Journal of the Nigerian Association of Mathematical Physics
***Volume* 16 (May, 2010), pp 199 – 204**

An Efficient Algorithm for Solving the Telegraph Equation.

[1]***Olayiwola , M. O,*** [1]***Gbolagade, A .W &*** [2]***Adesanya , A. O***
[1]Department of Mathematical & Physical Sciences
College of Science, Engineering & Technology
Osun State University, Osogbo, Nigeria.
[2]Department of Pure & Applied Mathematics
Ladoke Akintola University of Technology
Ogbomoso, Nigeria.
Corresponding author: Tel. M. O. O. ; Tel. +2348028063936

Abstract

In this work, we propose a numerical scheme to solve telegraph equations using modified variational iteration method. The numerical results are compared with analytical solutions to confirm the efficiency of the method.

Keyword: lagrange multiplier, series method, variational iteration method, telegraph equation.

1.0 Introduction

In the present work we are dealing with the numerical approximation of the following second order partial differential equation:

$$\alpha \frac{\partial^2 u}{\partial t^2} + \beta \frac{\partial u}{\partial t} + \gamma u = \frac{\partial^2 u}{\partial x^2} + f(x,t) \tag{1.0}$$

Where α, β, γ are constants related to resistance, inductance, capacitance and conductance of the cable.

This equation appears in the propagation of electrical signals along a telegraph line, digital image processing, telecommunication, signals and systems [2,3,20].

Equation (1.0), referred to as second order telegraph equation with constant coefficients, models mixture of between diffusion and wave propagation by introducing a term that accounts for effects of finite velocity to a standard heat or mass transport equation [16].

The existence of time-bounded solutions of nonlinear bounded perturbations of telegraph equation with Neumann boundary conditions has recently been considered in [12]. The approach is based upon Galerkin method combined with the use of some Lyapunov functional.

Finite difference methods are known as the first techniques for solving partial differential equations [13-15]. Even though these methods are very effective for solving various kinds of partial differential equations, conditional stability of explicit finite difference procedures and the need to use large amount of CPU time limits the applicability of these methods.

The basic motivation of this paper is the application of Modified initial guess Variational Iteration Method (MigVIM) for solving telegraph equations. The Variational Iteration Method (VIM) was developed and formulated by He J H for solving various problems [4-7,9)]. The method has been extensively useful for diversified initial and boundary value problems and has potential to cope with the versatility of the complex nature of physical problems [2-3].

This article presents a new algorithm to solve the second order telegraph equation using Modified initial guess Variational Iteration Method.

2.0 Variational Iteration Method (VIM)

To illustrate the basic concept of the technique, we consider the following general nonlinear partial differential equation.

$$Lu(x,t)+Ru(x,t)+Nu(x,t)=g(x,t) \tag{2.1}$$

where L is a linear time derivative operator, R is a linear operator which has partial derivative with respect to x, N is a nonlinear operator and g is an inhomogeneous term. According to VIM, we can construct a correct fractional as follows:

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda\left[Lu_n+R\tilde{u}_n+N\tilde{u}_n-g\right]\, d\tau \tag{2.2}$$

where λ is a Lagrange multiplier which can be identified optimally via variational iteration method. The subscript n denote the nth approximation, $\tilde{u}_n$ is considered as a restricted variation i.e, $\delta\tilde{u}_n=0$. The successive approximation $u_{n+1}, n\geq 0$ of the solution u will be readily obtained upon using the determined Lagrange multiplier and any selective function u_0, consequently, the solution is given by:

$$u=\lim_{n\to\infty} u_n \tag{2.3}$$

3.0 Derivation of λ for VIM

Consider equation (3.1) of the form

$$mu''(x,t)+Ru(x,t)+Nu(x,t)=g(x,t) \tag{3.1}$$

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \lambda\left[mu''(x,\tau)+R\tilde{u}_n(x,\tau)+N\tilde{u}_n(x,\tau)-g(x,\tau)\right]d\tau \tag{3.2}$$

Making (3.2) stationary, we have:

$$\delta u_{n+1}=\delta u_n+\delta\left(\lambda m u'_n-mu_n\lambda'+\int mu_n\lambda'' d\tau\right) \tag{3.3}$$

This yields the following stationary condition

$$1-m\lambda'\big|_{\tau=t}=0 \tag{3.4a}$$

$$\lambda m\big|_{\tau=t}=0 \tag{3.4b}$$

$$m\lambda''=0 \tag{3.4c}$$

Solving (3.4a-3.4c), we have

$$\lambda=\frac{1}{m}(\tau-t) \tag{3.5}$$

Equation (3.2) becomes

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \frac{1}{m}(\tau-t)\left[mu''_n(x,\tau)+R\tilde{u}_n(x,\tau)+N\tilde{u}_n(x,\tau)-g(x,\tau)\right]d\tau \tag{3.6}$$

In Modified initial guess Variational Iteration Method (MigVIM), (3.6) becomes

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^t \frac{1}{m}(\tau-t)\left[m u_n''(x,\tau)+R\tilde{u}_n(x,\tau)+N\tilde{u}_n(x,\tau)-g(x,\tau)\right]d\tau$$

(3.7)

Where $u_0(x,t)=u(x_0,t)+u_x(x_0,t)x+k_i(t)x^i$, $\quad 2(i)n \quad , \quad x_0=0$

(3.8)

where $k_i(t)$ can be found by substituting for $u_0(x,t)$ in (1.0) at $x=0$.

4.0 Numerical Applications

In this section we present MigVIM for solving the telegraph equations with different values of α, β, γ. α, β *and* γ are constants related to resistance, inductance, capacitance and conductance of the cable.

Example 4.1: Consider (1.0) with $\alpha=1, \beta=1, \gamma=-1, f(x,t)=0$

$$\frac{\partial^2 u}{\partial t^2}+\frac{\partial u}{\partial t}-u=\frac{\partial^2 u}{\partial x^2}$$

(4.1)

With boundary conditions $u(0,t)=e^{-2t}, \frac{\partial u(0,t)}{\partial x}=e^{-2t}$

and the initial condition $u(x,0)=e^{x}, \frac{\partial u(x,0)}{\partial t}=-2e^{x}$

the correction functional becomes

$$u_{n+1}(x,t)=u_n(x,t)+\int_0^x(\tau-x)\left[\frac{\partial^2 u}{\partial \tau^2}-\frac{\partial^2 u}{\partial t^2}-\frac{\partial u}{\partial t}+u\right]d\tau$$

(4.2)

$$u_1(x,t)=(1+x+\frac{x^2}{2})e^{-2t}+\int_0^x(\tau-x)\begin{bmatrix} e^{-2t}-4(1+\tau+\frac{\tau^2}{2})e^{-2t}+ \\ 2(1+\tau+\frac{\tau^2}{2})e^{-2t}+ \\ (1+\tau+\frac{\tau^2}{2})e^{-2t} \end{bmatrix}d\tau$$

$$=(1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\ldots)\ e^{-2t}$$

(4.3)

As $n\to\infty$

$$u(x,t)=\sum_{n=0}^{\infty}\frac{x^n}{n!}e^{-2t}$$

(4.4)

In a closed form:

$$u(x,t)=e^{x-2t}$$

(4.5)

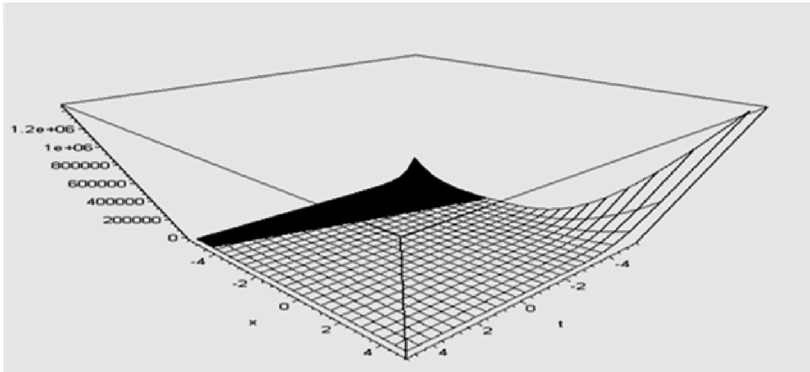

Fig I.: 3D Graph of Example 5.1

Example 4.2: Consider (1.0) with $\alpha = 1, \beta = 4, \gamma = 4, f(x,t) = 0$

$$\frac{\partial^2 u}{\partial t^2} + 4\frac{\partial u}{\partial t} + 4u = \frac{\partial^2 u}{\partial x^2} \tag{4.6}$$

With boundary conditions $u(0,t) = 1 + e^{-2t}, \dfrac{\partial u(0,t)}{\partial x} = 2$

The initial condition $u(x,0) = 1 + e^{2x}, \dfrac{\partial u(x,0)}{\partial t} = -2$

The correction functional becomes;

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^x (\tau - x)\left[\frac{\partial^2 u}{\partial \tau^2} - \frac{\partial^2 u}{\partial t^2} - 4\frac{\partial u}{\partial t} - 4u\right] d\tau \tag{4.7}$$

At n=0 we have

$$u_1(x,t) = e^{-2t} + 1 + 2x + 2x^2 + \int_0^x (\tau - x)\begin{bmatrix} \dfrac{\partial^2 u}{\partial \tau^2} - \dfrac{\partial^2 u}{\partial t^2} \\ -4\dfrac{\partial u}{\partial t} - 4u \end{bmatrix} d\tau \tag{4.8}$$

This gives

$$u_1(x,t) = e^{-2t} + 1 + 2x + 2x^2 + \frac{8x^3}{3} + \frac{8x^4}{12} + .. + \tag{4.9}$$

$$= e^{-2t} + 1 + \frac{(2x)}{1!} + \frac{(2x)^2}{2!} + \frac{(2x)^3}{3!} + \frac{(2x)^4}{4!} + \ldots + = e^{-2t} + \sum_{n=0}^{\infty} \frac{(2x)^n}{n!} \tag{4.10}$$

In a closed form:

$$u(x,t) = e^{-2t} + e^{2x} \tag{4.11}$$

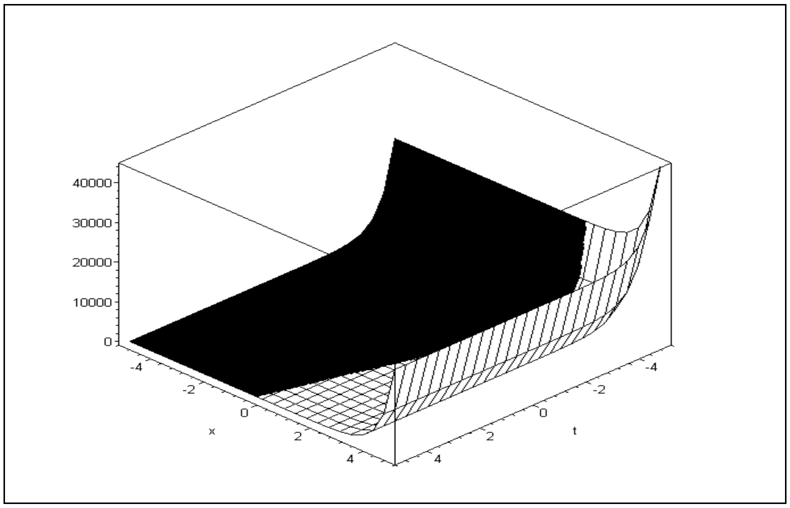

Fig II.: 3D Graph of Example 5.2

Example 4.3: Consider (1.0) with $\alpha = 1, \beta = 1\ \gamma = 1,\ f(x,t) = x^2 + t - 1$

$$\frac{\partial^2 u}{\partial t^2} + \frac{\partial u}{\partial t} + u = \frac{\partial^2 u}{\partial x^2} + x^2 + t - 1$$

(4.12)

The initial condition $u(x,0) = x^2, \dfrac{\partial u(x,0)}{\partial t} = 1,\ \ 0 \le x \le 1$

The correction functional becomes;

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t (\tau - x) \left[\begin{array}{l} \dfrac{\partial^2 u}{\partial \tau^2} + \dfrac{\partial^2 u}{\partial \tau} + u \\ -\dfrac{\partial^2 u}{\partial x^2} - x^2 \quad - \tau + 1 \end{array} \right] d\tau \tag{4.13}$$

At n=0 we have

$$u_1(x,t) = x^2 + t + \int_0^x (\tau - x) \left[\begin{array}{l} \dfrac{\partial^2 u}{\partial \tau^2} + \dfrac{\partial^2 u}{\partial \tau} + u \\ -\dfrac{\partial^2 u}{\partial x^2} - x^2 \quad - \tau + 1 \end{array} \right] d\tau \tag{4.14}$$

This gives

$$u_1(x,t) = x^2 + t \tag{4.15}$$

which is the exact solution.

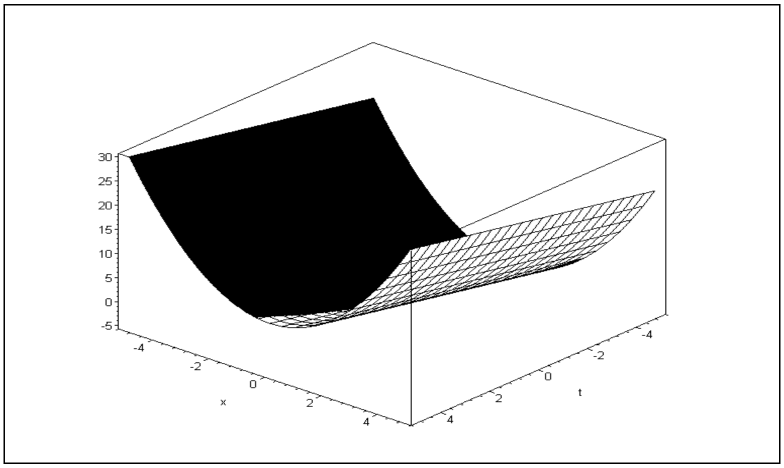

Fig III.: 3D Graph of Example 5.3

5.0 Conclusion

The MigVIM is a powerful method. It has provided an efficient potential for the solution of physical applications modeled by partial differential equation. The main goal of this article has been to derive an approximation to the solution of telegraph equation. The main advantage of the new method over VIM is in the improvement on the initial guess which in turn reduces the number of iterations when compared with [17,18]. This method can be extended to solve nonlinear partial differential equations of physical significance.

REFERENCES

[1] Abbasbandy, S. (2007). Numerical solutions of nonlinear Klein-Gordon equation by variational iteration method. Internat. J. Numer. Meth. Engrg. 70, pp. 876-881.

[2] Abdou, M. A. and Soliman, A. A. (2005). Variational iteration method for solving Burger's and coupled Burger's equations. J. Comput. Appl. Math. 181, pp. 245-251.

[3] Abdou, M. A. and Soliman, A. A. (2005). New applications of variational iteration method. Phys. D 211 (1-2), pp (2005), 1-8.

[4] He, J. H. (1999). Variational iteration method, A kind of non-linear analytical technique, some examples. Internat. J. Nonlin. Mech. 34 (4), pp. 699-708.

[5] He, J. H. (2000). Variational iteration method for autonomous ordinary differential systems. Comput. Math. Appl. 114 (2-3), pp. 115-123.

[6] He, J. H. and Wu, X. H (2006). Construction of solitary solution and compaction-like solution by variational iteration method. Chaos, Solitons & Fractals, 29 (1) pp. 108-113.

[7] He, J. H. (2007). Variational iteration method — some recent results and new interpretations. J. Comput. Appl. Math. 207, pp. 3-17.

[8] He, J. H. and Wu, X. H. (2007). Variational iteration method: New development and applications. Comput. Math. Appl. 54, pp. 881-894.

[9] He, J. H. (2007). The variational iteration method for eighth-order initial boundary value problems, Phys. Scr. 76(6), pp. 680-682.

[10] He, J. H. (2008). An elementary introduction of recently developed asymptotic methods and nanomechanics in textile engineering. Int. J. Mod. Phys. B 22 (21), pp. 3487-4578.

[11] He, J. H. (2006). Some asymptotic methods for strongly nonlinear equation. Int. J. Mod. Phys. 20 (20)10, pp. 1144-1199.

[12] J.M.Alonso, J. Mawhin, and R.Ortega (1999), Bounded solutions of second order semilinear evolution equations and applications to the telegraph equation, J Math Pures Appl 78, 49–63.

[13] M.Dehghan (2005), Parameter determination in a partial differential equation from the overspecified data, Math Comput Model 41,196–213.
[14] M.Dehghan (2006), Implicit collocation technique for heat equation with non-classic initial condition, IntJ Non-Linear Sci Numer Simul 7, 447–450.
[15] M.Dehghan (2006), Finite difference procedures for solving a problem arising in modeling and design of certain optoelectronic devices, Math Comput Simulation 71, 16–30.
[16] M.S.El-Azab and M.El-Gamel (2007), A numerical algorithm for the solution of telegraph equations,Appl Math Comput 190, 757–764
[17] Medi Dehghan, Ali Shokri.(2007). A Numerical Method for Solving the Hyperbolic Telegraph Equation. Wiley InterScience. Pp1080-1093.
[18] Mohyud-Din, S. T., Noor, M. A. and Noor, K. I. (2009). Variational Iteration Method for Solving Telegraph Equations. Appl & Applied Math Vol.4 No.1, pp 114-121
[19] Wazwaz, A. M. (2006). The modified decomposition method for analytic treatment of differential equations. Appl. Math. Comput. Pp. 165-176.
[20] Wazwaz, A. M. (2002). Partial differential equations, methods and applications. A. A. Balkema Publishers.
[21] Xu, L. (2007). The variational iteration method for fourth-order boundary value problems. Chaos, Solitons & Fractals, in press.