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L_2 norm error estimates for Crank-Nicolson-Galerkin procedure for stochastic wave equation

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Abstract

In this study, we presented the fully discrete scheme for the Galerkin finite element analysis of a stochastic wave equation driven by space-time white noise. The finite element discretization in time was obtained by applying the Crank-Nicolson time stepping finite difference method. Error estimates in the L_2 norm was obtained by using L_2-projections of the initial data as starting values.

1.0 Introduction

Our equation of study in this paper is the strongly damped stochastic wave equation driven by space-time white noise

$$u_{tt} + \alpha A u_t + Au = \sigma(u)dW \quad in \;\; \Omega\times[0,T]$$
$$u(\cdot,t) = 0 \;\; on \;\; \partial\Omega$$
$$u(0,\cdot) = \phi,\; u_t(0,\cdot) = \varphi \quad in \quad \Omega\times[0,T]$$

(1.1)

where Ω is a bounded domain in $\mathbf{R}^d$; $d \leq 3$, with smooth boundary $\partial\Omega$, $A = -\Delta$ is selfadjoint, positive definite linear elliptic partial operator of second order with smooth coefficients, Δ denotes the Laplacian. W is a Wiener process defined on a filtered probability space $\left(\Omega, F, P, \{F_t\}_{t\geq 0}\right)$ with covariance operator $Q: L_2(\Omega) \to L_2(\Omega)$ and the operator $\sigma = I$.

Finite element analysis of both the semidiscrete and completely discrete schemes of the deterministic form of (1.1) (i.e., σ (u) = 0) were studied by [5] in the L_2 norms and by [6] in the maximum norm. [3] gave the rate of strong convergence of the semi-discretized solution of (1.1) with $\alpha = 0$ in the L_2-norm and in [4] studied the finite element error estimate of the fully discrete scheme which was obtained by the backward Euler time stepping method. Also [1] discussed the finite element method for (1.1) and proved error estimates for both the semidiscrete and fully discrete solutions. Here the fully discrete scheme was obtained by applying the backward Euler time stepping finite difference method. The authors extended their work and in [2] proved error estimates for the finite element solution of (1.1) in the maximum norm.

In this study, we wish to extend the results in [4] and [2] by applying the Crank-Nicolson time stepping finite difference method to discretize completely (1.1) in time and obtain convergence results for the approximate solution.

2.0 Finite Element Analysis

In discussion the finite element analysis of (1.1), we shall first formulate it as a first order system in time, i.e., as a 2 × 2 system, by setting

$$x = \begin{bmatrix} u \\ u_t \end{bmatrix} \quad and \quad B = \begin{bmatrix} 0 & -I \\ A & \alpha A \end{bmatrix}$$

(2.1)

where the functions lie in the domain $D(A) = H_0^1 \cap H^2$, and thus the components of those in D(B) vanishes on $\partial\Omega$. Hence (1.1) becomes

$$x_t + Bx = dW, \qquad t > 0$$
$$x(0) = x_0 = (\phi, \varphi)$$

(2.2)

Let $E(t) = e^{-tA}$, denote the analytic semigroup of H generated by A, then –B generates an analytic semigroup in the Hilbert space $L_2 \times L_2$, and (2.2) has a unique solution

$$x(t) = e^{-tB} x_0 \text{ for any } x_0 \in L_2 \times L_2$$

2.1 Discretization in Space and Time

Let V_h be a family of finite elements spaces, which consists of continuous piecewise linear finite elements that vanish on the boundary with respect to the triangulation T_h of $\Omega_h \subset \Omega$ with boundary nodes of h on Ω_h on $\partial\Omega$. We also have that $\{V_h\} \subset H_0^1$. From the standard finite element method, the spatially semidiscrete problem of (1.1) is to find $u_h(t) \in V_h$, such that,

$$u_{h,tt} + \alpha A_h u_{h,t} + A_h u_h = P_h dW, \quad t > 0, \quad u_h(0) = \phi_h,\ u_t(0) = \varphi_h \qquad (2.3)$$

where ϕ_h and φ_h are approximations of V_h of ϕ and φ, P_h denotes the L_2-projection onto V_h . And as in (2.1) we have we have

$$x_h = \begin{bmatrix} u_h \\ u_{h,t} \end{bmatrix} \quad and \quad B_h = \begin{bmatrix} 0 & -I \\ A_h & \alpha A_h \end{bmatrix}$$

(2.4)

our semidiscrete problem now becomes

$$x_{h,t} + B_h x_h = P_h dW, \qquad t > 0, \quad x_h(0) = x_{0h} = (\phi_h, \varphi_h)^T$$

(2.5)

We finally turn to the fully discrete scheme which will be obtained by discretization of (2.5) in time. Thus, let r(z) be a rational function approximating e^{-z} to order p, i.e., such that

$$r(z) = e^{-z} + O(z^{p+1}), \quad for \quad z \to 0, \ where \ p \ge 1$$

(2.6)

and which is A-stable, so that

$$|r(z)| \le 1, \quad for \ \ \mathrm{Re}(z) \ge 0$$

(2.7)

We then define an approximation $X_n = (U_n, V_n)^T$ to the solution of (1.1) at time $t_n = nk$; where k is the time step, by

$$X_n = r(kB_h) X_{n-1}, \quad for \ \ n \ge 1, \quad with \quad X_0 = x_{0h} = (\phi_h, \varphi_h)^T$$

(2.8)

In this work we consider the case of the Crank-Nicolson approximations. This corresponds to the rational function $r(z) = (1-\frac{1}{2}z)/(1+\frac{1}{2}z)$ which gives

$$(1+\frac{1}{2}kB_h)X_n = (1-\frac{1}{2}kB_h)X_{n-1}, \quad for\ n \geq 1, \ with \ X_0 = x_{0h} = (\phi_h, \varphi_h)^T$$

Express in terms of the components of $X_n = (U_n, V_n)^T$ these may be written as

$$(1+(\frac{1}{2}\alpha k + \frac{1}{4}k^2)A_h)V_n = (1-(\frac{1}{2}\alpha k + \frac{1}{4}k^2)A_h)V_{n-1} - kA_h U_{n-1} + \frac{1}{2}\int_{t_{n-1}}^{t_n} P_h dW(s),$$

$$U_n = U_{n-1} + \frac{1}{2}k(V_n + V_{n-1}), \quad for \ \ n \geq 1$$

$$U_0 = \phi_h, V_0 = \varphi_h$$

Eliminating V_n we find that, with $\bar{\partial}U_n = (U_n - U_{n-1})/2k$ and $\hat{U}_n = \frac{1}{2}(U_n + 2U_{n-1} + U_{n-2})$,

$$(\bar{\partial}U_n, \chi) + \alpha A(\tilde{\partial}U_n, \chi) + A(\hat{U}_n, \chi) = P_h \Delta \hat{W}, \qquad \forall \quad \chi \in V_h, \quad n \geq 2 \tag{2.9}$$

$$U_0 = \phi_h, \ \ \bar{\partial}U_1 = \frac{1}{2}\varphi_h + \frac{1}{2}(1+\frac{1}{2}(\alpha k + k^2)A_h)^{-1}(\varphi_h - kA_h\phi_h + \int_{t_{n-1}}^{t_n} P_h dW(s))$$

3.0 Convergence results

If we define $P_h : L_2 \to V_h$ and $R_h : H_0^1 \to V_h$ as L_2 and Ritz projections respectively, then we have $\|P_h v\| \leq C\|v\| \quad and \quad \|R_h v\| \leq C\|v\|$

such that by the standard finite element analysis

$$\|P_h v - v\| \leq Ch^{\beta}\|v\|_{\beta} \quad for \ v \in H^{\beta} \cap H_0^1 \tag{3.1}$$

and

$$\|R_h v - v\| \leq Ch^{\beta}\|v\|_{\beta} \quad for \ v \in H^{\beta} \cap H_0^1 \tag{3.2}$$

To present our error estimate, we need the following results from Njoseh and Ayoola (2008).

Lemma 3.1

Let $\phi, \varphi \in \dot{H}^{\beta}$ and assume that $\|\phi_h - \phi\| \leq Ch^{\beta}\|\phi\|_{\beta} \ and \ \|\varphi_h - \varphi\| \leq Ch^{\beta}\|\varphi\|_{\beta}$

Then there is a constant C such that

$$\|u_h(t) - u(t)\| + \|u_{h,t}(t) - u_t(t)\| \leq Ch^{\beta}(|\phi|_{\beta} + |\varphi|_{\beta} + \|A^{-(1-\beta)/2}\|_{L_2})$$

and

$$\|x_h(t) - x(t)\| \leq Ch^{\beta}(|x_0|_{\beta\times\beta} + \|A^{-(1-\beta)/2}\|_{L_2\times L_2})$$

Lemma 3.2

Let $E_h(t) = e^{-tA_h}$ be the semigroup generated by A_h. Let $F_h(t) = E_h(t)P_h - E(t)$. Then

$$\|F_h(t)\|_{L_\infty([0,T];H)} \leq Ch^{\beta}|v|_{\beta}, \quad for \ \ v \in \dot{H}^{\beta}, \ 0 \leq \beta \leq 1$$

and

$$\|F_h(t)\|_{L_2([0,T];H)} \leq Ch^{\beta}|v|_{\beta-1}, \quad for \ \ v \in \dot{H}^{\beta-1}, \ 0 \leq \beta \leq 2$$

Theorem 3.1 Assume that $r(\lambda)$ satisfies (2.6) and (2.7). Let $\phi, \varphi \in \dot{H}^{s}, \quad s = \max(\beta, p)$, and assume that $\|\phi_h - \phi\| \leq Ch^{\beta}\|\phi\|_{\beta} \ and \ \|\varphi_h - \varphi\| \leq Ch^{\beta}\|\varphi\|_{\beta}$ (3.3)

Then for the solution of (1.1) and of (2.9) we have

$$\|U^n - u(t_n)\| + \|V^n - u_t(t_n)\| \leq Ch^{\beta}(|\phi|_{\beta} + |\varphi|_{\beta} + \|A^{-(1-\beta)/2}\|_{L_2}) + Ck^p(|\phi|_p + |\varphi|_p + \|A^{-(1-p)/2}\|_{L_2}), \quad t_n \geq 0 \tag{3.4}$$

This is equivalent to

$$\|X_n - x(t)\| \le Ch^{\beta}(|x_0|_{\beta} + \|A^{-(1-\beta)/2}\|_{L_2}) + Ck^{p}(|x_0|_{p} + \|A^{-(1-p)/2}\|_{L_2}),\ t_n \ge 0 \qquad (3.5)$$

Proof: We have $\|X_n - x(t)\| \le \|X_n - x_h(t_n)\| + \|x_h(t_n) - x(t)\|$

By lemma 3.1 $\|x_h(t) - x(t)\| \le Ch^{\beta}(|x_0|_{\beta\times\beta} + \|A^{-(1-\beta)/2}\|_{L_2\times L_2})$

Hence we need to estimate

$$X_n - x_h(t_n) = [r(kB_h)^n x_{0h} + r(kB_h)^n P_h \Delta\hat{W}(t) - e^{-(t_n B_h)} x_{0h}]$$

$$= [r(kB_h)^n x_{0h} - e^{-(t_n B_h)} x_{0h} + r(kB_h)^n P_h \Delta\hat{W}(t)]$$

We assume $x_{0h} = P_h x_0$, where P_h is also the orthogonal projection of $L_2 \times L_2$ onto $V_h \times V_h$ and if we let

$$F_n = F_n(kB_h) = r(kB_h)^n - e^{-(t_n B_h)}$$

such that

$$\|X_n - x_h(t_n)\| \le \underbrace{\|F_n(kB_h)P_h x_0\|}_{I} + \underbrace{\|r(kB_h)^n P_h \Delta\hat{W}(t)\|}_{II}$$

The bounds for (I) we will get from Lemma 3.1 and Lemma 3.2

$$\|I\| = \|F_n(kB_h)P_h x_0\| \le Ch^{\beta}(|\phi|_{\beta} + |\varphi|_{\beta})$$

From lemma 3.1, 3.2, Ito isometry and Paserval's relation, we obtain

$$\|II\| = \|r(kB_h)^n P_h \Delta\hat{W}(t)\| = \left\|r(kB_h)^n \int_{t_{n-1}}^{t_n} P_h dW(s)\right\|$$

$$\le C\left\|\int_{t_{n-1}}^{t_n} P_h dW(s)\right\|$$

$$= C\left\|P_h \sum_{j=1}^{\infty} \eta_j^{\frac{1}{2}} (\beta_j(t_n) - \beta_j(t_{n-1}))\xi_j\right\|$$

$$\le Ch^{\beta}\|A^{-(1-\beta)/2}\|_{L_2}^2 + Ck^{p}\|A^{-(1-p)/2}\|_{L_2}^2$$

This completes the proof ▪

Conclusive Remark 3.1

Since ϕ_h, $\varphi_h \in \dot{H}^{\beta}$, and both ϕ_h and φ_h are arbitrary initial approximations of optimal order, the error bound (3.4) shows that it is sufficient to have initial data in $\dot{H}^{\beta}$ in order to have optimal order convergence, for both u_h and $u_{h,t}$, uniformly for $t \ge 0$.

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