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Oscillatory Flow of solutions of Random and Stochastic Delay Differential Equations with Multi-Variable Delays Generated by Noise Perturbation

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Abstract

The paper studies the influence of noise perturbation on the oscillatory behaviour of solutions of the stochastic version of the first order delay differential equation

$$\left.\begin{aligned} x^{|}(t) &= ax(t) + \sum_{i=1}^{n} b_i x(t - r_i(t)),\ t \ge 0 \\ x(t) &= \phi(t),\ t \in [-r,\ 0] \end{aligned}\right\} \qquad (*)$$

where $0 < r_i(t) \le r$ ***, having several variable delays. By means of the Lisei type conjugation relation, it is proved that the addition of a multiplicative noise perturbation of Ito type to the delay differential equation (*) will generate oscillation in the solution of the resulting stochastic delay differential equation (SDDE) irrespective of the length of the delays. This can never happen in the non-stochastic case (*) which can admit a non-oscillatory solution due to the absence of noise.***

1.0 Introduction

Stochastic delay differential Equations (SDDEs) and their deterministic counterparts are adequate mathematical models of processes studied in theoretical physics, chemical technology, population dynamics and economics. New applications which involve these classes of equations continue to arise with increasing frequency in the modeling of diverse phenomena. This is why delay differential equations are objects of intensive investigation.

In the recent decades, the number of the investigations of the oscillatory and non-oscillatory behaviour of the solutions of delay differential equations is constantly growing. For instance, in monographs [9] and [8] published in 1987 and 1992 respectively, the properties of oscillation and asymptotic behaviour of different classes of deterministic delay differential equations were systematically studied. Although the literature concerned with oscillation and non-oscillation of solutions of deterministic delay differential equations is quite extensive, it appears that the contribution of noise perturbation to their oscillatory characteristics has not received much attention. It is well-known that oscillation in solutions of delay differential equations (DDEs) (both deterministic and stochastic) is caused by the presence of the delay or retarded arguments [see 2, 8, 9]. The first article which studied the influence of noise on the almost sure oscillatory properties of scalar linear SDDE with a single variable delay is that of [2]. The authors established that the presence of noise would induce oscillation in the solution of the SDDE under certain assumptions. For the investigation of noise contribution to the oscillatory behaviour of solutions of first order SDDEs with fixed delays, we refer to the papers of [5,6].

In the present paper, we study the contribution of noise perturbation to the oscillatory behaviour of solutions of a more general scalar first order SDDE with multi-variable delays of the form:

$$
\left.\begin{aligned}
dX(t) &= \left[aX(t) + \sum_{i=1}^{n} b_i X(t - r_i(t))\right]dt + \mu X(t)dB(t), \;\; t \ge 0 \\
X(t) &= \phi(t), \qquad\qquad t \in [-r,\, 0]
\end{aligned}\right\}
$$

where $a,\, b_i,\; 0 \le r_i(t) \le r,\;$ *for* $i = 1, 2, \ldots, n,\;\; r_i(t) > 0$ *are continuous functions also called multi-variable delays which satisfy for* $i = 1, 2, \ldots, n$

$$\lim_{t \to \infty} r_i(t) = 0$$

(1.1)

$r_i(t) \le r,$ *where* $-r = \inf_{t \ge 0}\{t - r_i(t)\}$ *and* $t \to t - r_i(t)$ *is* increasing, μ is a positive number which measures the average impact of the fast fluctuating internal noise and $\{B(t)\}_{t \ge 0}$ is a one-dimensional Brownian motion.

By solution of the SDDE (1.1), we mean a stochastic process $\{X(t)\}_{t \ge 0}$ defined on the probability space $(\Omega,\, F,\, P)$ and with continuous sample paths, which satisfies Eq. (1.1) almost certainly as well as its initial function. We shall carry out the study by using some oscillatory results in the oscillatory theory of deterministic delay differential equations, a method of solution transformation in [11] and a technique originally employed in [2]. We will always contrast the oscillatory and non-oscillatory results of the solution of the SDDE (1.1) with those of a comparable classical delay differential equation with several variable delays, which satisfies the same initial function of the type

$$x^{|}(t) = ax(t) + \sum_{i=1}^{n} b_i x(t - r_i(t))$$

(1.2)

By solution of (1.2), we mean a unique function $x \in C([t_0 - \rho, \infty), \Re)$ which satisfies (1.2) for $t_0 \ge 0,$ *where* $\rho = \max_{1 \le i \le n}(r_i(t))$. Under certain assumptions, the SDDE (1.1) and its corresponding classical DDE (1.2) are suitable models for describing the rate of production/distribution of several products from a central source as manufactured products are distributed to different destinations.

The paper is organized in three sections. In the second section, we present the preliminaries which contain certain assumptions and lemmas. The third section contains the main result.

2.0 Preliminaries:

Definition 1:

The solution x(t) of the classical delay differential equation (1.2) defined on the interval $[T_x, \infty)$ *and satisfies* $Sup\{|x(t)| : t > T\} > 0,$ *for every* $T \ge T_x$, *that is* $|x(t)| \ne 0$ on any infinite interval $[T_x, \infty)$ is called a regular or non-trivial solution. This is also true of the solution of the SDDE.

The non-trivial solution x(t) of the DDE (1.2) is said to be eventually or almost certainly positive if there exists $t_1 > 0,$ *such that* $x(t) > 0,$ *for all* $t \ge t_1$. The non-trivial solution x(t) of the DDE (1.2) is said to be eventually or almost certainly negative if there exists $t_1 > 0,$ *such that* $x(t) < 0,$ *for all* $t \ge t_1$.
The solution x(t) of the DDE (1.2) (also of the SDDE) is called an equilibrium or zero solution if $x(t) \equiv 0,$ (*also* $X(t) = 0$) whenever the initial function $\phi(t) \equiv 0$

Definition 2:

As it is customary for the deterministic DDE, a non-trivial solution x(t) of the DDE (1.2) is said to be oscillatory around the equilibrium solution if it has arbitrarily large zeros. That is, for $t \ge t_0$ there exists a sequence of zeros $\{t_n : x(t_n) = 0\}$ *of* $x(t)$ such that $\underset{n\to\infty}{Lim}\, t_n = +\infty$, otherwise x(t) is said to be non-oscillatory.

In 2005, Appleby and Buckwar [2] introduced this definition into stochastic processes as below:

Definition 3:

A non-trivial continuous function $y:[t_0,\infty)\to\Re$ is called oscillatory if the set $Z_y = \{t \ge t_0 : y(t) = 0\}$ *satisfies* $SupZ_y = \infty$. If a function is not oscillatory, it is said to be non-oscillatory. The authors extended this definition to stochastic processes in the following intuitive manner:

A stochastic process $\{X(t)\}_{t\ge 0}$ defined on the probability space (Ω, F, P) and with continuous sample paths is said to be almost surely (a.s.) oscillatory if there exists a subset $\Omega^* \subseteq \Omega$ *with* $P[\Omega^*] = 1$ such that for all $w \in \Omega^*$, the path X(.,w) is oscillatory. Otherwise it is said to be non-oscillatory.

2.1. Strategy:

The method of our proof involves building a conjugation relation between the solution $\{X(t)\}_{t\ge 0}$ of the SDDE (1.1) and a continuously differentiable solution $\{Z(t)\}_{t\ge 0}$ of a non-autonomous random delay differential equation

$$Z^{|}(t) = -\sum_{i=1}^{n} P_i(t) Z(t - r_i(t)) \tag{2.1}$$

Where the coefficients $P_i(t) \ge 0$ are continuous non-negative random functions. Moreover, the coefficients are defined on some subset $\Omega^* \subseteq \Omega$, *for* $w \in \Omega$ by

$$P_i(t,w) = \begin{cases} -b_i e^{-\lambda r_i(t)} e^{(-\mu(B(t)(w) - B(t = r_i(t)(w))))}, & t > \bar{t} \\ -b_i e^{-\lambda t - \mu B(t)(w)}, & t \le \bar{t} \end{cases} \tag{2.2a}$$

Where $\lambda = \left(a - \frac{\mu^2}{2}\right)$, $\bar{t} = \inf\{t \ge 0 : t - r_i(t) = 0\}$, $w \in \Omega$ *such that for all* $t \ge \bar{t}$, $t - r_i(t) \ge 0$.

By property (2.2a), the P_i depend upon the increments of a standard Brownian motion $\{B(t)\}_{t\ge 0}$. The large deviations in these increments ensure that the P_i are sufficiently large to generate oscillation in the random delay differential equation (2.1).

Also by property (2.2a), of the $P_{is,}$ we observe that

$$\int_t^{t+r_i(t)} P_i(s)ds = \int_t^{t+r_i(t)} - b_i \exp\left(-\left(a - \frac{\mu^2}{2}\right)\right) r_i(s) \exp(-\mu(B(s) - B(s - r_i(s))))ds$$

$$\ge -b_i \max\left(1, \exp\left(-\left(a - \frac{\mu^2}{2}\right)\right)\right) r_i(t) \int_{t-r_i(t)}^{t} \exp(-\mu(B(s) - B(s - r_i(s))))ds$$

It is observed (See [2]) that the event $\Omega^* \subseteq \Omega$ as defined above exists eventually whenever

$$\underset{t\to\infty}{Lim}\, Sup \int_{t-r_i(t)}^{t} \exp(-\mu(B(s) - B(s - r_i(s))))ds = \infty \tag{2.2b}$$

Lemma 1 [11]:

Consider the stochastic functional differential equation driven by a continuous helix special semi-martingale of the kunita type

$$\left.\begin{array}{l} dX(t) = H(t, X(t), X_t)dt + M(dt, X(t)),\ t \ge 0 \\ X(0) = v \in \Re^d,\ X_0 = \psi \in L^2([-r,0], \Re^d) \end{array}\right\}$$
$$where\ X_t(., w)(u) = X(t+u, w),\ u \in [-r,0], t \ge 0,\ w \in \Omega$$

(2.3)

Also consider a random functional differential equation of the form

$$\left.\begin{array}{l} dY(t) = G(t, Y(t), Y_t)dt,\ t \ge 0 \\ X(0) = v \in \Re^d, Y_0 = \psi \in L^2([-r,0], \Re^d) \end{array}\right\}$$

(2.4)

Let $\{\Lambda(t,.)\}_{t\in\Re}$ be a random stationary coordinate change or a process satisfying the following properties:

$(i)\ \{\Lambda(t,.)\}_{t\in\Re}$ *is a continuous* $C^{k+1.\in}$ *semi* $=$ *martingale* $(with\ 0 <\in< \delta)$ *such that for all* $w \in \Omega, \Re^d \ni v \to \Lambda(t,v) \in \Re^d$ *is a* C^{k+1} *diffeomorphism of* $\Re^d$ *and* $\{\Gamma(t,.)\}_{t\in\Re}$ *is a continuous* $C^{k,\in}$ *semi* $-$ *martingale* $(with\ 0 <\in< \delta)$

(ii) *For all* $t \ge s, v \in \Re^d$ *and a.e.* $w \in \Omega$

$$\Lambda(t,u) = \Lambda(s,v) + \int_s^t M(du, \Lambda(u,v)) + \int_s^t \Gamma(u,v)du$$

(iii) *The processes* $\{\Lambda(t,.)\}_{t\in\Re}$, $\{\Gamma(t,.)\}_{t\in\Re}$ *are perfectly stationary, i.e,*

$$\Lambda(t,vw) = \Lambda(0.v, \theta(t,w))\ and\ \Gamma(t,vw) = \Gamma(0.v, \theta(t,w))\ for\ all\ t \in \Re, v \in \Re^d,\ w \in \Omega$$

Let $\{X(t)\}_{t\ge\Re}$ be the solution of (2.3). Also let $\{Y(t)\}_{t\ge\Re}$ be the solution of (2.4). Then the following conjugation relation holds:

$$X(t,., w) = \Lambda(0,., \theta(t,w)) \mathrm{o} Y(t,., w) \mathrm{o} \Lambda^{-1}(t,., w)$$

The difficulty encountered in obtaining direct necessary and sufficient conditions for oscillation of the solution of the SDDE (1.1) due to the disturbances of the noise perturbation is overcome by recalling for use on a path-wise basis (i.e. for each $w \in \Omega$) certain existing oscillatory and non-oscillatory results in the deterministic theory of oscillation which apply directly to equation (2.1). The following result pertaining to oscillatory solutions is found in [9] (Theorem 2.7.2).

Proposition 1:

Assume that $r_i(t) = r_i\ \ for\ i \in I_n = 1,2,\ldots n$ and non-decreasing and there exists $r_i^* < r_i,\ for\ i \in I_n\ such\ that$

$$\underset{t\to\infty}{Lim}\inf \int_t^{t+r_i^*} P_i(s)ds > 0$$

Then there exists N > 0 such that $\dfrac{x(t)}{x(t-r)} \ge N\ \ and\ \ for\ any\ r \ge 0$ where x(t) is any positive solution.

In addition, suppose that

$$P_{ij}^{*} = \underset{t\to\infty}{Lim}\inf \int_{h(t)}^{t} P_i(s)ds > e^{-1} \quad for \quad i, j = 1,2,.......,n \tag{2.5}$$

Then every solution of

$$x^{|}(t) = -\sum_{i=1}^{n} P_i(t) x_i(t - r_i(t)) \tag{2.6}$$

oscillate

We also have results concerning non-oscillation of solutions of (2.6). The following is a special case of the result found in Elabbasy et al [7] (Lemma 1.4). It originally appeared in Li [10] (Lemma 2).

Proposition 2:

Assume that $r_i(t) = r_i$, $i = 1,2,.........,n$. If

$$\int_{t}^{t+r_i} \sum_{i=1}^{n} P_i(s) ds < 1, \quad i = 1,2,...,n \tag{2.7}$$

Then equation (2.6) has an eventually positive solution x(t) and hence non-oscillatory.

In the main result, we establish that once the function $t \to t - r_i(t)$ satisfies the condition (2.5) of proposition 1 for negative feedback intensities, then the SDDE (1.1) has an oscillatory solution. We also comment that by the condition (2.7) of proposition 2, the corresponding classical delay differential equation can admit a non-oscillatory solution, when the magnitudes of the delays are sufficiently small.

2.2 Solution Transformation:

The proof of the existence of oscillatory and non-oscillatory solutions of the SDDE (1.1) depends on the following transformation: We introduce a nowhere differentiable random bijective process $\{\eta(t,w)\}_{t\ge -r}$ which is strictly positive and satisfies properties (i) – (iii) of Lemma 1. Using the stationary coordinate change, we transform the solution of the SDDE into a conjugation relation with a process which has continuously differentiable sample paths and solves the random delay differential equation (2.1). To this end, we let $\{X(t)\}_{t\ge 0}$ be the solution of the equation

$$X(t) = \phi(0) + \int_{0}^{t}\left(aX(s) + \sum_{i=1}^{n} b_i X(s - r_i(s))\right) ds + \int_{0}^{t} \mu X(s) dB(s) \tag{2.8}$$

and let $\{Z(t)\}_{t\ge 0}$ *be the solution of the equation*

$$Z(t) = Z(0) - \int_{0}^{t} \sum_{i=1}^{n} P_i(s) Z(s - r_i(t)) ds \tag{2.9}$$

Also let $\{\eta(t)\}_{t\ge -r}$ be a random process satisfying properties (i) – (iii) of Lemma 1. We see by the Lemma that the following conjugation relation holds for all $w \in \Omega$

$$X(t,w) = Z(t - r_i(t), w) \,\mathrm{o}\, \eta(t - r_i(t), w) \,\mathrm{o}\, \eta^{-1}(t,w) \tag{2.10}$$

The conjugation relation (2.10) is important in our analysis because it builds a relationship between the process Z(t) and the solution X(t) of the SDDE (1.1). by this, the zeros of the process Z corresponds to the zeros of the process X. Hence it is sufficient to analyze the oscillatory behaviour of the process Z in order to obtain information about the oscillatory properties of the solution X of the SDDE.

The approach is of great benefit in the sense that there is a pair of deterministic oscillatory and non-oscillatory results (as in proposition 1 and proposition 2 in this case) that apply directly to the sample paths of the solution Z(t) of the random delay differential equation (2.1).

3.0 The Main Result:

In the main result, we establish that for negative feedbacks from the multi-delays and any selection of initial datum, oscillation in the solution of the SDDE (1.1) is generated by the presence of the multiplicative noise perturbation. This occurs in the SDDE even if the comparable deterministic DDE has a non-oscillatory solution.

Theorem 1:

Assume that the $b_i < 0$ *for* $i = 1,2,\ldots\ldots, n.$ *Let* $r_i(t) > 0$ be continuous functions which satisfy $\lim_{t\to\infty} r_i(t) = 0,\ h(t) = t - r_i(t)$ is strictly monotone increasing on $[0,\infty)$.. Then the SDDE (1.1) has an oscillatory solution on $[0,\infty]$, almost certainly for any choice of initial datum ϕ.

Proof:

By the relation (2.10) and the properties of the process $\eta(t)$, *the set* $W = \{t \geq 0 : X(t) = 0\}$ can only satisfy $SupW = \infty$ if and only if the set

$W^* = \{t \geq 0 : Z(t) = 0\}$ *satisfies* $SupW^* = 0$. Since $\{X(t)\}_{t\geq 0}$ is defined on an appropriate probability triple (Ω, F, P), we define for $t \geq 0,\ w \in \Omega$

$$P_i(t,w) = -b_i \eta(t - r_i(t), w)\eta^{-1}(t,w)$$

Then the $P_i(.)$ are almost surely positive continuous functions on $[0,\infty]$. Moreover, Z satisfies the equation

$$Z^{|}(t,w) = -\sum_{i=1}^{n} P_i(t,w) Z(t - r_i(t), w),\ t \geq 0$$

(3.1)

Hence Z satisfies condition (2.5) of proposition 1 and thus almost certainly oscillatory. If not, we may assume that equation (2.1) has a non-oscillatory solution, $\{Z(t)\}_{t\geq 0}$ which is in fact non-decreasing for the sake of contradiction. Without loss in generality, let Z(t) be a positive solution of the form

$$Z(t) = \exp\left(-\int_{t_0}^{t} \alpha(s) ds\right)$$

(3.2)

Substituting (3.2) into (2.6), we have

$$\alpha(t) - \sum_{i=1}^{n} P_i(t) \exp\left(\int_{t - r_i(t)}^{t} \alpha(s) ds\right) = 0$$

(3.3)

which may have no solution $\alpha(t)$ if all the solutions of (2.6) are oscillatory. By the first condition of proposition 1, α_i^*, $i = 1, 2, ..., n$ are bounded. From (3.3) we have for j = 1, 2, 3,….,n

$$\alpha_j^* = \underset{t\to\infty}{Lim}\inf \int_{t-r_j(t)}^{t} \alpha(s)ds = \underset{t\to\infty}{Lim}\inf \int_{t-r_j(t)}^{t} \sum_{i=1}^{n} P_i(s)\int_{t-r_i(s)}^{s} \alpha(\theta)d\theta)ds$$
$$\geq \sum_{i=1}^{n} P_{ij} \exp \alpha_i^*$$

(3.4)

It is true from the properties of exponential function that for $y \geq 0. \max y\exp(-y) = e^{-1}$

Hence from (3.4), we obtain

$$P_{ij} \leq \alpha_i^* \exp(\alpha_i^*) \leq e^{-1}$$

(3.5)

which contradicts (2.5). Hence Z is almost certainly oscillatory. Suppose that there exists a subset $\Omega^* \subseteq \Omega$ such

that

$$\Omega^* = \left\{ w \in \Omega : \underset{t\to\infty}{Lim}\inf \int_{h(t)}^{t} P_i(s)ds > e^{-1} \right\} \qquad with\ P\left[\Omega^*\right]$$

(3.6)

Then as P_i and $h(t) = t - r_i(t)$ satisfy the hypothesis of proposition 1, it follows that the trajectory Z(., w) is oscillatory and so the path X(.,w) is oscillatory and hence as the subset $\Omega^* \subseteq \Omega$ exists, it follows that the solution X(t) of the SDDE (1.1) is almost surely oscillatory.

Remark 3.1:

We observe that in the stochastic delay differential equation (1.1), under theorem 1, the important factor that generates oscillation in the solution is equation (2.2b) which must always occur in the stochastic case as a result of the presence of the multiplicative noise perturbation. If the $r_i(t)$ are small enough, the integral in (2.2b) is made so small that condition (2.7) of proposition 2 holds in the deterministic case (1.2) and at that instance, a non-oscillatory solution occurs in (1.2). However, this cannot happen in the SDDE (1.1) as result of equation (2.2b). Hence the multiplicative noise sustains oscillation in the solution of the SDDE (1.1) even when the non- stochastic equation (1.2) has a non-oscillatory solution.

The following result shows that the crucial factor (2.2b) which ensures oscillation in the solution must always hold in the stochastic case. It is a special case of the result found in [2] (Lemma 1).

Lemma 2:

Assume that $r_i(t) \in C(\Re_+, \Re_+)$ *and that* $0 < r_i(t) \leq r < \infty$. *If* $\mu \neq 0$, then

$\underset{t\to\infty}{Lim\ Sup} \int_{t-r_i(t)}^{t} \exp(-\mu(B(s) - B(s - r_i(s))))ds = \infty$ almost certainly holds.

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