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Transverse Vibrations Of Axially Tensioned - Uniform Beam Subjected To Harmonic Moving Loads And Resting On Variable Foundation

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Abstract

This paper investigates the transverse deflection of a uniform beam resting on a non-uniform elastic foundation and traversed by a harmonic load of variable magnitude. The mode-superposition method is used to obtain the approximate analytical solution of the differential equation of motion of the dynamical beam problem. Analytical and numerical solutions reveal that increase in the axial force leads to decrease in the amplitude of deflection of the undamped elastic beam for fixed foundation stiffness. Also for fixed value of axial force, the amplitude of deflection of the elastic beam structure decreases with an increase in the value of foundation stiffness. Furthermore, the various resonance conditions and the corresponding critical speeds are obtained. These show that the critical speed of the dynamical system increases with an increase in the effect of axial force suggesting that the structural designs are more stable and reliable for higher values of axial force. In unlike manner, the critical speed decreases with an increase in the effect of foundation stiffness, indicating a reduction in the attendant risk of resonance.

Keywords: Transverse deflection, elastic foundation, mode-superposition method, axial force, foundation stiffness, critical speed, resonance condition.

1.0 Introduction

Moving loads have great effect on dynamic stresses in traversed beam or beam-like structures, thereby subjecting them to intensive vibrations especially at high velocities. This phenomenon is of considerable interest to Applied Mathematicians, Physicists, Transport Engineers, Construction Engineers and Designers of structures like railway and highway bridges, suspension bridges, rails sleepers, roadways and airport runways, pipelines, etc [1]. For a beam resting on a non-linear elastic foundation, the force exerted by the foundation on the beam opposes displacements of the beam's centroidal axis. As a result, various investigations have been carried out on this subject. [2] considered the problem of elastic beam under the action of moving loads. In his study, the mass of the beam was considered much smaller than the mass of the moving load. [3] later considered the problem of simply supported finite beams lying on an elastic foundation and traversed by moving loads. In his analysis he assumed that the loads were moving with constant velocities along the beam. Recently, [4] considered the response of a thick beam under the action of harmonic variable concentrated force moving at a uniform velocity. The method of integral transformation was used, in particular, the finite Fourier sine transform is used for length co-ordinate and the Laplace transform for the time co-ordinate. Solution, which converges, was obtained for the deflection of simply supported thick beam. The effect of an elastic foundation on the transverse displacement of the beam was analyzed

for the problem. [5] investigated the dynamic response of finite elastic beam under the influence of a dynamic load moving with constant speed. [6] analyzed the influence of foundation stiffness and axial force on the vibration of thin beam under variable harmonic moving load using the method of integral transformation. In all the aforementioned, no consideration has been given to the effect of variable foundation on the transverse displacement of undamped thin beam subjected to moving loads. Thus, this work is set to investigate the influence of exponentially decaying foundation and axial force on the transverse deflection of undamped simply-supported thin beam (Bernoulli-Euler beam) under harmonic load of variable magnitude.

2.0 BASIC EQUATION OF MOTION

The motion of a concentrated force P(x, t) moving at a uniform speed v along a uniform beam of length l in the positive x- direction is considered. Neglecting the influence of damping and rotatory inertia correction factor, the effect of the axial force on the dynamic deflection V(x, t) under moving load of the beam structure resting on a non-uniform elastic foundation K(x) is examined by solving the governing fourth-order partial differential equation of motion of the form

$$EI\frac{\partial^4 V(x,t)}{\partial x^4} - N\frac{\partial^2 V(x,t)}{\partial x^2} + \mu\frac{\partial^2 V(x,t)}{\partial t^2} + K(x)V(x,t) = P(x,t)$$

(2.1)

where x = spatial coordinate, t = time, $\frac{\partial^n}{\partial x^n}$ = nth partial derivative with respect to x,
EI = flexural rigidity of the beam, I = constant moment of inertia of the beam's cross section about the neutral axis, E= Young's modulus, P(x, t) =applied concentrated load, N= axial force, μ= mass per unit length of the beam, K(x) = non-uniform foundation function.

Furthermore, the ends of the beam x = 0 and x = l are assumed to be simply supported. Thus, the boundary conditions are

$$V(0,t) = 0 = V(l,t); \qquad \frac{\partial^2 V(0,t)}{\partial x^2} = 0 = \frac{\partial^2 V(l,t)}{\partial x^2}$$

(2.2)

Without loss of generality, one can consider the initial conditions of the form

$$V(x,0) = 0 = \frac{\partial V(x,0)}{\partial t}$$

(2.3)

On one hand, the exciting force moving on the beam is taken to be of the form

$$P(x,t)\begin{cases} P_f(t)\delta(x-ct), 0 \le ct \le l \\ 0 \qquad ,ct > l \end{cases}$$

(2.4)

where P_f (t) is assumed to be a harmonic forcing function given by

$$P_f(t) = P_0 \cos\Omega t$$

(2.5)

P_o is the amplitude and Ω is the driving circular frequency and $\delta(\cdot)$ is the unit impulse or Dirac delta function defined as

$$\delta(x-\zeta)\begin{cases} 0, & x \neq \zeta \\ \infty, & x = \zeta \end{cases}$$

(2.6)

with the property;

$$\int_0^L \delta(x-\xi)f(x) = \begin{cases} 0, & \xi < 0 \\ f(\xi), & 0 < \xi < l \\ 0, & \xi > l \end{cases}$$

(2.7)

while on the other hand, the elastic foundation is taking to be of the form

$$K(x) = K_0 e^{-\alpha x}, \qquad \alpha > 0,$$

(2.8)

where K_0 is the bedding constant of the foundation material.

Introducing equations (2.4), (2.5) and (2.8) into (2.1) yields

$$EI\frac{\partial^4 V(x,t)}{\partial x^4} - N\frac{\partial^2 V(x,t)}{\partial x^2} + \mu\frac{\partial^2 V(x,t)}{\partial t^2} + K_0 e^{-\alpha x}V(x,t) = P_0\cos\Omega t\delta(x-ct)$$

(2.9)

3.0 SOLUTION PROCEDURE

In order to obtain the analysis of the dynamic displacement response V(x, t) of the beam structure under concentrated load, the normal-mode method of dynamic analysis [7 - 11] is employed. This technique assumes that any displacement function V(x, t) for the beam or beam-like structure can be developed by super-imposing suitable amplitudes of the modes of vibration. When the beam performs one of its natural modes of vibration, say $V_n(x,t)$, it is given as

$$V_n(x,t) = \varphi_n(x)\Theta_n(t)$$

(3.1) thus the total displacement is seen as the sum of the modal components which is given by

$$V(x,t) = \sum_{n=1}^{N}\varphi_n(x)\Theta_n(t)$$

(3.2)

where $\Theta_n(t)$ are the modal amplitudes representing the time dependency of the displacements and $\varphi(x)$ are the normal-mode shape functions of the beam. Assuming the beam structure is undergoing harmonic oscillations, the shape function of the beam in free transverse vibration can be written as

$$\varphi_n(x) = \sin\lambda_n x + A_n\cos\lambda_n x + B_n\sinh\lambda x + C_n\cosh\lambda_n x$$

(3.3)

The constants A_n, B_n, C_n and the mode frequencies λ_n can be determined by using the boundary conditions associated with the beam structure. For simply supported end conditions, it can be shown that

$A_n = B_n = C_n = 0,$ and $\lambda_n = \frac{n\pi}{l},$ (n=0,1,2,…) (3.4)

which results in

$$\varphi_n(x) = \sin\frac{n\pi x}{l}$$

(3.5)

For simply supported edges, X=0 and X=L, the shape function of vibration takes the form

$$V(x,t) = \sum_{n=1}^{N}\Theta_n(t)\sin\frac{n\pi x}{l}$$

(3.6)

Substituting equation (3.6) into equation (2.9), one obtains

$$\sum_{n=1}^{\infty}\left\{\left[EI\left(\frac{n\pi}{l}\right)^4\theta(t) + N\left(\frac{n\pi}{l}\right)^2\theta(t) + \mu\ddot{\theta}(t) + K_0 e^{-\alpha x}\theta(t)\right]\sin\frac{n\pi x}{l}\right\} - P_0\cos\Omega t\delta(x-ct) = 0$$

(3.7) where $\dot{\theta}$ is the first derivative of θ with respect to t

Assuming $\varphi_n(x)$ and $\varphi_m(x)$ are any two mode-shape functions of the vibrating system corresponding to the circular frequencies λ_n and λ_m respectively, the mode superposition procedure requires that for any such two mode-shape functions, the orthogonality relationship for transverse vibration is

$$\int_0^l \varphi_n(x)\varphi_m(x)dx = \delta_{nm} \tag{3.8}$$

where the symbol δ_{nm} is the kronecker delta defined as

$$\delta_{nm} = \begin{cases} 0, & n \neq m \\ 1, & n = m \end{cases} \tag{3.9}$$

Multiplying equation (3.7) by $\sin\frac{m\pi x}{l}$ and integrating over the beam span l with respect to x, one obtains

$$\int_0^l \sum_{n=1}^{\infty} \left\{ \left[EI\left(\frac{n\pi}{l}\right)^4 \theta_n(t) + N\left(\frac{n\pi}{l}\right)^2 \theta_n(t) + \mu\ddot{\theta}_n(t) + K_0 e^{-\alpha x}\theta_n(t) \right] \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l} \right\} - \int_0^l P_0 \cos\Omega t \sin\frac{m\pi x}{l}\delta(x - c_n t)dx = 0 \tag{3.10}$$

Equation (3.10) can be rewritten as

$$\sum_{n=1}^{\infty}[G_1(t) + G_2(t) + G_3(t) + G_4(t)] = Q(t) \tag{3.11}$$

where

$$\left.\begin{aligned} G_1(t) &= \ddot{\theta}_n(t)\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx & (a)\\ G_2(t) &= \frac{EI}{\mu}\left(\frac{n\pi}{l}\right)^4 \theta_n(t)\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx & (b)\\ G_3(t) &= \frac{N}{\mu}\left(\frac{n\pi}{l}\right)^2 \theta_n(t)\int_0^l \sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx & (c)\\ G_4(t) &= \frac{K_0}{\mu}\theta_n(t)\int_0^l e^{-\alpha x}\sin\frac{n\pi x}{l}\sin\frac{m\pi x}{l}dx & (d)\\ Q(t) &= \frac{P_0}{\mu}\cos\Omega t\int_0^l \delta(x - c_n t)\sin\frac{m\pi x}{l}dx & (e) \end{aligned}\right\} \tag{3.12}$$

In view of Equations (3.8) and (3.9) and the property of Dirac delta function in equation (2.7), the evaluation of integrals in (3.12) yields

$$\sum_{n=1}^{\infty}\left[\ddot{\theta}_n(t) + H_1\theta_n(t) + H_2\theta_n(t) + H_3\theta_n(t)\right] = F\cos\Omega t \sin\frac{m\pi c_n t}{l} \tag{3.13}$$

where

$$F = \frac{P_0}{\mu}. \tag{3.14}$$

After some rearrangements, equation (3.13) becomes

$$\sum_{n=1}^{\infty}\left[\ddot{\theta}_n(t) + \gamma^2\theta_n(t)\right] = F\cos\Omega t \sin\frac{m\pi c_n t}{l} \tag{3.15}$$

where

$$\gamma^2 = H_1 + H_2 + H_3 \tag{3.16}$$

$$H_1 = \frac{EI}{\mu}\left(\frac{n\pi}{l}\right)^4, \quad H_2 = \frac{N}{\mu}\left(\frac{n\pi}{l}\right)^2, \quad H_3 = \frac{K_0}{\mu}\left[\frac{(2n\pi)^2(e^{-\alpha l} - 1)}{(2n\pi\alpha)^2 + l^2}\right] \tag{3.17}$$

In what follows, the n^th mode of vibration of the n^th particle is considered. The non-homogeneous second order ordinary differential equation (3.15) is now subjected to Laplace transformation defined as

$$\theta(s) = \int_0^{\infty} \theta(t)\, e^{-st}\, dt \tag{3.18}$$

with the inverse,

$$\theta(t) = \frac{1}{2\pi i}\int_{a+i\infty}^{a+i\infty} \theta(s) e^{st}\, ds \tag{3.19}$$

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where the path of integration is a line parallel to the imaginary axis at Real s=a and extending from $-\infty$ to $+\infty$ and

$$\frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \theta(s) e^{st}\, ds = \lim_{s\to s_k}(s - s_k)\,\theta(s) \tag{3.20}$$

Introducing equation (3.18) into equation (3.15) and setting the initial conditions equal to zero, one obtains

$$(S^2 + \gamma^2)\theta(s) = F\left[\frac{\beta_1}{s^2+\beta_1^2} + \frac{\beta_2}{s^2+\beta_2^2}\right] \tag{3.21}$$

where

$$\beta_1 = \left(\frac{m\pi c}{l} + \Omega\right),\ \beta_2 = \left(\frac{m\pi c}{l} - \Omega\right) \tag{3.22}$$

Thus

$$\theta(s) = \frac{1}{(S^2+\gamma^2)} \cdot F\left[\frac{\beta_1}{s^2+\beta_1^2} + \frac{\beta_2}{s^2+\beta_2^2}\right] \tag{3.23}$$

It remains to obtain the Laplace inversion of equation (3.23). Thus, equation (3.23) can be rewritten as

$$\theta(s) = F\left[\frac{\beta_1}{(S^2+\gamma^2)(s^2+\beta_1^2)} + \frac{\beta_2}{(S^2+\gamma^2)(s^2+\beta_2^2)}\right] \tag{3.24}$$

Introducing equation (3.19) into equation (3.24), the Laplace inversion of (3.24) as defined in [7 - 11] is

$$\theta(t) = F[A_1(t) + A_2(t)] \tag{3.25}$$

Where

$$A_1(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{1}{s^2+k^2} \cdot \frac{\beta_1 e^{st}}{s^2+\beta_1^2} ds, \qquad A_2(t) = \frac{1}{2\pi i}\int_{a-i\infty}^{a+i\infty} \frac{1}{s^2+\gamma^2} \cdot \frac{\beta_2 e^{st}}{s^2+\beta_2^2} ds \tag{3.26}$$

In order to evaluate the above integrals in (3.26), the residue theorem defined in [9, 12] is employed.

The singularities in the integrals are poles. The denominators of the integrands of A_1 (t) and A_2(t) have simple poles at

$s = \pm\gamma,\quad s = \pm\beta_1,\quad s = \pm\beta_2$

Evidently, it is straight forward to show that

$$A_1(t) = \frac{\beta_1}{\gamma^2-\beta_1^2}\sin\gamma t - \frac{\beta_1}{\gamma^2-\beta_1^2}\sin\beta_1 t \tag{3.27}$$

$$A_2(t) = \frac{\beta_2}{\gamma^2-\beta_2^2}\sin\gamma t - \frac{\beta_2}{\gamma^2-\beta_2^2}\sin\beta_2 t \tag{3.28}$$

Substituting equations (3.27) and (3.28) into equation (3.25) yields

$$\theta(t) = F\left[\frac{\beta_1}{\gamma^2-\beta_1^2}(\sin\gamma t - \sin\beta_1 t) + \frac{\beta_2}{\gamma^2-\beta_2^2}(\sin\gamma t - \sin\beta_2 t)\right] \tag{3.29}$$

Thus, in view of equation (3.27), the displacement function in equation (3.6) can be given as

$$V(x,t) = \sum_{n=1}^{\infty} F\left[\frac{\beta_1}{\gamma^2-\beta_1^2}(\sin\gamma t - \sin\beta_1 t) + \frac{\beta_2}{\gamma^2-\beta_2^2}(\sin\gamma t - \sin\beta_2 t)\right]\sin\frac{n\pi x}{l} \tag{3.30}$$

The expression in (3.30) represents the transverse deflection (or displacement response) of the tensioned-beam under the action of a harmonic moving load with uniform velocity and resting on exponentially decaying foundation.

4.0 Discussion Of Analytical Soluton

In a beam vibration problem such as this, with damping neglected, one is interested in the resonance conditions of the dynamical system. These are conditions under which any of the exciting circular frequencies coincides with one of the natural frequencies of the dynamical system, thereby making the amplitudes of displacement grow without bounds.

In this case however, equation (3.30) reveals that the axially-tensioned thin beam under harmonically moving load deflects continuously with time when

$$\gamma^2 = \beta_1^2 \ \ and\ \ \gamma^2 = \beta_2^2 \tag{4.1}$$

The expressions in equation (4.1) show that the state of resonance of the thin beam is dependent on axial force and foundation stiffness. At this juncture, one seeks the critical speeds at which these resonance conditions occur. The critical speeds at the respective states of resonance are

$$C_{r_1} = \frac{1}{\varphi}[\gamma - \varphi] \quad (4.2)$$

$$C_{r_2} = \frac{1}{\varphi}[\gamma + \varphi] \quad (4.3)$$

where $$\varphi = \frac{m\pi}{l} \quad (4.4)$$

From the equations (4.2) and (4.3), it is straightforward to examine the effects of the various parameters, such as axial force and foundation stiffness on the beam dynamical system.

In what follows, the effects of various pertinent parameters on the displacement response of the beam and the critical speeds of the dynamical system are analysed.

5.0 NUMERICAL ILLUSTRATION

For the purpose of numerical illustration of analytical results obtained for the dynamical beam problem, the uniform beam of length 12.192m with flexural rigidity $EI = 6.068 \times 10^6 Kgm^3/sec$ is considered. The transverse deflection profile at various times t when the foundation stiffness is fixed and the axial force (N) is varied between 0N and 2,000,000N and equally too, for fixed axial force, the transverse deflection profile at various times when the foundation stiffness which is varied between 0N and 40000N are shown in plotted curves as presented in Figures 5.1 and 5.2 respectively. Figure 5.1 shows that for fixed values of K, the displacement response of the beam is in the form of an exponentially decaying harmonics which eventually disappear as the effect of axial force is increased, while figure 5.2 indicates that for fixed N, the transverse deflection of the thin beam decreases with increases in the values of the foundation stiffness. In figure 5.3 and 5.4, the respective graphs of the critical speed (C_{r_1} or C_{r_2}, both follow like patterns) against various values of K and N are displayed. Figure 5.3 shows that the critical speed decreases as the axial force effect increases, indicating that the structural designs are more stable and reliable for higher values of axial force while figure 5.4 shows that critical speed increases as values of foundation stiffness increases, thereby reducing the risk of resonance.

6.0 CONCLUSION

This paper examined the transverse deflection of a highly tensioned-uniform thin beam resting on an exponentially decaying foundation and traversed by harmonically varying moving load.

The governing differential equation of motion of the beam, assumed to be simply-supported and resting on variable Winkler elastic foundation, is a non-homogeneous fourth-order partial differential equation with variable and single coefficients. The solution of the dynamical beam problem is obtained, using the method of normal-mode analysis in conjunction with the method of integral transformations and the Cauchy residue theorem. This solution is analysed and two distinct resonance conditions of the dynamical system emerged. Numerical analysis is carried out and the study revealed the following results.

For fixed values of K, the displacement response of the beam is in the form of an exponentially decaying harmonics which eventually disappear as the effect of axial force increases

For fixed values of N, the transverse deflection of the thin beam decreases with increases in the values of the foundation stiffness.

Critical speed decreases as the axial force effect increases, which suggests that the structural designs are more stable and reliable for higher values of axial force.

Critical speed increases as the foundation stiffness increases, thereby reducing the risk of resonance.

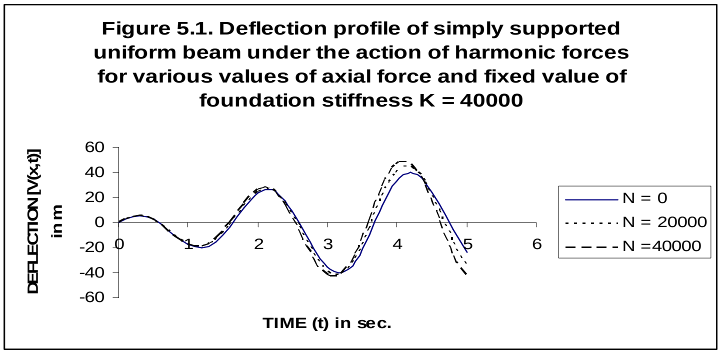


Figure 5.1. Deflection profile of simply supported uniform beam under the action of harmonic forces for various values of axial force and fixed value of foundation stiffness K = 40000
DEFLECTION [V(x,t)] in m
TIME (t) in sec.
N = 0
N = 20000
N =40000

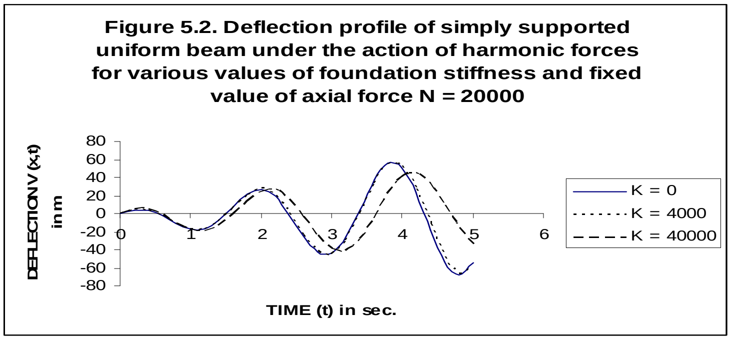


Figure 5.2. Deflection profile of simply supported uniform beam under the action of harmonic forces for various values of foundation stiffness and fixed value of axial force N = 20000
DEFLECTION V (x,t) in m
TIME (t) in sec.
K = 0
K = 4000
K = 40000

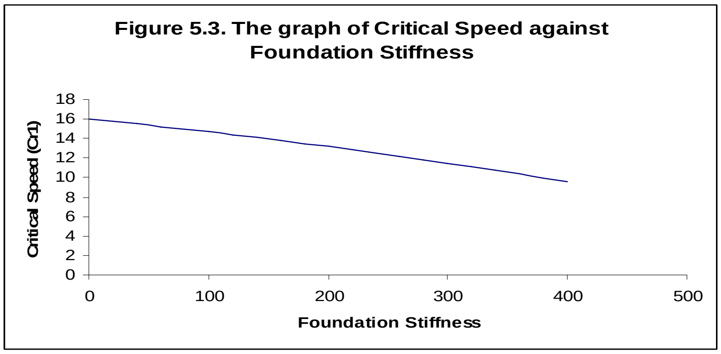
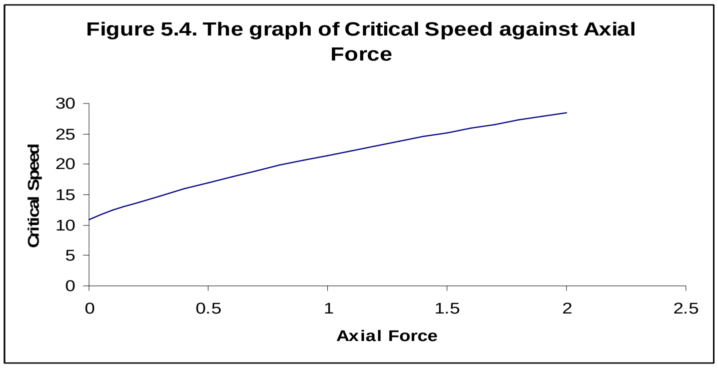


Figure 5.3. The graph of Critical Speed against Foundation Stiffness
Critical Speed (Cr1)
Foundation Stiffness



References

[1] Fryba L., (1972): *Vibration of solids structures under moving loads*. Noordhoff International Publishing Groningen, Netherlands.

[2] Willis R. et al.: Preliminary essay to the appendix B.: Experiments for determining the effects produced by causing weights to travel over bars with different velocities. In: Barlow P., (1851): Treatise on the strength of timber, cast iron and malleable iron. London.

[3] Timoshenko, S. P., (1922): On the forced vibration of bridges. *Philosophy Magazine ser*. Vol. 6(43), Pg 1018.

[4] Oni, S. T., (1990): On thick beams under the action of a variable travelling transverse load. *Abacus Journal of Mathematical association of Nigeria*.

[5] Kenny, J., (1954). Steady state vibration of a beam on an elastic foundation for a moving load. *Journal of Appl. Mech. Vol.* 78 pp 359 – 364.

[6] Awodola, T. O., (2005). Influence of foundation and axial force on the vibration of thin beam under variable harmonic moving load. *Journal of the Nigerian association on Mathematical Physics*, vol. 9, pp 143 -150.

[7] Odman, S. T. A., (1948): Differential equation for calculation of vibrations produced in load-bearing structures by moving loads, Preliminary Publication. *International Association for Bridge and Construction Engineering*, 3rd congress, Liege, Pg 669 – 680.

[8] Weaver, Jr., Timoshenko, P. and Young, D. H.,(1990): *Vibration Problems in Engineering*. 5ed., John Wiley & Sons, Inc. New York.

[9] Meirovitch, L., (1969): *Analytical Methods in Vibrations*. The Macmillan company, Collier-Macmillan. Canada.

[10] Olunloyo, V.O.S and Hutter, K., (1977). The response of anisotropically prestressed thick rectangular membrane to dynamic loading. *Acta mechanica*, 28, 293-311.

[11] Omolofe, B. and Oseni B. M., (2008). Effect of damping and exponentially decaying foundation on the motion of a finite thin beam subjected to travelling loads. *Journal of Nigerian association of mathematical physics* vol. 13 pp 119 - 126.
[12] Oni, S.T. and Tolorunshagba, J.M., (2005). Rotatory inertia influence on the highly prestressed rectangular plates under travelling loads. *Journal of Nigerian association of mathematical physics* vol. 9 pp 103 – 126