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Influence of Elastic Foundations On The Motion And Critical Velocity of Timoshenko Beam Subjected To Fast Traveling Load.

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Abstract

In this present study, the classical problem of the dynamic behaviour of a prismatic Timoshenko beam resting on an elastic foundation and subjected to a variable magnitude moving load, which is harmonic, is investigated using analytical approach. This problem is treated by considering three different types of elastic foundations of which the rigidities are assumed to be functions of the position coordinate in the axial direction of the beam. In particular, rigidities of constant, linear and quadratic functions are used. This is aimed at investigating the dynamic stability of the forced vibrating Timoshenko beam and to know whether this depends on the form and magnitude of the stiffness of the elastic foundation used in the beam model. The versatile Generalized Galerkin's method and Integral transform techniques are used to handle the coupled second order partial differential equations governing the motion of the vibrating system. Analytical solutions are obtained for both the transverse displacement and the rotation of the elastic deep beam. Analytical and Numerical results depict that for all the different forms of foundation considered in this study, as the values of the foundation modulus increases, the transverse displacement response of the beam decreases. It is also found that the critical velocity of the vibrating system increases with an increase in the values of foundation modulus. It is equally observed that when an elastic foundation whose rigidity is of the quadratic form is used as a bearing member for a Timoshenko beam subjected to a fast traveling load the risk of resonance is sufficiently reduced.

Keywords: Dynamic behaviour, moving load, elastic foundation, vibrating system, resonance.

1.0 Introduction

The study of the dynamic behaviour of structural members in the field of Engineering, Mathematical Physics and Applied Mathematics is quite interesting and is of great technological and economical importance, as some of the results obtained may be applicable in understanding the dynamic behaviour of roadways and runways. Therefore, the complex practical problem of the dynamic response of engineering structures to moving concentrated loads has been given considerable attention during the past years.
These structural members are often modeled as one-dimensional element (e.g. beams, rods and membrane), two-dimensional elements (e.g. plates) or three-dimensional elements (e.g. shells) on or without elastic foundation. Scholarly publications on the vibration analyses of plates on elastic foundation or beams and beams on elastic foundation under the action of moving loads are numerous in literature [1]. Among

several authors who have worked extensively in structural dynamics are [2] studied the correction for shear of the differential equation for transverse vibration of prismatic bars. [3] studied the transverse oscillations of beams under the actions of moving

variable loads. [4] studied the critical speeds and the response of a tensional beam on an elastic foundation to repetitive moving loads. [5] studied the dynamic behaviour of multi-span beams under moving loads. [6] who the dynamic response of rectangular plate with moving mass. [7] studied Dynamic behaviour of non-uniform Bernoulli-Euler beam resting on elastic foundation and traverse by loads moving with non-uniform velocity. [8] studied the effect of an added mass to the dynamic response of a prestressed Rayleigh beam traversed by moving masses to mention, but few.

However, in all the aforementioned works, a Bernoulli-Euler beam is the model often employed. It has been reported in the literature that when such beam model is used to study the beam behaviour, the wave velocity solution in the beam becomes unreasonable within the high frequency range and this phenomenon implies that the theory may lead to erroneous results when a beam is subjected to a fast traveling load [9,10]. Thus, to obtain a more reliable and accurate result the effects of shear deformation and rotatory inertia factor must be incorporated into the beam model and this is termed Deep or Timoshenko beam model. The effects due to the rotatory inertia and the deviation of the beam cross-section after deformation are taken into account in the governing equations. These, effects lead to significant improvement in comparison with the classical Bernoulli-Euler beam model [11]

Until recently, the effects of shear deformation and rotatory inertia on both the dynamic response and the critical velocity of a Timoshenko beam were rarely discussed. The problem of vibration of multi-span Timoshenko beams has been studied by [10]. His study shows that the effects of rotatory inertia and shear deformation cause the modal frequencies of the Timoshenko beam to be less than those of the Bernoulli-Euler beam. In the same vein, the problem of thick beams under the action of a variable traveling transverse load was taken up by [12] and in his study, he found that the transverse response of a deep beam decreases as the moving load frequency increases. Nevertheless, their methods of solutions are not suitable for cases where the governing equations of motion possess variable coefficients [10, 12] and also in these studies, the effect of elastic foundation on the deflection and critical velocities of the Tomoshenko beam were not investigated. To the best of author's knowledge, studies on the influence of elastic foundation on the critical velocity of a vibrating system involving a Timoshenko beam resting on elastic foundation has not been reported in literature. The specific aims of this study therefore, are to determine the effects of elastic foundations on the critical velocities and transverse displacement response of uniform Timoshenko beam, to clarify whether the critical velocity of a moving load problem involving Timoshenko beam model depends on the types and form of the bearing member incorporated into the beam model and to deduce which of the three types of the foundations in this study could be regarded as the most efficient bearing member.

2.0 PROBLEM DEFINITION

A uniform Timoshenko beam resting on elastic foundation and traversed by variable magnitude moving load is considered. The beam's properties such as moment of inertia I and the mass per unit length μ of the beam do not vary along the span L of the beam. The beam is assumed to maintain contact with subgrade reaction modulus K. Further, there are no friction forces at the interface. The deflection $W(x,t)$ from the equilibrium and the rotation $U(x,t)$ of the beam under the action of a variable magnitude moving load is described by the system of partial differential Equations [11]

$$\mu W_{tt}(x,t) = K^* GA\left(W_{xx}(x,t) - U_x(x,t)\right) + P(x,t) \tag{2.1a}$$

and

$$I\rho U_{tt}(x,t) = EIU_{xx}(x,t) + K^* GA\left(W_x(x,t) - U(x,t)\right) \tag{2.1b}$$

If the elastic foundation is incorporated into the beam Equations above, then after some rearrangements one obtains

$$\mu \frac{\partial^2 W(x,t)}{\partial t^2} - K^* GF\left[\frac{\partial^2 W(x,t)}{\partial x^2} - \frac{\partial U(x,t)}{\partial x}\right] + K(x)V(x,t) = P(x,t)$$

(2.2a)

and

$$EI\frac{\partial^2 U(x,t)}{\partial x^2} + K^* GF\left[\frac{\partial W(x,t)}{\partial x} - U(x,t)\right] - I\rho \frac{\partial^2 U(x,t)}{\partial t^2} = 0 \qquad (2.2b)$$

as the coupled second order partial differential equation governing the motion of Timoshenko beam resting on elastic foundation and subjected to variable magnitude moving load.

In the above equations, μ is the constant mass m of the beam per unit length L, K* is a constant dependent on the shape of the cross-section, G is the modulus of elasticity in the shear, F is the cross-sectional area, P(x,t) is the harmonic moving force, E is the Young's modulus of the beam, I is the constant moment of inertia of the beam cross-section and ρ is the mass of the beam per unit volume and K(x) is the rigidity of the elastic foundation.

The boundary conditions at the end x = 0 and x = L is given by

$$W(0,t) = 0; \qquad U(0,t) = 0$$

$$\frac{\partial W(L,t)}{\partial x} = 0; \qquad \frac{\partial U(L,t)}{\partial x} = 0 \qquad (2.3)$$

and the initial conditions are

$$W(x,0) = 0 = \frac{\partial W(x,0)}{\partial t} \text{ and } U(x,0) = 0 = \frac{\partial U(x,0)}{\partial t} \qquad (2.4)$$

The variable magnitude moving force $P(x,t)$ which is harmonic acting on the beam is given by

$$P(x,t) = PCos\omega t\delta(x - c_i t) \qquad (2.5)$$

where ω is the frequency of the load c_i is the velocity of the i[th] particle of the system and $\delta(\bullet)$ is the dirac-delta function.

Furthermore, in this present study, we shall consider elastic foundations of three different form of rigidities [7,13] namely,

a) $K(x) = K_o$

b) $K(x) = K_o(bx+1)$

c) $K(x) = K_o\left(ax - bx^2 + cx^3\right)$ (2.6)

Evidently, a closed form solutions to the simultaneous second order partial differential Equations (2.2a) and (2.2b) do not exist. Consequently, an approximate analytical solution is sought to obtain some vital information about the vibrating system.

PROBLEM AND SOLUTION

In order to solve the beam problem above, we shall use the versatile solution technique called Galerkin's method extensively discussed in [7] often used in solving diverse problems involving mechanical vibrations. This solution technique involves solving equations of the form

$$\Theta(V) - P = 0 \qquad (3.1)$$

where,

Θ is the differential operator, V is the structural displacement and P is the traverse load acting on the structure. To this effect, the solutions of the system of equations (2.2b) and (2.2b) are expressed as

$$V_i(x,t) = \sum_{i=1}^{n} P_i(t)Q_i(x) \tag{3.2}$$

and

$$U_i(x,t) = \sum_{i=1}^{n} Y_i(t)X_i(x) \tag{3.3}$$

where the functions $Q_i(x)$ and $X_i(x)$ are chosen to satisfy the pertinent boundary conditions. Thus, substituting equations (3.2) and (3.3) into the simultaneous ordinary differential equations (2.2a) and (2.2b) we obtain

$$\mu\sum_{i=1}^{n} \ddot{P}_i(t)Q_i(x) - K*GF\left[\sum_{i=1}^{n} P_i(t)Q_i''(x) - \sum_{i=1}^{n} Y_i(t)X_i'(x)\right] + K(x)\sum_{i=1}^{n} P_i(t)Q_i(x) = P_o Cos\,\omega t\,\partial(x - c_i t) \tag{3.4}$$

and

$$EI\sum_{i=1}^{n} Y_i(t)X_i''(x) - K*GF\left[\sum_{i=1}^{n} P_i(t)Q_i'(x) - \sum_{i=1}^{n} Y_i(t)X_i(x)\right] - I\rho\sum_{i=1}^{n} \ddot{Y}_i(t)X_i(x) = 0 \tag{3.5}$$

Equations (3.4) and (3.5) after some rearrangements yield

$$\sum_{i=1}^{n} \left\{\mu\ddot{P}_i(t)Q_i(x) - K*GF\left[P_i(t)Q_i''(x) - Y_i(t)X_i'(x)\right] + K(x)P_i(t)Q_i(x)\right\} = P_o Cos\,\omega t\,\delta(x - v_i t) \tag{3.6}$$

and

$$\sum_{i=1}^{n} \left\{EIX_i''(x)Y_i(t) + K*GF\left[P_i(t)Q_i'(x) - Y_i(t)X_i(x)\right] - I\rho\ddot{Y}_i(t)X_i(x)\right\} = 0 \tag{3.7}$$

To determine $P_i(t)$ and $Y_i(t)$, the expressions on the left hand sides of the equations (3.6) and (3.7) are required to be orthogonal to the functions $Q_k(x)$ and $X_k(x)$ respectively. Thus,

$$\int_0^L \sum_{i=1}^{n} \left(\left\{\mu\ddot{P}_i(t)Q_i(x) - K*GF\left[P_i(t)Q_i''(x) - Y_i(t)X_i'(x)\right] + K(x)P_i(t)Q_i(x)\right\} - P_o Cos\,\omega t\,\delta(x - c_i t)\right)Q_j(x)dx = 0 \tag{3.8}$$

and

$$\int_0^L \sum_{i=1}^{n} \left\{EIX_i''(x)Y_i(t) + K*GF\left[P_i(t)Q_i'(x) - Y_i(t)X_i(x)\right] - I\rho\ddot{Y}_i(t)X_i(x)\right\}X_j(x)dx = 0 \tag{3.9}$$

Considering only the i[th] concentrated moving force, Equations (3.8) and (3.9) after some rearrangements and simplifications yield

$$\theta_a(i,j)\ddot{P}_i(t) + \theta_b(i,j)P_i(t) + \theta_c(i,j)Y_i(t) = P_o Cos\,\omega t\,Q_j(c_i t) \tag{3.10}$$

and

$$\gamma_a(i,j)\ddot{Y}_i(t) + \gamma_b(i,j)P_i(t) + \gamma_c(i,j)Y_i(t) = 0 \tag{3.11}$$

Where

Integrals Q_a, Q_b, Q_c, γ_a, γ_b, γ_c and their solutions are stated under appendix.

In view of the boundary conditions (2.4) at ends x = 0 and x = L, functions $Q_i(x)$ and $X_i(x)$ can be chosen as

$$Q_i(x) = Sin\frac{i\pi x}{L} \tag{3.12}$$

and

$$X_i(x) = Cos\frac{i\pi x}{L}.$$

(3.13)

Thus, equations (3.10) and (3.11) are further written to be of the form

$$\theta_A(i,j)\ddot{P}_i(t) + \theta_B(i,j)P_i(t) + \theta_C(i,j)Y_i(t) = P_0 Cos\,\omega t Sin\frac{j\pi c_i t}{L}$$

(3.14)

and

$$\gamma_A(i,j)\ddot{Y}_i(t) + \gamma_B(i,j)P_i(t) + \gamma_C(i,j)Y_i(t) = 0$$

(3.15)

which after some simplifications and rearrangements yield

$$\theta_A(i,j)\ddot{P}_i(t) + \theta_B(i,j)P_i(t) + \theta_C(i,j)Y_i(t) = \frac{P_0}{2}\left[Sin\frac{(\omega + j\pi c_i)t}{L} - Sin\frac{(\omega - j\pi c_i)t}{L}\right]$$

(3.16)

and

$$\gamma_A(i,j)\ddot{Y}_i(t) + \gamma_B(i,j)P_i(t) + \gamma_C(i,j)Y_i(t) = 0$$

(3.17)

Subjecting the system of ordinary differential Equations (3.16) and (3.17) to a Laplace transform defined as

$$(\tilde{\bullet}) = \int_0^\infty (\bullet)e^{-st}dt$$

(3.18)

where s is the Laplace parameter. Applying the initial condition (2.4), we thus obtain the following algebraic simultaneous equations

$$\left(\theta_A(i,j)s^2 + \theta_B(i,j)\right)P_i(s) + \theta_C(i,j)Y_i(s) = \frac{P_0}{2}\left[\frac{\omega + \frac{j\pi c_i}{L}}{Sin\left(\omega + \frac{j\pi c_i}{L}\right)} - \frac{\omega - \frac{j\pi c_i}{L}}{Sin\left(\omega - \frac{j\pi c_i}{L}\right)}\right]$$

(3.19)

and

$$\left(\gamma_A(i,j)s^2 + \gamma_C(i,j)\right)Y_i(s) + \gamma_B(i,j)P_i(s) = 0$$

(3.20)

Solving the simultaneous Equations (3.19) and (3.20) one obtains

$$P_i(s) = \frac{P_0\left(\frac{\omega + \frac{j\pi c_i}{L}}{s^2 + \left(\omega + \frac{j\pi c_i}{L}\right)^2} - \frac{\omega - \frac{j\pi c_i}{L}}{s^2 + \left(\omega - \frac{j\pi c_i}{L}\right)^2}\right)\left(\gamma_A(i,j)s^2 + \gamma_C(i,j)\right)}{2\left[\theta_A(i,j)\gamma_A(i,j)s^4 + \left(\theta_A(i,j)\gamma_C(i,j) + \theta_B(i,j)\gamma_A(i,j)\right)s^2 - \theta_C(i,j)\gamma_B(i,j)\right]}$$

(3.21)

and

$$Y_i(s)=\frac{P_0\left(\frac{\omega+\frac{j\pi c_i}{L}}{s^2+\left(\omega+\frac{j\pi c_i}{L}\right)^2}-\frac{\omega-\frac{j\pi c_i}{L}}{s^2+\left(\omega-\frac{j\pi c_i}{L}\right)^2}\right)\gamma_B(i,j)}{2\left[\theta_A(i,j)\gamma_A(i,j)s^4+\left(\theta_A(i,j)\gamma_C(i,j)+\theta_B(i,j)\gamma_A(i,j)\right)s^2-\theta_C(i,j)\gamma_B(i,j)\right]} \tag{3.22}$$

Equations (3.21) and (3.22) after some simplifications and rearrangements lead to

$$P_i(s)=\frac{P_0}{2R_A}\left[\frac{\omega+\frac{j\pi c_i}{L}}{s^2+\left(\omega+\frac{j\pi c_i}{L}\right)^2}-\frac{\omega-\frac{j\pi c_i}{L}}{s^2+\left(\omega-\frac{j\pi c_i}{L}\right)^2}\right]\frac{\left(\gamma_A(i,j)s^2+\gamma_C(i,j)\right)}{\left(s^2+\alpha_1^2\right)\left(s^2+\beta_1^2\right)} \tag{3.23}$$

and

$$Y_i(s)=\frac{P_0}{2R_A}\left[\frac{\omega+\frac{j\pi c_i}{L}}{s^2+\left(\omega+\frac{j\pi c_i}{L}\right)^2}-\frac{\omega-\frac{j\pi c_i}{L}}{s^2+\left(\omega-\frac{j\pi c_i}{L}\right)^2}\right]\frac{\gamma_B(i,j)}{\left(s^2+\alpha_1^2\right)\left(s^2+\beta_1^2\right)} \tag{3.24}$$

where α and β are clearly defined on the appendix page.

Furthermore, we note that,

$$\frac{\gamma_A(i,j)s^2+\gamma_C(i,j)}{\left(s^2+\alpha_1^2\right)\left(s^2+\beta_1^2\right)}=\frac{\gamma_C(i,j)-\gamma_A(i,j)\alpha_1^2}{\left(\beta_1^2-\alpha_1^2\right)\left(s^2+\alpha_1^2\right)}+\frac{\gamma_A(i,j)\beta_1^2-\gamma_C(i,j)}{\left(\beta_1^2-\alpha_1^2\right)\left(s^2+\beta_1^2\right)} \tag{3.25}$$

and

$$\frac{\gamma_B(i,j)}{\left(s^2+\alpha_1^2\right)\left(s^2+\beta_1^2\right)}=\frac{\gamma_B(i,j)}{\left(\beta_1^2-\alpha_1^2\right)\left(s^2+\alpha_1^2\right)}-\frac{\gamma_B(i,j)}{\left(\beta_1^2-\alpha_1^2\right)\left(s^2+\beta_1^2\right)} \tag{3.26}$$

Thus in view of (3.25) and (3.26), equations (3.23) and (3.24) can be rewritten as

$$P_i(s)=\frac{P_0}{2R_A}\left(\frac{\lambda_1}{s^2+\lambda_1^2}-\frac{\lambda_2}{s^2+\lambda_2^2}\right)\left[\frac{\gamma_A(i,j)\beta_1^2-\gamma_C(i,j)}{\left(\beta_1^2-\alpha_1^2\right)}\left(\frac{1}{\left(s^2+\beta_1^2\right)}\right)-\frac{\gamma_A(i,j)\alpha_1^2-\gamma_C(i,j)}{\left(\beta_1^2-\alpha_1^2\right)}\left(\frac{1}{\left(s^2+\alpha_1^2\right)}\right)\right] \tag{3.27}$$

and

$$Y_i(s)=-\frac{P_0}{2R_A}\left(\frac{\lambda_1}{s^2+\lambda_1^2}-\frac{\lambda_2}{s^2+\lambda_2^2}\right)\left[\frac{\gamma_B(i,j)}{\left(\beta_1^2-\alpha_1^2\right)}\left(\frac{1}{\left(s^2+\alpha_1^2\right)}\right)-\frac{\gamma_B(i,j)}{\left(\beta_1^2-\alpha_1^2\right)}\left(\frac{1}{\left(s^2+\beta_1^2\right)}\right)\right] \tag{3.28}$$

which after some rearrangements and simplifications give

$$P_i(s)=\left[\frac{P_0\gamma_A(i,j)\beta_1}{2R_A(\beta_1^2-\alpha_1^2)}-\frac{P_0\gamma_C(i,j)}{2\beta_1R_A(\beta_1^2-\alpha_1^2)}\right]\left(\frac{\lambda_1}{s^2+\lambda_1^2}\right)\left(\frac{\beta_1}{s^2+\beta_1^2}\right)$$
$$-\left[\frac{P_0\gamma_A(i,j)\alpha_1}{2R_A(\beta_1^2-\alpha_1^2)}-\frac{P_0\gamma_C(i,j)}{2\alpha_1R_A(\beta_1^2-\alpha_1^2)}\right]\left(\frac{\lambda_1}{s^2+\lambda_1^2}\right)\left(\frac{\alpha_1}{s^2+\alpha_1^2}\right)$$
$$-\left[\frac{P_0\gamma_A(i,j)\beta_1}{2R_A(\beta_1^2-\alpha_1^2)}-\frac{P_0\gamma_C(i,j)}{2\beta_1R_A(\beta_1^2-\alpha_1^2)}\right]\left(\frac{\lambda_2}{s^2+\lambda_2^2}\right)\left(\frac{\beta_1}{s^2+\beta_1^2}\right)$$
$$+\left[\frac{P_0\gamma_A(i,j)\alpha_1}{2R_A(\beta_1^2-\alpha_1^2)}-\frac{P_0\gamma_C(i,j)}{2\alpha_1R_A(\beta_1^2-\alpha_1^2)}\right]\left(\frac{\lambda_2}{s^2+\lambda_2^2}\right)\left(\frac{\alpha_1}{s^2+\alpha_1^2}\right)$$
(3.29)

and

$$Y_j(S)=-\frac{P_0\gamma_B(i,j)}{2R_A(\beta_1^2-\alpha_1^2)}\left[\frac{\lambda_1}{S^2+\lambda_1^2}\cdot\frac{1}{S^2-\beta_1^2}-\frac{\lambda_1}{S^2+\lambda_1^2}\cdot\frac{1}{S^2-\alpha_1^2}-\frac{\lambda_2}{S^2+\lambda_2^2}\cdot\frac{1}{S^2-\beta_1^2}-\frac{\lambda_2}{S^2+\lambda_2^2}\cdot\frac{1}{S^2-\alpha_1^2}\right]$$
(3.30)

where

$$\lambda_1=\omega+\frac{k\pi c_i}{L},\quad \lambda_2=\omega-\frac{k\pi c_i}{L}$$
(3.31)

In what follows , the Laplace inversion of equations (3.29) and (3.30) is sought. To this effect, we employ the following representations

$$g_1(s)=\frac{\lambda_1}{s^2+\lambda_1^2},\quad f_1(s)=\frac{\beta_1}{s^2+\beta_1^2}$$
(3.32)

$$g_2(s)=\frac{\lambda_2}{s^2+\lambda_2^2},\quad f_2(s)=\frac{\alpha_1}{s^2+\alpha_1^2}$$
(3.33)

so that the Laplace inversion of the equation (3.29) is given as the convolution of f_i 's and g_i 's defined as

$$f_i*g_j=\int_0^t f_i(t-u)g_j(u)du \quad \text{where} \quad \begin{matrix} i=1,2,3......\\ j=1,2,3......\end{matrix}$$
(3.34)

Thus the Laplace inversion of (3.29) is given by

$$P_i(s)=\left[\frac{P_0\gamma_A(i,j)\beta_1}{2R_A(\beta_1^2-\alpha_1^2)}-\frac{P_0\gamma_C(i,j)}{2\beta_1R_A(\beta_1^2-\alpha_1^2)}\right]H_A-\left[\frac{P_0\gamma_A(i,j)\alpha_1}{2R_A(\beta_1^2-\alpha_1^2)}-\frac{P_0\gamma_C(i,j)}{2\alpha_1R_A(\beta_1^2-\alpha_1^2)}\right]H_B$$
$$-\left[\frac{P_0\gamma_A(i,j)\beta_1}{2R_A(\beta_1^2-\alpha_1^2)}-\frac{P_0\gamma_C(i,j)}{2\beta_1R_A(\beta_1^2-\alpha_1^2)}\right]H_C+\left[\frac{P_0\gamma_A(i,j)\alpha_1}{2R_A(\beta_1^2-\alpha_1^2)}-\frac{P_0\gamma_C(i,j)}{2\alpha_1R_A(\beta_1^2-\alpha_1^2)}\right]H_D$$
(3.35)

where

$$H_A=\int_0^t Sin\beta_1(t-u)Sin\lambda_1u\,du,\qquad H_B=\int_0^t Sin\alpha_1(t-u)Sin\lambda_1u\,du$$

$$H_C = \int_0^t Sin\beta_1(t-u)Sin\lambda_2 u\,du\,, \qquad H_D = \int_0^t Sin\alpha_1(t-u)Sin\lambda_2 u\,du \tag{3.36}$$

where solutions of integrals (3.36) are presented under the appendix.
Thus in view of equation (3.2) taking into account (3.35) one obtains

$$W_i(x,t) = \sum_{i=1}^{n} \left\{ \left[\frac{P_0\gamma_A(i,j)\beta_1}{2R_A(\beta_1^2-\alpha_1^2)} - \frac{P_0\gamma_C(i,j)}{2\beta_1 R_A(\beta_1^2-\alpha_1^2)} \right] H_A - \left[\frac{P_0\gamma_A(i,j)\alpha_1}{2R_A(\beta_1^2-\alpha_1^2)} - \frac{P_0\gamma_C(i,j)}{2\alpha_1 R_A(\beta_1^2-\alpha_1^2)} \right] H_B \right.$$
$$\left. - \left[\frac{P_0\gamma_A(i,j)\beta_1}{2R_A(\beta_1^2-\alpha_1^2)} - \frac{P_0\gamma_C(i,j)}{2\beta_1 R_A(\beta_1^2-\alpha_1^2)} \right] H_C + \left[\frac{P_0\gamma_A(i,j)\alpha_1}{2R_A(\beta_1^2-\alpha_1^2)} - \frac{P_0\gamma_C(i,j)}{2\alpha_1 R_A(\beta_1^2-\alpha_1^2)} \right] H_D \right\} \times Sin\frac{i\pi x}{L} \tag{3.37}$$

which represents the transverse displacement response of the prismatic Timoshenko beams resting on elastic foundation and subjected to variable magnitude moving load.
Similarly,

$$Y_i(t) = -\frac{P_0\gamma_B(i,j)}{2R_A(\beta_1^2-\alpha_1^2)} \left[\frac{1}{\beta_1}H_A - \frac{1}{\alpha_1}H_B - \frac{1}{\beta_1}H_C + \frac{1}{\alpha_1}H_D \right] \tag{3.38}$$

which on inversion yields

$$U_i(x,t) = \sum_{i=1}^{n} -\left\{ \frac{P_0\gamma_B(i,j)}{2R_A(\beta_1^2-\alpha_1^2)} \left[(H_A - H_C)\frac{1}{\beta_1}H_A - (H_B - H_D)\frac{1}{\alpha_{1\,D}} \right] \right\} \times Cos\frac{i\pi x}{L} \tag{3.39}$$

which is the rotation of the uniform deep beam under the action of variable magnitude moving load.
Following the same arguments as those presented in [15], it is can be shown that the series solutions (3.37) and (3.39) converge rapidly.

4.0 DISCUSSION OF THE SOLUTION

In studying an undamped system such as this, it is desirable to examine resonance phenomena, because the deflection of a Timoshenko beam under the actions of traveling load may grow without bounds. The velocity of the load which brings about resonance effect in the vibrating system is termed critical velocity.
Equation (3.33) clearly shows that the prismatic elastic deep beam resting on elastic foundation will experience resonance effects whenever

$$4[\theta_B(i,j)\gamma_C(i,j) - \theta_C(i,j)\gamma_B(i,k)] = [\theta_A(i,j)\gamma_C(i,j) + \theta_B(i,j)\gamma_A(i,j)]^2\,, \tag{4.1}$$

$$\beta_1^2 = \lambda_1^2\,,\quad \beta_1^2 = \lambda_2^2\,, \tag{4.2}$$

$$\alpha_1^2 = \lambda_1^2\,,\quad \alpha_1^2 = \lambda_2^2 \tag{4.3}$$

It is also observed that for all the forms and types of the foundation rigidities considered in this study, as the foundation modulli increases the critical velocity of the dynamical system increases thereby reducing the risk of resonant effects. This implies that elastic foundations of the form of rigidities considered in this study would produce a stabilizing effects on a Timoshenko beam constantly subjected to a fast traveling loads. This result agrees with what has been reported earlier [14]

5.0 COMMENTS ON THE NUMERICAL RESULTS

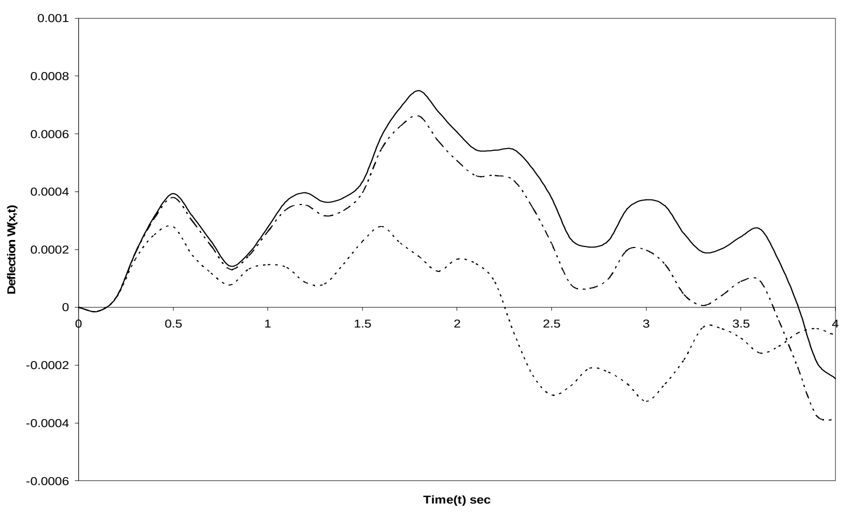
In order to present some typical results in this study, an elastic beam of length L = 12.192m is considered. The velocity c_i of the moving load is taken to be 8.128 meters/ second. The values of foundation moduli are varied between 0 N/m^3 and 10000 N/m^3 and the value of the natural frequency ω is taken to be $\frac{2\pi}{3}$.

Figure 1 displays the deflection profile of an elastic deep beams resting on variable elastic foundation of the form 'a' and subjected to variable magnitude moving load. The figure shows that as the value of foundation stiffness K_o increases the deflection of the beam at various time t decreases. Similar results are displayed in Figures 2 and 3; respectively for the elastic foundation of the form 'b' and form 'c'.
For the purpose of comparison, the deflection profile of the timoshenko beam for all the elastic foundation types considered in this study is illustrated in figure 4. It is clearly seen from the figure that the response amplitude of the beam is highest for the elastic foundation of the form 'a', while the transverse displacement response of the beam is the lowest for the elastic foundation of the form 'c' which implies that an elastic foundation of the quadratic form acts as the most efficient bearing member.
Figures 5 and 6 which are the graphs of the critical velocity for resonant conditions (4.2) and (4.3) have been plotted against the foundation stiffness K_o for all foundation types considered. The graphs show that as the K_o increases, the critical velocities of the dynamical system increases. It is equally observed that the foundation stiffness of the form 'c' produces the highest critical velocity. Thus, the risk of resonance is sufficiently reduced if the foundation stiffness of the form 'c' is used as the bearing member to support an elastic structures under the actions of fast traveling load.

6.0 CONCLUSIONS

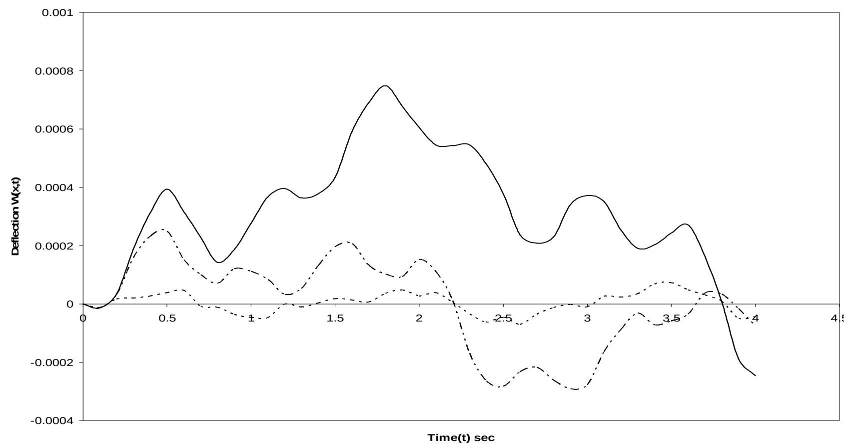
The Generalized Galerkin's method and Integral Transform technique has been used to solve the problem of an elastic deep beam resting on elastic foundation and subjected to a harmonic variable magnitude moving load. Analytical solutions have been obtained for both the deflection and the rotation of the beam. The objective is to study the behaviour of the Timoshenko beam when subjected to the moving load. In particular, the effect of foundation stiffness on the dynamic stability of the vibrating system is investigated. Analytical solution and Numerical result in plotted curves show that, as the value of foundation stiffness K_o increases the deflection profile of the deep beam decreases. It is equally observed from figures 5 and 6 that the critical velocities of the dynamical systems increase with an increase in the values of foundation stiffness K_o for all the types of the elastic foundation considered. This, suggests that in general, higher values of foundation stiffness K_o reduce the risk of resonance in a

dynamical system involving uniform beam under the action of a moving load irrespective of the foundation type considered. Finally, it is clearly seen that the critical velocity of a prismatic deep beam resting on elastic foundation and traversed by fast traveling load depends to large extent on the type of the elastic foundation on which the beam is supported.



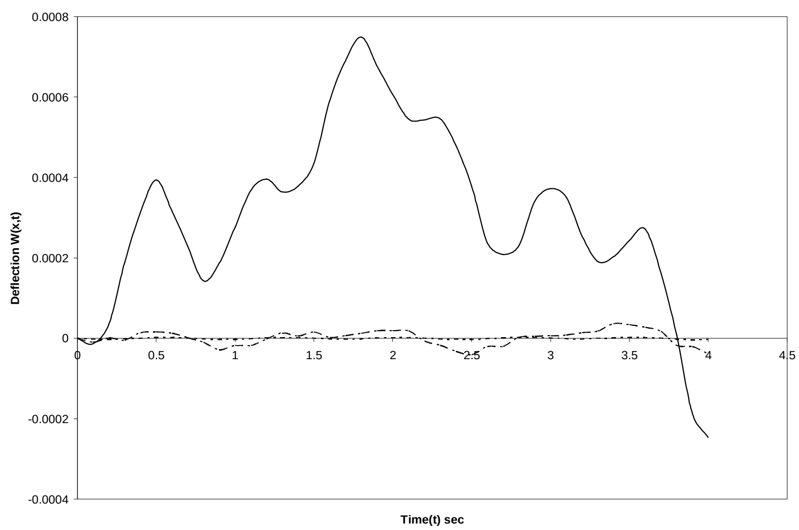
$K_0 = 0$, $K_0 = 10000$, $K_0 = 100000$

Fig 1: The transverse displacement response of Timoshenko beam subjected to variable magnitude traveling load for various values of foundation moduli K_0 for the foundation type $K(x) = K_0$



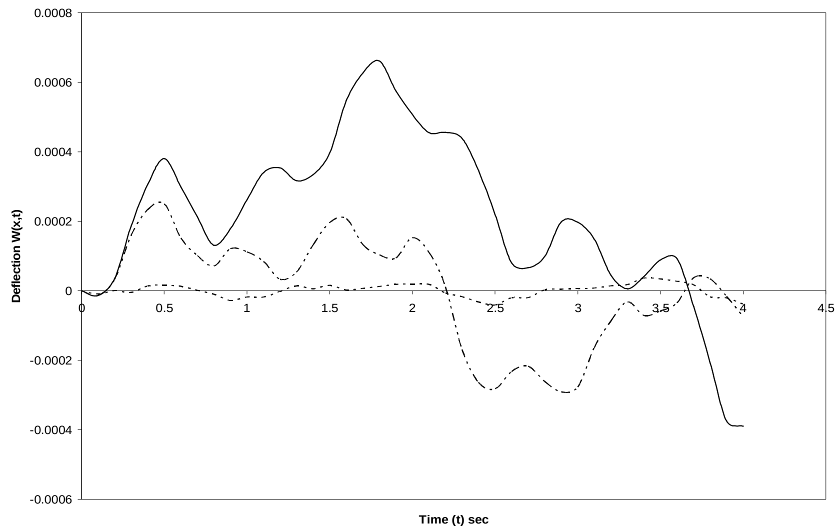
$K_0 = 0$, $K_0 = 10000$, $K_0 = 100000$

Fig 2: The transverse displacement response of Timoshenko beam subjected to variable magnitude traveling load for various values of foundation moduli K_0 for the foundation having rigidity of the form $K(x) = K_o(bx+1)$



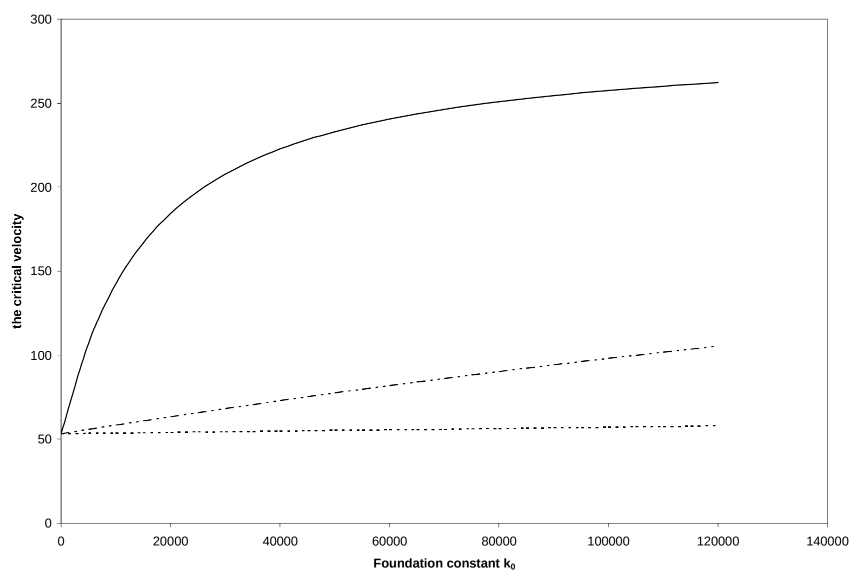
$K_0 = 0$, $K_0 = 10000$, $K_0 = 100000$

Fig 3: The transverse displacement response of Timoshenko beam subjected to variable magnitude traveling load for various values of foundation moduli K_0 for the foundation having rigidity of the form $K(x) = K_o\left(ax - bx^2 + cx^3\right)$



foundation type a, foundation type b, foundation type c

Fig 4: Comparison of the deflection profile of Timoshenko beam subjected to variable magnitude traveling load for all the three types of foundation for $K_0 = 10000$



foundation type c, foundation type b, foundation type c

Fig 5: Comparison of the critical velocities of Timoshenko beam subjected to variable magnitude traveling load for all the three types of foundation for various values of K_0 when $\beta_1^2 = \lambda_1^2$

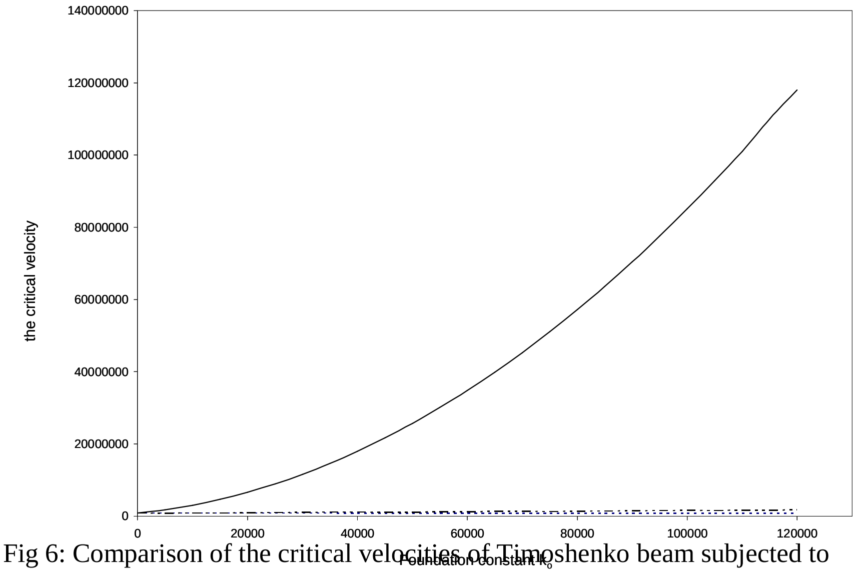


Fig 6: Comparison of the critical velocities of Timoshenko beam subjected to variable magnitude traveling load for all the three types of foundation for various values of K_0 when $\alpha_1^2 = \lambda_1^2$

APPENDIX

The following are the evaluated integrals which are made use of in this work

$$\theta_a(i,j) = \mu\int_0^L Q_i(x)Q_j(x)dx \tag{A1}$$

$$\theta_b(i,j) = \int_0^L \left[-K^*GFQ_i''(x) + K(x)Q_i(x)\right]Q_j(x)dx \tag{A2}$$

$$\theta_c(i,j) = K^*GF\int_0^L X_i'(x)Q_j(x)dx \tag{A3}$$

$$\gamma_a(i,j) = I\rho\int_0^L X_i(x)X_j(x)dx \tag{A4}$$

$$\gamma_b(i,j) = K^*GF\int_0^L QX_i(x)X_j(x)dx \tag{A5}$$

and

$$\gamma_c(i,j) = \int_0^L EIX_i''(x)X_j(x)dx - \int_0^L K^*GFX_i(x)X_j(x)dx \tag{A6}$$

In view of (3.12) and (3.13), integrals (A1) to (A6) become

$$\theta_A(i,j) = \int_0^L \mu Sin\frac{i\pi x}{L}Sin\frac{j\pi x}{L}dx \tag{A7}$$

$$\theta_B(i,j) = \int_0^L \left[K(x)Sin\frac{i\pi x}{L}Sin\frac{j\pi x}{L} + \left(\frac{i\pi}{L}\right)^2 K*GF\,Sin\frac{i\pi x}{L}Sin\frac{j\pi x}{L}\right]dx \tag{A8}$$

$$\theta_C(i,j) = \left(\frac{i\pi}{L}\right)\int_0^L Sin\frac{i\pi x}{L}Sin\frac{j\pi x}{L}dx \tag{A9}$$

$$\gamma_A(i,j) = -I\rho\int_0^L Cos\frac{i\pi x}{L}Cos\frac{j\pi x}{L}dx \tag{A10}$$

$$\gamma_B(i,j) = \frac{i\pi}{L}K*GF\int_0^L Cos\frac{i\pi x}{L}Cos\frac{j\pi x}{L}dx \tag{A11}$$

$$\gamma_C(i,j) = -\int_0^L \left[EI\left(\frac{i\pi}{L}\right)^2 Cos\frac{i\pi x}{L}Cos\frac{j\pi x}{L} + K*GF\,Cos\frac{i\pi x}{L}Cos\frac{j\pi x}{L}\right]dx \tag{A12}$$

which are thus evaluated to give

$$\theta_A(i,j) = \begin{cases} 0 & i \neq j \\ \frac{\mu L}{2} & i = j \end{cases} \tag{A13}$$

$$\theta_B(i,j) = \begin{cases} \frac{K_0 L}{2} + \frac{i^2\pi^2}{L^2}K*GF\cdot\frac{L}{2}; & K(x) = K_0 \\ \frac{K_0 L}{2} + K_1\frac{L^2}{4} + \frac{i^2\pi^2}{L^2}K*GF\cdot\frac{L}{2}; & K(x) = K_1 x + K_0 \\ K_0 L^2\left(1 - \frac{L}{2} + \frac{L^2}{8}\right) + \frac{i^2\pi^2}{L^2}K*GF\cdot\frac{L}{2}; & K(x) = K_0(4x - 3x^2 - x^3) \end{cases}; \tag{A14}$$

$$\theta_C(i,j) = \begin{cases} 0 & i \neq j \\ \frac{i\pi}{L}k*GF\cdot\frac{L}{2} & i = j \end{cases} \tag{A15}$$

$$\gamma_A(i,j) = -I\rho\frac{L}{2} \tag{A16}$$

$$\gamma_B(i,j) = \frac{i\pi}{L}K*GF\cdot\frac{L}{2} \tag{A17}$$

$$\gamma_C(i,j) = -\left(EI\frac{i^2\pi^2}{L^2} + K*GF\right)\cdot\frac{L}{2} \tag{A18}$$

In equations (3.23) and (3.24),

$$\alpha_1^2 = \frac{R_B}{2R_A} - \sqrt{\frac{R_B^2}{4R_A^2} - \frac{R_C}{R_A}}, \qquad \beta_1^2 = \frac{R_B}{2R_A} + \sqrt{\frac{R_B^2}{4R_A^2} - \frac{R_C}{R_A}} \tag{A19}$$

where

$$R_A = \theta_A(i,j)\gamma_A(i,j);\ R_B = \theta_A(i,j)\gamma_C(i,j) + \theta_B(i,j)\gamma_A(i,j),\ R_C = \theta_B(i,j)\gamma_C(i,j) - \theta_C(i,j)\gamma_B(i,j) \tag{A20}$$

It is straightforward to show that

$$\int_0^t SinB(t-u)SinAudu = \frac{BSinBt}{B^2 - A^2}\left[SinBtCosAt + \frac{A}{B}(CosAtCosBt - 1)\right] - \frac{BCosBt}{B^2 - A^2}\left[SinAtCosBt - \frac{A}{B}SinBtCosAt\right]$$

(A21)

In view of (A21), integrals (3.36) are thus evaluated and one obtains,

$$H_A = \frac{\beta_1 Sin\beta_1 t}{\beta_1^2 - \lambda_1^2}\left[Sin\beta_1 tCos\lambda_1 t + \frac{\lambda_1}{\beta_1}(Cos\lambda_1 tCos\beta_1 t - 1)\right] - \frac{\beta_1 Cos\beta_1 t}{\beta_1^2 - \lambda_1^2}\left[Sin\beta_1 tCos\lambda_1 t - \frac{\lambda_1}{\beta_1}Sin\beta_1 tCos\lambda_1 t\right]$$

(A22)

$$H_B = \frac{\alpha_1 Sin\alpha_1 t}{\alpha_1^2 - \lambda_1^2}\left[Sin\alpha_1 tCos\lambda_1 t + \frac{\lambda_1}{\alpha_1}(Cos\lambda_1 tCos\alpha_1 t - 1)\right] - \frac{\alpha_1 Cos\alpha_1 t}{\alpha_1^2 - \lambda_1^2}\left[Sin\alpha_1 tCos\lambda_1 t - \frac{\lambda_1}{\alpha_1}Sin\alpha_1 tCos\lambda_1 t\right]$$

(A23)

$$H_C = \frac{\beta_1 Sin\beta_1 t}{\beta_1^2 - \lambda_2^2}\left[Sin\beta_1 tCos\lambda_2 t + \frac{\lambda_2}{\beta_1}(Cos\lambda_2 tCos\beta_1 t - 1)\right] - \frac{\beta_1 Cos\beta_1 t}{\beta_1^2 - \lambda_2^2}\left[Sin\beta_1 tCos\lambda_2 t - \frac{\lambda_2}{\beta_1}Sin\beta_1 tCos\lambda_2 t\right]$$

(A24)

$$H_D = \frac{\alpha_1 Sin\alpha_1 t}{\alpha_1^2 - \lambda_2^2}\left[Sin\alpha_1 tCos\lambda_2 t + \frac{\lambda_2}{\alpha_1}(Cos\lambda_2 tCos\alpha_1 t - 1)\right] - \frac{\alpha_1 Cos\alpha_1 t}{\alpha_1^2 - \lambda_2^2}\left[Sin\alpha_1 tCos\lambda_2 t - \frac{\lambda_2}{\alpha_1}Sin\alpha_1 tCos\lambda_2 t\right]$$

(A25)

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