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A Generalised Mixed Boundary Value Problem For A Cylinder Under Anti- Plane Strain

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Abstract

An anti-plane strain problem leading to a generalized mixed boundary value problem is solved using the method of conformal transformation and Mellin transform to obtain the only non-vanishing displacement field $\psi(r,\theta)$. the strain and stress fields are calculated.

1.0 Introduction

The anti-plane strain problem plays a useful role as a pilot problem, within which various aspects of solutions in solid mechanics may be examined in a particularly simple setting. In recent years, considerable attention has been paid to the analysis of anti-plane strain deformations within the context of various constitutive theories (linear and non linear) of solid mechanics [1], [2]. Most of these studies were concerned for the cracked infinite-strip problem, whereas only few considered a problem involving non-straight boundaries [3], [4], [5], [6]. Georgiadis [7] in his paper, studied the elastic anti-plane shear problem of a by cylinder within a cracked elliptical cross-section from the viewpoint of fracture mechanics. Successive conformal mappings and techniques of the analytic function theory were employed in order to obtain the crack-plane stresses. Ibem [8] in his work studied an anti-plane strain problem for an elastic cylinder leading to a mixed boundary value problem which is solved using the method of conformal mapping and Mellin transform to obtain the displacement, strain and stress.

In this project work, we have studied a some what related problem to that of [8]. This is an anti-plane strain problem for an elastic cylinder leading to a generalized mixed boundary value problem. Obviously, our problem is more generalized than Ibem's problem, which is a special case of the problem we studied. However, by using the conformal mapping technique and Mellin transform we succeeded in obtaining an integral solution for the mixed boundary value problem from which the strain and stress fields can be calculated.

2.0 Mixed Boundary Value Problem

Ibem in his project work solved the problem

$$\nabla^2\psi(r,\theta) = 0, \qquad r \le a$$
$$\psi(a,\theta) = 0, \qquad -\pi < \theta < 0 \tag{2.1}$$
$$\psi_r(a,\theta) = \frac{\gamma}{\mu} \qquad 0 < \theta < \pi$$

which is generalized as

$$\nabla^2\psi(r,\theta) = 0 \quad , \qquad r \le a$$
$$\psi(a,\theta) = f(\theta), \qquad -\frac{\pi}{2} < \theta < \frac{\pi}{2} \tag{2.2}$$
$$\psi_r(a,\theta) = g(\theta) \qquad \frac{\pi}{2} < \theta < \frac{3\pi}{2}$$

Physically the problem is represented by fig. 1

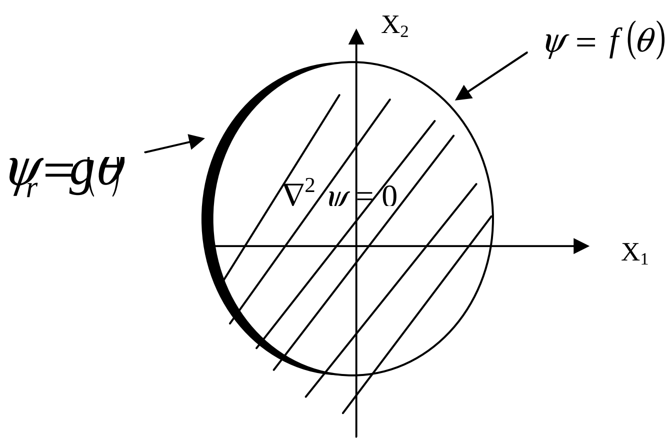


Fig. 1: Diagram illustrating problem domain.

The problem (2.2) can be transformed into a mixed boundary value problem in the upper half plane $\text{Im} \quad w = 0$ by conformal transformation using the mapping function.

$$w = -i\,\frac{z - ai}{z + ai} \tag{2.3}$$

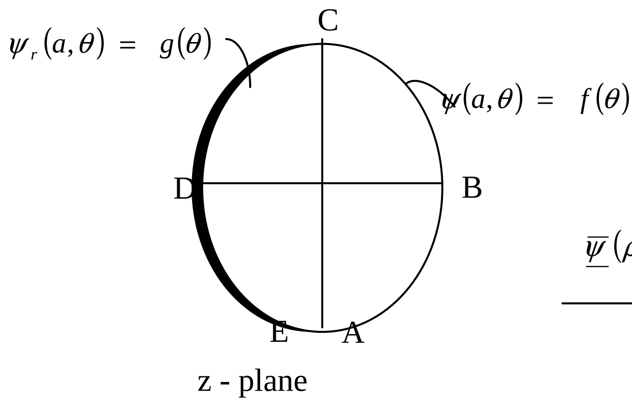
where

$$w = u + iv \qquad and \qquad z = x_1 + ix_2$$

From this transformation $\psi(a,\theta) = f(\theta)$ is equivalent to $\underline{\overline{\psi}}(\rho,\pi) = h(\rho)$ in the w – plane.

To find the boundary condition in the w – plane corresponding to $\psi_r(a,\theta) = g(\theta)$ in the z – plane.
We let

$$z = re^{i\theta} \quad and \quad w = \rho e^{i\phi}$$



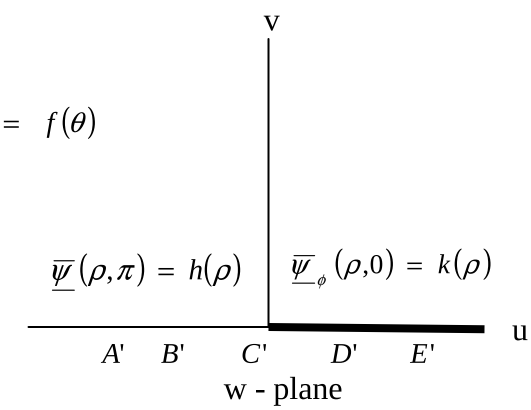


Fig. 2: *The problem in the z – plane*

Fig. 3: The transformed problem in w – plane.

such that

$$\overline{\psi}(\rho,\phi) = \psi(r,\theta)$$

From the mapping function (2.3)
we have that

$$\rho e^{i\phi} = -i\,\frac{r e^{i\theta} - ai}{re^{i\theta} + a i} = -i\,\frac{\left(re^{i\theta} - ai\right)\left(re^{-i\theta} - ai\right)}{\left(re^{i\theta} + ai\right)\left(re^{-i\theta} - ai\right)}$$

$$= -i\,\frac{r^2 - a^2 - 2air\cos\theta}{r^2 + a^2 + 2ar\sin\theta}$$

$$= \frac{-2ar\cos\theta - i\left(r^2 - a^2\right)}{r^2 + a^2 + 2ar\sin\theta}$$

$$\rho = \left|\frac{-2ar\cos\theta - i\left(r^2 - a^2\right)}{r^2 + a^2 + 2ar\sin\theta}\right|$$

$$= \frac{1}{r^2 + a^2 + 2ar\sin\theta}\left[\left(r^2 - a^2\right)^2 + 4a^2r^2\cos^2\theta\right]^{1/2} \qquad (2.4)$$

$$\phi = \tan^{-1}\frac{r^2 - a^2}{2ar\cos\theta} \qquad (2.5)$$

Thus when $r = a$, $\phi = 0$

$$\rho = \frac{2\cos\theta}{2(1+\sin\theta)} = \frac{\cos\theta}{1+\sin\theta} = \frac{\left(1-\sin^2\theta\right)}{1+\sin\theta}$$

$$\rho^2 = \frac{1-\sin\theta}{1+\sin\theta}$$

$$\rho^2 + \rho^2 \sin\theta = 1-\sin\theta$$

$$\left(\rho^2 + 1\right)\sin\theta = 1-\rho^2$$

$$\sin\theta = \frac{1-\rho^2}{1+\rho^2} \tag{2.6}$$

$$\theta = \sin^{-1}\left(\frac{1-\rho^2}{1+\rho^2}\right)$$

$$\cos\theta = \sqrt{1-\left(\frac{1-\rho^2}{1+\rho^2}\right)^2} = \sqrt{\frac{\left(1-\rho^2\right)^2-\left(1-\rho^2\right)^2}{\left(1+\rho^2\right)^2}}$$

$$= \sqrt{\frac{4\rho^2}{\left(1+\rho^2\right)^2}} = \frac{2\rho}{1+\rho^2} \tag{2.7}$$

$$\frac{\partial\psi}{\partial r} = \frac{\partial\overline{\psi}}{\partial r} = \frac{\partial\overline{\psi}}{\partial\rho}\frac{\partial\rho}{\partial r} + \frac{\partial\overline{\psi}}{\partial\phi}\frac{\partial\phi}{\partial r}$$

$$= \left[\frac{\tfrac{1}{2}\left(a^2+r^2+2ar\sin\theta\right)\left((r^2-a^2)^2+4a^2r^2\cos^2\theta\right)^{-\frac{1}{2}}\left(4r(r^2-a^2)+8a^2r\cos^2\theta\right)-\left((r^2-a^2)^2+4a^2r^2\cos^2\theta\right)^{\frac{1}{2}}2\left(r+a\sin\theta\right)}{\left(r^2+a^2+2ra\sin\theta\right)^2}\right]\frac{\partial\overline{\psi}}{\partial\rho}$$

$$+ \left\{\frac{1}{1+\left(\frac{r^2-a^2}{2ra\cos\theta}\right)^2}\left(\frac{4ar^2\cos\theta-\left(r^2-a^2\right)2a\cos\theta}{\left(2ra\cos\theta\right)^2}\right)\right\}\frac{\partial\overline{\psi}}{\partial\phi}$$

Thus

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=a} = \frac{1}{a\cos\theta}\frac{\partial\overline{\psi}}{\partial\phi}$$

$$\frac{\partial\overline{\psi}}{\partial\phi} = a\cos\theta\left.\frac{\partial\psi}{\partial r}\right|_{r=a} = a\cos\theta\, g(\theta)$$

$$= \frac{2a\rho}{1+\rho^2}\, g\left[\cos^{-1}\left(\frac{2\rho}{1+\rho^2}\right)\right] = k(\rho) \tag{2.8}$$

Hence the normal derivative

$$\left.\frac{\partial\psi}{\partial r}\right|_{r=a} = g(\theta)$$

in the z-plane is thus transformed to

$$\frac{\partial\overline{\psi}}{\partial\phi}(\rho,0) = k(\rho)$$

in the w-plane.

Now the corresponding BVP in the upper half plane is

$$\nabla^2\ \underline{\overline{\psi}}(\rho,\phi) = 0 \quad \rho > 0 \quad 0<\phi<\pi$$

$$\underline{\overline{\psi}}_{\phi}(\rho,0) = k(\rho) \qquad (2.9)$$

$$\underline{\overline{\psi}}(\rho,\pi) = h(\rho)$$

Applying Mellin transform ([9]Sneddon (1979)) to (2.9) we obtain

$$s^2\underline{\hat{\overline{\psi}}}(s,\phi) + \underline{\hat{\overline{\psi}}}_{\phi\phi}(s,\phi) = 0$$

$$\underline{\hat{\overline{\psi}}}(s,\pi) = H(s) \qquad (2.10)$$

$$\underline{\hat{\overline{\psi}}}_{\phi}(s,0) = K(s)$$

where

$$\underline{\hat{\overline{\psi}}}(s,\phi) = M\left[\underline{\overline{\psi}}(\rho,\phi)\right] = \int_0^{\infty}\rho^{s-1}\ \underline{\overline{\psi}}(\rho,\phi)\ d\rho \qquad (2.11)$$

Equation (2.10) has the general solution

$$\underline{\hat{\overline{\psi}}}(s,\phi) = H(s)\ \frac{\cos(\phi s)}{\cos(\pi s)} - \frac{K(s)}{s}\ \frac{\sin(\pi-\phi)s}{\cos(\pi s)} \qquad (2.12)$$

we invert by making use of the convolution theorem to get

$$\underline{\overline{\psi}}(\rho,\phi) = \frac{\rho^{1/2}}{\pi}\cos\phi/2 \int_0^{\infty}\frac{u^{-1/2}(\rho+u)\,h(u)}{u^2 + 2\rho u\cos\phi+\rho^2}\,du$$

$$-\frac{1}{\pi}\int_0^{\infty}u^{-1}\tanh^{-1}\left(\frac{2(\rho u)^{1/2}\cos\phi/2}{\rho+u}\right)k(u)\ du \qquad 0<\phi<\pi \qquad (2.13)$$

Which converge uniformly for $k(u) = O(u^{-1})$ *and* $h(u) = O(1)$ *in* $(0,\infty)$

i.e $\frac{k(u)}{u}$ *and* $h(u)$ are bounded in $(0,\infty)$

The integral solution (2.13) gives,using Ibem's boundary values,the result

$$\overline{\psi}(\rho,\phi)=\frac{2a\gamma}{\mu}\begin{cases}\displaystyle\sum_{m=0}^{\infty}\frac{(-1)^{m+1}}{2m+1}\rho^{1+2m}\sin(2m+1)\phi+\sum_{m=0}^{\infty}\frac{\alpha_m}{2m+1}\rho^{\frac{1}{2}+m}\cos\left(m+\frac{1}{2}\right)\phi & 0<\rho<1\\ \displaystyle\sum_{m=1}^{\infty}\frac{(-1)^{m}}{2m-1}\rho^{1-2m}\sin(2m-1)\phi+\sum_{m=1}^{\infty}\frac{\beta_m}{2m-1}\rho^{\frac{1}{2}-m}\cos\left(m-\frac{1}{2}\right)\phi & \rho>1\end{cases}$$

(2.14)

Which is Ibem's result.

We next transform back the solutions (2.13) and (2.14) to the original (r,θ) coordinate using the mapping function (2.3) to obtain respectively the solutions

$$\psi(r,\theta)=\frac{A^{\frac{1}{2}}}{\pi}\cos{}^{B}\!/_{2}\int_0^{\infty}\frac{u^{-1/2}(A+u)\,h(u)}{u^2+2Au\cos B+A^2}du\quad-\frac{1}{\pi}\int_0^{\infty}u^{-1}\tanh^{-1}\left(\frac{2(Au)^{1/2}\cos{}^{B}\!/_{2}}{A+u}\right)k(u)du$$

$$\theta<\sin^{-1}\left[-\frac{(r^2+a^2)}{2ar}\right]\ \ and\ \ \sin^{-1}\left[-\frac{(r^2+a^2)}{2ar}\right]<\theta<\sin^{-1}\left[\frac{r^2+a^2}{2ar}\right] \tag{2.15}$$

$$\psi(r,\theta)=\frac{2a\gamma}{\mu}\begin{cases}\displaystyle\sum_{m=0}^{\infty}\frac{(-1)^{m+1}A^{1+2m}}{2m+1}\sin(2m+1)B+\sum_{m=0}^{\infty}\frac{\alpha_m}{2m+1}A^{\frac{1}{2}+m}\cos\left(m+\frac{1}{2}\right)B, & \cos^{-1}\left[-\frac{a^2+r^2}{2ar}\right]<\theta<\cos^{-1}\left[\frac{a^2+r^2}{2ar}\right]\\ \displaystyle\sum_{m=1}^{\infty}\frac{(-1)^{m}}{2m-1}A^{1-2m}\sin(2m-1)B+\sum_{m=1}^{\infty}\frac{\beta_m}{2m-1}A^{\frac{1}{2}-m}\cos\left(m-\frac{1}{2}\right)B, & \theta<\cos^{-1}\left[-\frac{a^2+r^2}{2ar}\right]\end{cases}$$

(2.16)

Where

$$H(r)=\frac{r^2-a^2}{2ar}$$

$$A(r,\theta)=\rho=\left[H(r)+\frac{a}{r}+\sin\theta\right]^{-1}\left[H^2(r)+\cos^2\theta\right]^{\frac{1}{2}}$$

$$B(r,\theta)=\phi=\tan^{-1}\left[H(r)\sec\theta\right]$$

3.0 Strain And Stress Fields

In this section we obtain the strain and stress fields corresponding to the problem (2.2).We note that for the anti-plane strain problem the relevant strains are the ε_{i3} and the stresses σ_{i3} .We compute therefore ε_{13} and ε_{23}

$$\varepsilon_{13}=\frac{1}{2}\left(\frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x_1}+\frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial x_1}\right) \tag{3.1}$$

Using (2.16) in (3.1) we have

$$\varepsilon_{13} = -a\frac{\gamma}{\mu}\begin{cases} \sum_{m=0}^{\infty}(-1)^{m+1}\left\{A^{1+2m}\cos(2m+1)B\left(\cos\theta\frac{\partial B}{\partial r} - \frac{\sin\theta}{r}\frac{\partial B}{\partial\theta}\right)\right. \\ \left. + A^{2m}\sin(2m+1)B\left(\cos\theta\frac{\partial A}{\partial r} - \frac{\sin\theta}{r}\frac{\partial A}{\partial\theta}\right)\right. & 0 < A < 1 \\ \frac{-1}{2}\sum_{m=0}^{\infty}\alpha_m\left\{A^{\frac{1}{2}+m}\sin\left(m+\tfrac{1}{2}\right)B\left(\cos\theta\frac{\partial B}{\partial r} - \frac{\sin\theta}{r}\frac{\partial B}{\partial\theta}\right)\right. \\ \left. - A^{m-\frac{1}{2}}\cos\left(m+\tfrac{1}{2}\right)B\left(\cos\theta\frac{\partial A}{\partial r} - \frac{\sin\theta}{r}\frac{\partial A}{\partial\theta}\right)\right\} \\ \sum_{m=1}^{\infty}(-1)^{m}\left\{A^{1-2m}\cos(2m-1)B\left(\cos\theta\frac{\partial B}{\partial r} - \frac{\sin\theta}{r}\frac{\partial B}{\partial\theta}\right)\right. \\ \left. - A^{-2m}\sin(2m-1)B\left(\cos\theta\frac{\partial A}{\partial r} - \frac{\sin\theta}{r}\frac{\partial A}{\partial\theta}\right)\right\} & A > 1 \\ -\frac{1}{2}\sum_{m=1}^{\infty}\beta_m\left\{A^{\frac{1}{2}-m}\sin\left(m-\tfrac{1}{2}\right)B\left(\cos\theta\frac{\partial B}{\partial r} - \frac{\sin\theta}{r}\frac{\partial B}{\partial\theta}\right)\right. \\ \left. \qquad + A^{-m-\frac{1}{2}}\cos\left(m-\tfrac{1}{2}\right)B\left(\cos\theta\frac{\partial B}{\partial r} - \frac{\sin\theta}{r}\frac{\partial B}{\partial\theta}\right)\right\} \end{cases} \tag{3.2}$$

And $\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial\psi}{\partial r}\frac{\partial r}{\partial x_2} + \frac{\partial\psi}{\partial\theta}\frac{\partial\theta}{\partial x_2}\right)$

$$= -\frac{a\gamma}{\mu}\begin{cases} \sum_{m=0}^{\infty}(-1)^{m+1}\left\{A^{1+2m}\cos(2m+1)B\left(\sin\theta\frac{\partial B}{\partial r} + \frac{\cos\theta}{r}\frac{\partial B}{\partial\theta}\right)\right. \\ \left. + A^{2m}\sin(2m+1)B\left(\sin\theta\frac{\partial A}{\partial r} + \frac{\cos\theta}{r}\frac{\partial A}{\partial\theta}\right)\right\} & 0 < A < 1 \\ -\frac{1}{2}\sum_{m=0}^{\infty}\alpha_m\left\{A^{\frac{1}{2}+m}\sin\left(m+\tfrac{1}{2}\right)B\left(\sin\theta\frac{\partial B}{\partial r} + \frac{\cos\theta}{r}\frac{\partial B}{\partial\theta}\right)\right. \\ \left. - A^{m-\frac{1}{2}}\cos\left(m+\tfrac{1}{2}\right)B\left(\sin\theta\frac{\partial A}{\partial r} + \frac{\cos\theta}{r}\frac{\partial A}{\partial\theta}\right)\right\} \\ \sum_{m=1r}^{\infty}(-1)^{m}\left\{A^{1-2m}\cos(2m-1)B\left(\sin\theta\frac{\partial B}{\partial r} + \frac{\cos\theta}{r}\frac{\partial B}{\partial\theta}\right)\right. \\ \left. - A^{-2m}\sin(2m-1)B\left(\sin\theta\frac{\partial A}{\partial r} + \frac{\cos}{r}\frac{\partial A}{\partial\theta}\right)\right\} & A > 1 \\ -\frac{1}{2}\sum_{m=1}^{\infty}\beta_m\left\{A^{\frac{1}{2}-m}\sin\left(m-\tfrac{1}{2}\right)B\left(\sin\theta\frac{\partial B}{\partial r} + \frac{\cos\theta}{r}\frac{\partial B}{\partial\theta}\right)\right. \\ \left. + A^{-m-\frac{1}{2}}\cos\left(m-\tfrac{1}{2}\right)B\left(\sin\theta\frac{\partial A}{\partial r} + \frac{\cos\theta}{r}\frac{\partial A}{\partial\theta}\right)\right\} \end{cases} \tag{3.3}$$

Where $\dfrac{\partial r}{\partial x_1} = \cos\theta,\ \dfrac{\partial \theta}{\partial x_1} = \dfrac{-\sin\theta}{r} \quad and\ \dfrac{\partial r}{\partial x_2} = \sin\theta,\ \dfrac{\partial\theta}{\partial x_2} = \dfrac{\cos\theta}{r}$

The corresponding stress fields are given as

$$\sigma_{13} = -a\gamma \begin{cases} 2\sum_{m=0}^{\infty}(-1)^{m+1}\left\{A^{1+2m}\cos(2m+1)B\left(\cos\theta\dfrac{\partial B}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial B}{\partial\theta}\right)\right. \\ \left. +A^{2m}\sin(2m+1)B\left(\cos\theta\dfrac{\partial A}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial A}{\theta}\right)\right\} & 0 < A < 1 \\ -\sum_{m=0}^{\infty}\alpha_m\left\{A^{\frac{1}{2}+m}\sin\left(m+\tfrac{1}{2}\right)B\left(\cos\theta\dfrac{\partial B}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial B}{\partial\theta}\right)\right. \\ \left. -A^{m-\frac{1}{2}}\cos\left(m+\tfrac{1}{2}\right)B\left(\cos\theta\dfrac{\partial A}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial A}{\partial\theta}\right)\right\} \\ 2\sum_{m=1}^{\infty}(-1)^{m}\left\{A^{1-2m}\cos(2m-1)B\left(\cos\theta\dfrac{\partial B}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial B}{\partial\theta}\right)\right. \\ \left. -A^{-2m}\sin(2m-1)B\left(\cos\theta\dfrac{\partial A}{\partial r} - \dfrac{\sin}{r}\dfrac{\partial A}{\partial\theta}\right)\right\} & A > 1 \\ -\sum_{m=1}^{\infty}\beta_m\left\{A^{\frac{1}{2}-m}\sin\left(m-\tfrac{1}{2}\right)B\left(\cos\theta\dfrac{\partial B}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial B}{\partial\theta}\right)\right. \\ \left. +A^{-m-\frac{1}{2}}\cos\left(m-\tfrac{1}{2}\right)B\left(\cos\theta\dfrac{\partial B}{\partial r} - \dfrac{\sin\theta}{r}\dfrac{\partial B}{\partial\theta}\right)\right\} \end{cases} \tag{3.4}$$

$$\sigma_{23} = -a\gamma \begin{cases} \sum_{m=0}^{\infty} (-1)^{m+1} \left\{ A^{1+2m} \cos(2m+1) B \left(\sin\theta \frac{\partial B}{\partial r} + \frac{\cos\theta}{r} \frac{\partial B}{\partial \theta} \right) \right. \\ \left. + A^{2m} \sin(2m+1) B \left(\sin\theta \frac{\partial A}{\partial r} + \frac{\cos\theta}{r} \frac{\partial A}{\partial \theta} \right) \right\} \\ -\sum_{m=0}^{\infty} \alpha_m \left\{ A^{\frac{1}{2}+m} \sin\left(m+\tfrac{1}{2}\right) B \left(\sin\theta \frac{\partial B}{\partial r} + \frac{\cos\theta}{r} \frac{\partial B}{\partial \theta} \right) \right. & 0 < A < 1 \\ \left. -A^{m-\frac{1}{2}} \cos\left(m+\tfrac{1}{2}\right) B \left(\sin\theta \frac{\partial A}{\partial r} + \frac{\cos\theta}{r} \frac{\partial A}{\partial \theta} \right) \right\} \\ 2\sum_{m=1}^{\infty} (-1)^{m} \left\{ A^{1-2m} \cos(2m-1) B \left(\sin\theta \frac{\partial B}{\partial r} + \frac{\cos\theta}{r} \frac{\partial B}{\partial \theta} \right) \right. \\ \left. -A^{-2m} \sin(2m-1) B \left(\sin\theta \frac{\partial A}{\partial r} + \frac{\cos\theta}{r} \frac{\partial A}{\partial \theta} \right) \right\} & A > 1 \\ -\sum_{m=1}^{\infty} \beta_m \left\{ A^{\frac{1}{2}-m} \sin\left(m-\tfrac{1}{2}\right) B \left(\sin\theta \frac{\partial B}{\partial r} + \frac{\cos\theta}{r} \frac{\partial B}{\partial \theta} \right) \right. \\ \left. +A^{-m-\frac{1}{2}} \cos\left(m-\tfrac{1}{2}\right) B \left(\sin\theta \frac{\partial A}{\partial r} + \frac{\cos\theta}{r} \frac{\partial A}{\partial \theta} \right) \right\} \end{cases} \tag{3.5}$$

4.0 Discussion of result and conclusion

From the result obtained, we make the following observations;

1. For r = a in the generalized form of the above fields, the equations (2.4),(3.1),(3.2),(3.3) and (3.4) represent values at the boundary of the cylinder while $r < a$, the equations represent values within the radius of the cylinder.
2. At the boundary, the fields in the two domains coincide and depend only on the angle θ
3. We note that the integral solution (2.13) gives, using Ibem's boundary values, the result

$$\underline{\overline{\psi}}(\rho,\phi) = \frac{2a\gamma}{\mu} \begin{cases} \sum_{m=0}^{\infty} \frac{(-1)^{m+1}}{2m+1} \rho^{1+2m} \sin(2m+1)\phi + \sum_{m=0}^{\infty} \frac{\alpha_m}{2m+1} \rho^{\frac{1}{2}+m} \cos\left(m+\frac{1}{2}\right)\phi & 0 < \rho < 1 \\ \sum_{m=1}^{\infty} \frac{(-1)^{m}}{2m-1} \rho^{1-2m} \sin(2m-1)\phi + \sum_{m=1}^{\infty} \frac{\beta_m}{2m-1} \rho^{\frac{1}{2}-m} \cos\left(m-\frac{1}{2}\right)\phi & \rho > 1 \end{cases}$$

which is Ibem's result.

It is interesting to note that the same mixed boundary value problem is solved in [10] by means of systems of equations. This method gives the solution as infinite series which is of a more complex form and can only be applied to a few problems,for instance we can only apply it to Ibem's problems for $\psi(a,\theta) \neq 0, \ -\pi < \theta < 0$ .however the method of conformal mapping and Mellin transform can be applied to the mixed boundary value problem with a better result.

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