Banach space $C_0(\Omega)$ of continuous functions in $\overline{\Omega}$ vanishing on $\partial\Omega$, with norm

$$\|v\| = \max_{x\in\Omega}|v(x)| \tag{3.1}$$

and throughout this work, when the space is not specified as a subscript, the norm denotes the maximum-norm (3.1). The spectrum $\sigma(A)$ of A is located in a segment $\{\lambda : \lambda \geq C_0 > 0\}$ of the positive real axis, with C_0 the smallest eigenvalue of A. The following is then a special case of a result shown by Stewart (1974).

Lemma 3.1

For any $\varepsilon > 0$ there is a constant C = C_0 such that

$$\|(zI - A)^{-1}\| \leq C(1+|z|)^{-1}, \ for \quad z \notin \sum_{\varepsilon} = \{z : |\arg z|\} < \varepsilon \tag{3.2}$$

We require also the following results from Thomee and Wahlbin (2004) which we shall use in the proof of our main result here.

Lemma 3.2

Assume that A satisfies the resolvent estimate (3.2), and let B be the operator on $X \times X$ defined by

$$B = \begin{bmatrix} 0 & -I \\ A & \alpha A \end{bmatrix}, \quad with \ \alpha > 0$$

Then there exists $\theta \in (0, \frac{\pi}{2})$ such that

$$\|(zI - B)^{-1}\| \leq C(1+|z|)^{-1}, \ for \ \ z \notin \sum_{\theta}.$$

For the semidiscrete schemes [2] proved the following result which we shall apply in the proof of our main result.

Lemma 3.3

With $y_{0h} = (R_h x_0, R_h x_1)^T$ we have for the solutions of (1.1) and (2.3)

$$\|u_h(t) - u(t)\| + \|u_{h,t}(t) - u_t(t)\| \leq Ch^2 l_h^2 (\|Ax_0\| + \|Ax_1\|) + Ch^{\beta}\|A^{-(1-\beta)/2}\|_{L_2}, \ 0 \leq \beta \leq 1, \ t > 0$$

Using the notations for the semidiscrete problem

$$\tilde{y}_h(t) := \begin{bmatrix} I & 0 \\ 0 & A_h^{-1} \end{bmatrix} y_h(t) = F_h y_h(t) = \begin{bmatrix} u_h(t) \\ A_h^{-1} u_{h,t}(t) \end{bmatrix} \tag{3.3}$$

Where

$$\tilde{B}_h = \begin{bmatrix} 0 & -A_h \\ I & \alpha A_h \end{bmatrix}$$

we have a result which yields an error estimate for $u_h(t)$ of the same order as lemma 3.3 above under weaker regularity assumptions on x_1.

Lemma 3.4

With $y_{0h} = (R_h x_0, R_h x_1)^T$ we have for the solutions of (1.1) and (2.3)

$$\|u_h(t) - u(t)\| \le Ch^2 l_h^2 (\|Ax_0\| + \|x_1\|) + Ch^\beta \|A^{-(1-\beta)/2}\|_{L_2}, \ for \ \ t > 0$$

3.3.2 Convergence Result

First let us state this important results from Thomee and Wahlbin (2004).

Lemma 3.5

Let –B generate an analytic semigroup $E(t) = e^{-tB}$ in a Banach space X with norm $\|\cdot\|$, then

$$\|r(kB)^n v\| \le C\|v\| \quad (3.4)$$

and

$$\|(r(kB)^n - e^{-t_n B})v\| \le Ck^p \|B^p v\|, \ for \ \ v \in D(B^p), \ \ n \ge 0 \quad (3.5)$$

Lemma 3.6

We have for the L_2-projection,

$$\|P_h v - v\| \le Ch^2 l_h \|Av\| \quad for \ v \in C^2(\overline{\Omega}) \cap H_0^1(\Omega)$$

and, for the Ritz projection

$$\|R_h v - v\| \le Ch^2 l_h^2 \|Av\| \ , \ \ for \ \ v \in C^2(\overline{\Omega}) \cap H_0^1(\Omega)$$

The error estimate in the maximum norm for the fully discrete scheme is

Theorem 3.1

For $y_{0h} = (P_h x_0, P_h x_1)^T$ we have for the solutions of (1.1) and (2.8)

$$\|U_n - u(t_n)\| + \|V_n - u_t(t_n)\| \le Ch^2 l_h^2 (\|Ax_0\| + \|Ax_1\| + \|A^{-(1-\beta)/2}\|) + Ck^2 l_h^2 (\|A^2 x_0\| + \|A^2 x_1\| + \|A^{-(1-\beta)/2}\|), \ 0 \le \beta \le 1, \ t_n > 0 \quad (3.6)$$

and

$$\|U_n - u(t_n)\| \le C(h^2 l_h^2 + k^2)(\|Ax_0\| + \|Ax_1\| + \|A^{-(1-\beta)/2}\|),\ 0 \le \beta \le 1,\ t_n > 0$$
(3.7)

Proof:

Using Parseval's relation and Ito isometry, we have

$$\left\|P_h \Delta \hat{W}(s)\right\| \le Ch^2 \left\|A^{-(1-\beta)/2}\right\|$$

By lemma 3.5,

$$\begin{aligned}\|Y_n - y_h(t_n)\| &= \left\|r(kB_h)^n y_{0h} + P_h \Delta \hat{W}(s) - e^{-nkB_h} y_{0h}\right\| \\ &= \left\|(r(kB_h)^n - e^{-nkB_h}) y_{0h} + P_h \Delta \hat{W}(s)\right\| \\ &\le Ck^2(\|B_h^2 y_{0h}\| + \|A^{-(1-\beta)/2}\|)\end{aligned}$$

Next we show that

$$\|B_h^2 R_h y_0\| \le Cl_h^2(\|A^2 x_0\| + \|A^2 x_1\|)$$

With A_h as defined earlier, we have $A_h R_h v = P_h Av \quad for\ v \in D(A)$. Hence

$$B_h^2 R_h y_0 = \begin{pmatrix} -P_h Ax_0 - \alpha P_h Ax_1 \\ \alpha A_h P_h Ax_0 - P_h Ax_1 + \alpha^2 A_h P_h Ax_1 \end{pmatrix}$$

To estimate the norm of $A_h P_h Ax_0 = (A_h P_h A - A_h R_h A)x_0 + A_h R_h Ax_0$, we note that the last term equals $P_h A^2 x_0$. Using the global uniformity of the triangulation, i.e.,

$$\|A_h \phi\| \le Ch^{-2}\|\phi\|,\ \ for\ \phi \in S_h$$

and lemma 3.6, we have

$$\|A_h(P_h - R_h)Ax_0\| \le Ch^{-2}\|(P_h - R_h)Ax_0\| \le Cl_h^2\|A^2 x_0\|$$

Hence

$$\|A_h P_h Ax_0\| \le Cl_h^2 \|A^2 x_0\|$$

Since $\|P_h Ax_0\| \le C\|Ax_0\| \le C\|A^2 x_0\|$, and similar argument for terms in x_1. This proves (3.6).

For the prove (3.7), by lemma 3.5,

$$\|\tilde{Y}_n - \tilde{y}_h(t_n)\| \le Ck^2(\|\tilde{B}_h^2 \tilde{y}_{0h}\| + \|A^{-(1-\beta)/2}\|)$$

and using the notations of (3.3), we have

$$\tilde{B}_h^2 \tilde{y}_{0h} = \begin{pmatrix} 0 & -A_h \\ I & \alpha A_h \end{pmatrix}\begin{pmatrix} 0 & -1 \\ 1 & \alpha I \end{pmatrix}\begin{pmatrix} R_h x_0 \\ R_h x_1 \end{pmatrix}\begin{pmatrix} -P_h Ax_0 - \alpha P_h Ax_1 \\ \alpha A_h P_h Ax_0 - R_h x_1 + \alpha^2 P_h Ax_1 \end{pmatrix}$$

and

$$\|R_h x_1\| \le \|x_1\| + Ch^2 l_h^2 \|Ax_1\| \le C\|Ax_1\|$$

We have

$$\|\tilde{B}_h^2 \tilde{y}_{0h}\| = Ch^2 l_h^2 (\|Ax_0\| + \|Ax_1\|)$$

This concludes the proof of the theorem.

4.0 Conclusive Remark

The strong convergence rate in both the spatial and time steps can be obtained if the finite element solution derived on a very fine mesh is considered as the true solution and the finite element solutions obtained on the less finer meshes are compared with this numerically obtained "true solution". This is due to the fact that the true solution to the SPDE (1.1) itself is a random process and is not explicitly available.

References

[1]. Njoseh I. N. and Ayoola E. O. (2008); Finite Element Method for a strongly damped stochastic wave equation driven by space-time noise; *Journal of Mathematical Sciences*, Vol. 19, No. 1: 61-71.

[2]. Njoseh I. N. and Ayoola E. O. (2009); Maximum-Norm error estimate for a strongly damped stochastic wave equation; *Journal of Mathematical Sciences*, Vol. 20, No. 1: 21-30.

[3]. Njoseh I. N. (2009a); On the rate of strong convergence of the semi-discretized solution of hyperbolic stochastic equation; *Journal of Mathematical Sciences*, Vol. 20, No. 3: 301-306.

[4]. Njoseh I. N. (2009b); On the strong convergence rate of the fully discretized solution of hyperbolic stochastic equation; *Journal of Mathematical Sciences*, Vol. 20, No. 3: 287-292.

[5]. Larsson S., Thomee V. and Wahlbin L. B. (1991); Finite element methods for a strongly damped wave equation, *IMA J. Numer. Anal*. 11: 115-142.

[6]. Thomee V. and Wahlbin L. B. (2004); Maximum-norm estimates for finite element methods for a strongly damped wave equation. *Applied Mathematics Report*, BIT 44: 165-179.

[7]. Stewart B. (1974); Generation of Analytic Semigroups by strongly elliptic operators, *Trans. Amer. Math. Soc*. 199: 141-161.