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On Typical Elastic Problem of Green's Function For Rectangular Strip Using The Method of Separation Of Variables.

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Abstract

Green's function provides a wide range of methods for solving elastic problems. In this paper, our focus is on the method of the separation of variables. Here, the Green's function of the Neumann's problem for Poisson's equation is adopted. Unlike the problems of the full and half strips, the problem of the rectangular strip admits known formulae of the hyperbolic functions in addition to conventional trigonometric functions.

Keywords: Green's function, separation of variables, elastic materials.

1.0 Introduction

Many methods can be used to construct Green's functions. One of such methods is the separation of variables. As in, [1], it is shown that the incompressibility constraint allows one to describe standard thermoelastic effects. This approach is recently supported by the works of [4]. The basic idea of the last two authors is to decompose the motion of deformation into two parts: one due to traction free uniform heating and the other due to isothermal mechanical loading.

But in our method of separation of variables, the Green's function is being determined in a holistic form. This method had earlier been used by the same author [7, 8] to solve the problems of the full and half strips of elastic materials and is hereby extended to rectangular strip problem as shown in this paper. The major difference with the already solved problems is that hyperbolic functions are admissible in the computational procedure.

Seremet [6] solved the rectangular strip problem with reflection method, where the Green's function was constructed for Laplace's equation. But in our approach here, we solve the problem by constructing Green's function for Poisson's equation.

2.0 Mathematical Formulations.

We define the Green's function of the Neumann's problem for Poisson's equation in the form

$$\nabla_x^2 \; G(x,\xi) = \left(\frac{1}{a_1 a_2} \right) - \delta(x,\xi) \tag{2.1}$$

at the following intervals ($0 \le x_1 \le a_1, 0 \le x_2 \le a_2$) under the following conditions;

$$\frac{\partial G}{\partial x_1} = 0 \;\; ; \; x_1 = 0, a_1 ; \;\; 0 \le x_2 \le a_2 \tag{2.2}$$

$$\frac{\partial G}{\partial x_2} = 0 \;\; ; \; x_2 = 0, a_2 \;\; ; \;\; 0 \le x_1 \le a_1 \tag{2.3}$$

2.1 Computational Procedure

The solution is sought by using the method of separation of variables and by defining the following trigonometric series (see[2])

$$G = a_0 + \sum_{n=1}^{\infty} a_n \cos v_1 x_2 + \sum_{m=1}^{\infty} b_m \sin v_1 x_2 \tag{2.4}$$

where the coefficients a_0, a_n , b_m are the functions of the variable x_i . The boundary conditions of this problem simplify this series and reduce it to the following form

$$G = a_0 + \sum_{n=1}^{\infty} a_n \cos v_1 x_2 , \;\; v_1 = \frac{n\pi}{a_2} , \; n = 1,2,3, ... \tag{2.5}$$

Using equation (2.5) in equation (2.1), we get,

$$a_0'' + \sum_{m=0}^{\infty} \left[a_m'' - v_1^2 a_m \right] \cos v_1 x_2 = 1/a_1 a_2 - \delta(x_1 - \xi_1)(x_2 - \xi_2); \;\; \text{and} \;\; \lambda_1^2 = v_1^2 + \mu_1^2 \tag{2.6}$$

providing that $\delta(x - \xi) = \delta(x_1 - \xi_1)\, \delta(x_2 - \xi_2)$ in the method of separation of variables. We now integrate (2.6) with respect to x_2 and note that the following integral

$$\int_0^{a_2} a_0'' \, d x_2 = a_0'' a_2 \; ; \int_0^{a_2} \delta(x_1 - \xi_1)\, \delta(x_2 - \xi_2) \, dx_2 = \delta(x_1 - \xi_1) \tag{2.7}$$

are non zero.

Thus, we obtain the following ordinary differential equation

$$a_0'' = a_1^{-1} a_2^{-1} - a_2^{-1} \delta(x_1 - \xi_1) \tag{2.8}$$

and boundary conditions

$$a_0' \; (x_1 = 0) = 0; \qquad a_0' \, (x_1 = a_1) \tag{2.9}$$

to determine the function a_0 (x_1)

To construct Green's functions for 1D differential equations, we use the standard constructing procedure (see [2]). The general solution to the influence function is sought in the following form

$$a_0(x_1)=\begin{cases}\dfrac{x_1^2}{2a_1a_2}+c_1x_1+c_2, & x_1\le\xi_1\\ \dfrac{x_1^2}{2a_1a_2}+k_1x_1+k_2, & x_1\ge\xi_1\end{cases} \tag{2.10}$$

Then from the conditions of conjugality at the point $x_1 = \xi_1$,

$$a_0(x_1=\xi_1-0)=\bar{a}_n(x_1=\xi_1+0) \tag{2.11}$$

$$a_0'(x_1=\xi_1-0)-a_0'(x_1=\xi_1+0)=a_2^{-1}, \tag{2.12}$$

we obtain the following set of simultaneous algebraic equations

$$(c_1-k_1)\xi_1+(c_2-k_2)=0; \quad (c_1-k_1)=a_2^{-1}, \tag{2.13}$$

Then $$(c_2-k_2)=a_2^{-1}\xi_1; \quad (c_1-k_1)=a_2^{-1} \tag{2.14}$$

The boundary conditions give

$$a_0'(x_1=0)=0\Rightarrow c_1=0; \quad a_0'(x_1=a_1)=0\Rightarrow k_1=a_2^{-1}. \tag{2.15}$$

Therefore, in accordance with the obtained values of the coefficients

$$c_1=0,\ k_1=-a_2^{-1},\ c_2=k_2-a_2^{-1}\xi_1,$$

we come to the following expression for the function

$$a_0(x_1,\xi_1)=\begin{cases}\dfrac{x_1^2+\xi_1^2}{2a_1a_2}-a_2^{-1}+b, & x_1\le\xi_1\\ \dfrac{x_1^2+\xi_1^2}{2a_1a_2}-a_2^{-1}+b, & x_1\ge\xi_1\end{cases} \tag{2.16}$$

Then, having multiplied the equation (2.6)

$a_0''+\sum_{m=0}^{\infty}\left[a_m''-v_1^2a_m\right]\cos v_1x_2=1/a_1a_2-\delta(x_1-\xi_1)(x_2-\xi_2);$ and $\lambda_1^2=v_1^2+\mu_1^2$ by

$\cos v_2x_2,\ v_2=s\pi/a_2$, where s = 1,2,3,……, we take the integral with respect to the variable x_2, provided the orthogonality property of the trigonometric functions has been taken into account and come to the differential equations

$$a_m''-v_1^2a_m=-2a_2^{-1}\delta(x_1-\xi_1)\cos v_1\xi_2 \tag{2.17}$$

and the boundary conditions

$$a_m'(x_1=0)=0, \quad a_m'(x_1=a_1)=0, \tag{2.18}$$

to determine the function $a_m(x_1)$. Here, the following integrals are non-zero:

$$\int_0^{a_2}\cos v_2x_2\cos v_1x_2\,dx_2=\begin{cases}0, & v_1\ne v_2\\ a_0/2, & v_1=v_2\end{cases} \tag{2.19}$$

$$\int_0^{a_2}\delta(x_1-\xi_1)\,\delta(x_2-\xi_{21})\cos v_2x_2\,dx_2=\delta(x_1-\xi_1)\cos v_1\xi_2;\ v_1=v_2 \tag{2.20}$$

Then, by denoting $a_m=2a_2^{-1}\bar{a}_m\cos v_1\xi_2$, we reduce the above equations to the respective

boundary –value problem aimed at determining the function $\bar{a}_m$:

$$\bar{a}_m''-v_1^2\bar{a}_m=-\delta(x_1-\xi_1); \quad \bar{a}_m'(x_1=0); \quad \bar{a}_m'(x_1=a_1)=0 .$$

(2.21)

To solve this problem we use a standard method of Green's function construction for 1D differential equation (see[6]). The general solution to this differential equation is

$$\bar{a}_n = \begin{cases} c_1 e^{-v_1 x_1} + c_2 e^{v_1 x_1} \quad ; \quad x_1 \le \xi_1 \\ k_1 e^{-v_1 x_1} + k_2 e^{v_1 x_1} \quad ; \quad x_1 \ge \xi_1 \end{cases}$$

(2.22)

Then, from the conditions of conjugality at the point $x = \xi$

$$\bar{a}_m(x = \xi_1 - 0) = \bar{a}_m(x = \xi_1 + 0); \qquad \bar{a}_m(x = \xi_1 - 0) - \bar{a}'_m(x = \xi_1 + 0) = 1$$

(2.23)

we obtain the following set of simultaneous algebraic equations

$$(c_1 - k_1)\, e^{-v_1 \xi_1} + (c_2 - k_2)\, e^{v_1 \xi_1} = 0; \; -v_1\left[(c_1 - k_1)\, e^{-v_1 \xi_1} - (c_2 - k_2)\, e^{v_1 \xi_1}\right] = 1$$

(2.24)

Then, from the boundary conditions, it follows that

$$\bar{a}_m(x = 0) \Rightarrow c_2 - c_1 = 0; \quad \bar{a}_m(x_1 = a_1) \Rightarrow k_2\, e^{v_1 a_1} - k_1\, e^{-v_1 a_1} = 1$$

(2.25)

and for the coefficients, we obtain

$$c_1 = c_2 = \frac{e^{-v_1 \xi_1}}{2 v_1} + \frac{e^{v_1 \xi_1} + e^{-v_1 \xi_1}}{2 v_1 (e^{2 a_1} - 1)}; \quad k_1 = c_1 + \frac{e^{v_1 \xi_1}}{2 v_1}; \quad k_2 = k_1\, e^{-2 v_1 a_1}.$$

(2.26)

After some transformation, this makes it possible to deduce the expression for the Green's function as:

$$\bar{a}_m(x_1, \xi_1) = \frac{1}{2 v_1}\left(e^{v_1 (x_1 - \xi_1)} + e^{-v_1 (x_1 + \xi_1)}\right) + \frac{\operatorname{ch} v_1 x_1 \operatorname{ch} v_1 \xi_1}{v_1 e^{v_1 a_1} \operatorname{sh} v_1 a_1}; \quad x_1 \le \xi_1$$

(2.27)

where the known formulae for the hyperbolic functions were used

$$\operatorname{sh}\alpha = 2^{-1}(e^{\alpha} - e^{-\alpha}), \qquad \operatorname{ch}\alpha = 2^{-1}(e^{\alpha} + e^{-\alpha})$$

(2.28)

We note that the expression for the function $\bar{a}_m(x_1, \xi_1)$ at $x_1 \ge \xi_1$ follows from the given expression provided the change of the values x1 and ξ_1 is allowed for. So, the requirements for the symmetry of the Green's function, $\bar{a}_m(x_1, \xi_1) = \bar{a}_m(\xi_1, x_{11})$ is satisfied. Thus, after introducing the notation,

$$\bar{a}_m(x_1, \xi_1) = \Pi_m(x_1, \xi_1) = \frac{1}{2 v_1}\left(e^{v_1 (x_1 - \xi_1)} + e^{-v_1 (x_1 + \xi_1)}\right) + \frac{\operatorname{ch} v_1 x_1 \operatorname{ch} v_1 \xi_1}{v_1 e^{v_1 a_1} \operatorname{sh} v_1 a_1}; \quad x_1 \le \xi_1$$

(2.29)

we deduce the expression for the Green's function of the initial –value problem in a form of infinite series

$$G(x, \xi) = b + (2 a_1 a_2)^{-1}(x_1^2 + \xi_1^2) - a_2^{-1}\xi_1 + \frac{2}{a_2}\sum \Pi_m(x_1, \xi_1) \cos v_1 x_2 \cos v_1 \xi_2, \; x_1 \le \xi_1$$

(2.30)

However, by making use of the known sum (see [2, 3, and 5])

$$\sum_{n=1}^{\infty} \frac{p^n}{n} \cos n\alpha = -\ln\sqrt{1 - 2p\cos\alpha + p^2}\; ; \; p^2 < 1, \;\; 0 \le \alpha \le 2\pi; \quad \text{or} \;\; p^2 \le 1, \;\; 0 \le \alpha \le 2\pi$$

(2.31)

as well as the trigonometric formulae

$$\cos\beta \cos\gamma = \frac{\cos(\beta - \gamma) + \cos(\beta + \gamma)}{2}$$

(2.32)

we are able to sum ordinary infinite series.

The final expression for the Green's function of the Neuman's problem for the rectangular strip is written in the form of [3];

$$G^{(2)}(x,\xi) = b - \frac{\xi_1}{a_2} + \frac{x_1^2 + \xi_1^2}{2a_1 a_2} - \frac{1}{2\pi}\ln EE_1E_2E_{12}$$
$$+ \frac{2}{a_2}\sum_{m=1}^{\infty}\frac{\operatorname{ch} v_1 x_1 \operatorname{ch} v_1 \xi_1}{v_1 e^{v_1 a_1} \operatorname{sh} v_1 a_1}\cos v_1 x_2 \cos v_1 \xi_2;\ b = \text{constant};\ v_1 = \frac{m\pi}{a_2} \quad (2.33)$$

where the functions E, E_1, E_2, E_{12} are determined by the expressions

$$E = \sqrt{1 - 2e^{\frac{\pi}{a_2}(x_1 - \xi_1)}\cos \pi/a_2\,(x_2 - \xi_2) + e^{\frac{2\pi}{a_2}(x_1 - \xi_1)}} \quad (2.34)$$

$$E_1 = \sqrt{1 - 2e^{-\frac{\pi}{a_2}(x_1 + \xi_2)}\cos \pi/a_2\,(x_2 - \xi_2) + e^{\frac{-2\pi}{a_2}(x_1 + \xi_1)}} \quad (2.35)$$

$$E_2 = \sqrt{1 - 2e^{\frac{\pi}{a_2}(x_1 - \xi_1)}\cos \pi/a_2\,(x_2 + \xi_2) + e^{\frac{2\pi}{a_2}(x_1 - \xi_1)}} \quad (2.36)$$

$$E_{12} = \sqrt{1 - 2e^{-\frac{\pi}{a_2}(x_1 + \xi_2)}\cos \pi/a_2\,(x_2 + \xi_2) + e^{\frac{-2\pi}{a_2}(x_1 + \xi_1)}} \quad (2.37)$$

(Note: ch and sh stand for cos and sine hyperbolic respectively.)

4.0 Conclusion

In this paper, we used the methods of separation of variables in Green's function to solve the problem of the rectangular strip. Earlier, the same method had been used by this author to solve the problems of the full strip and half strip. The significant feature of the full strip was that the shearing stresses were completely zero. In the half strip problem, a feature of the conjugality conditions became prominent and the solution to this problem admits trigonometry functions in its principal stresses unlike that of the full strip.

In the rectangular strip problem, unlike the full and half-strips, the solution is amendable to the methods of hyperbolic functions. In particular, the introduction of the hyperbolic function preserves the symmetry of the Green's function and satisfies the conditions of conjugalities. Also, as a a great departure from the two previous problems, the rectangular strip problem admits the Green's function of Neumann's problem as well as the Green' function of the second kind for Poisson's equation, a situation which was not possible in the full and half strip problems. The advantage of this is that the solutions to the rectangular problem can explicitly be presented in hyperbolic functions instead of the conventional trigonometric functions. Consequently, the shearing stresses are symmetrical.

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