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A numerical study of the hemodynamics of stenosed artery

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Abstract

In this paper, the non-Newtonian flow of blood in large blood vessel is studied by using Eyring–Powell model. We also assumed a variable blood viscosity. The momentum equation for the flow is non-dimensionalized and the resulting non-linear dimensionless equation is then solved numerically under various flow conditions. Variations of different flow parameters are conducted and discussed.

Keywords
viscoelastic, hemodynamics, stenosis, non-Newtonian flow.

1.0 Introduction

Blood is a specialized body fluid that delivers necessary substances to the body cell such as; nutrients and oxygen it also transports waste products away from those same cells. However under disease condition like stenosis there is a reduced blood supply to the heart, this condition may be acquired like in atherosclerosis in which there is fatty deposit in the inner lumen of the coronary artery.

In a recent paper [1], gave an important model of blood viscosity in terms of hematocrit, here we use the earlier result by [2] to extend his work to accommodate the visco-elastic nature of blood in a deformed large vessel. Several authors who worked in this direction had used power law model to study blood flow due to stenosis but a major disadvantage of this model is that it does not account for the elastic nature of blood.

The objective of the present paper is to investigate the visco-elastic effect on the blood flow through stenosed artery with the assumption that the viscosity in not constant and the elastic parameter is not negligible.

The paper is organized in this form; in section 1 we give brief introduction and the statement of problem, in section 2 of the work, the problem is formulated and non-dimensionalized, in section 3, the problem is solved and in section 4, numerical results are presented and discussed. While section 5 gives some concluding remarks.

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2.0 Problem formulation

We consider the viscous, incompressible and laminar flow of blood flowing through large. The flow is driven by a pulsatile pressure gradient in the flow direction in the case of large vessels and driven by a constant pressure gradient in the case of blood flow through capillaries. The geometry of the flow under consideration in this present study is given below.

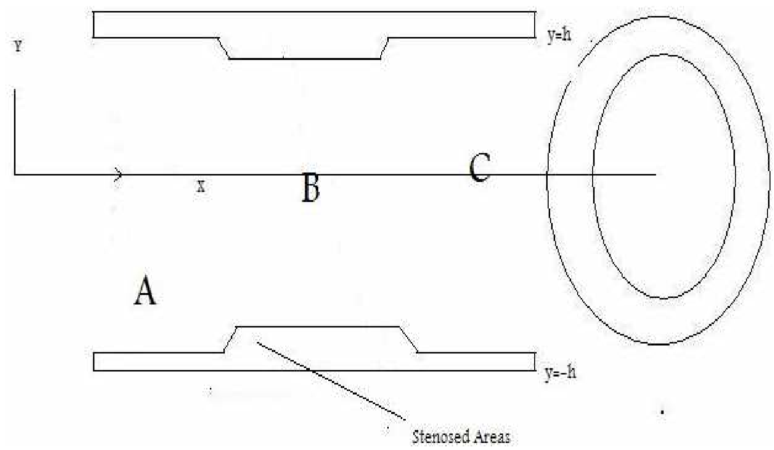

Figure 2.1: Flow geometry

To analyse the model, the following major assumptions are made: -

1. The walls are rigid and stationary, this become imperative due to the formation of bonny plaque in the inner wall of the lumen thus reducing its elastic response to the pulse wave
2. Under this configuration, we have assume that there is a reduction in the flow velocity and that the fluid after the stenosed region will be deficient in red blood cell but be rich in other blood constituents,

In region A, there is possibility of platelet activation if the degree of stenosis is very high due to reduction in flow velocity, causing Rolleaux formation due to low shear stress this implies increase in blood viscosity and the fluid becomes non-Newtonian

In region B, there is reduction in the artery diameter, as compared to a normal range of 0.05-0.3 cm but the diameter of normal and healthy red blood cell is approximately 8×10^{-4}, therefore the red blood cell capsule undergo high deformation to pass through the stenosed region B, hence under a disease condition the viscosity profile is a function of flow geometry

In region C, we assumed that there is possibility of coronary ischemia (insufficient oxygenated blood) and the fluid will be rich in plasma which contains dissolved proteins and electrolytes

3. A two dimension flow is considered
4. The flow is fully developed

The flow continuity equation is

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \tag{2.1}$$

The momentum equation is

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}\right)=-\frac{dp}{dx}+\frac{\partial}{\partial y}\tau_{xy} \tag{2.2}$$

Following [1], τ_{xy} represents the stress tensor in the case of the non-polar theory of fluids. The

model for describing the shear of a non Newtonian flow is derived from the theory of rate processes. This model can be used in some cases to describe the viscous behaviour of polymer solutions and viscoelastic suspensions over a wide range of shear rates. The stress tensor in the Eyring-Powell model for non-Newtonian fluids takes the form

$$\tau_{xy} = \mu \frac{\partial u_i}{\partial x_j} + \frac{1}{\alpha} \sinh^{-1}\left(\frac{1}{c} \frac{\partial u_i}{\partial x_j} \right) \tag{2.3}$$

where μ represents the dynamic viscosity, α, c are characteristics of the Eyring–Powel model, u = velocity, ρ = density, μ = viscosity, p = pressure, x = co-ordinate in the direction of flow, y = coordinate across the flow. Assuming a constant flow area we take

$$v = v_0 = 0 \tag{2.4}$$

The second term of (2.3) is expanded as

$$\sinh^{-1}\left(\frac{1}{c} \frac{\partial u}{\partial y} \right) \cong \frac{1}{c} \frac{\partial u}{\partial y} + \frac{1}{6}\left(\frac{1}{c} \frac{\partial u}{\partial y} \right)^3, \quad \left| \frac{1}{c} \frac{\partial u}{\partial y} \right| < 1 \tag{2.5}$$

Assuming that $\mu = \mu(y)$ alone,

$$\rho\left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} \right) + \mu \frac{\partial^2 u}{\partial x^2} + \frac{1}{c} \frac{\partial^2 u}{\partial x^2} + \frac{1}{c} \frac{\partial^2 u}{\partial y^2} \tag{2.6}$$

and

$$0 = -\frac{dp}{dy} \tag{2.7}$$

And in view of (2.4) the continuity equation (2.1) reduces to

$$\frac{du}{dx} = 0 \tag{2.8}$$

Then (2.8) and (2.4) equation (2.6) becomes

$$\rho \frac{\partial u}{\partial t} = -\frac{dp}{dx} + \frac{\partial}{\partial y}\left(\mu \frac{\partial u}{\partial y} \right) + \frac{1}{c} \frac{\partial^2 u}{\partial y^2} + \frac{1}{2c^3}\left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} \tag{2.9}$$

Introducing the following dimensionless parameters

$$u' = \frac{u}{v_0}, \quad x' = \frac{x}{h}, \quad y' = \frac{y}{h}, \quad t' = \frac{v_0 t}{h}, \quad p' = \frac{p}{\rho v_0^2}, \mathrm{Re} = \frac{v_0 h}{\nu}, \nu = \frac{\mu_0}{\rho} \tag{10}$$

Then (2.9) can be written in dimensionless form

$$\frac{\partial u'}{\partial t'} = \lambda + \frac{1}{\mathrm{Re}} \frac{\partial}{\partial y'}\left(\mu' \frac{\partial u'}{\partial y'} \right) + \frac{a}{\mathrm{Re}} \frac{\partial^2 u'}{\partial y'^2} + \frac{\in}{\mathrm{Re}}\left(\frac{\partial u}{\partial y} \right)^2 \frac{\partial^2 u}{\partial y^2} \tag{2.11}$$

Where $a = \frac{1}{\alpha c \mu_0}, \in = \frac{v_0}{2\alpha\rho c^3 h^2 \mu_0}, \lambda = -\frac{dp}{dx}$ and Re- Reynold's Number. The relationship between red blood cell, viscosity and distance between vessel [1]

$$\mu' = e^{\beta(1-y^2)} \tag{2.12}$$

Where β represents the viscosity variation parameter

Substituting (2.11) into equation (2.12) and dropping bars since there is no confusion, Re=1

$$\frac{\partial u}{\partial t}=\lambda+\frac{\partial}{\partial y}\left(e^{\beta\left(1-y^{2}\right)}\frac{\partial u}{\partial y}+a\frac{\partial u}{\partial y}+\in\left(\frac{\partial u}{\partial y}\right)^{3}\right) \tag{2.13}$$

With the appropriate initial and boundary conditions

$$\begin{aligned} &t\le 0:\ u=Sin(\pi y)\ \forall\ y\\ &t>0:\ u(-1)=0=u(1)\end{aligned} \tag{2.14}$$

2.1 Existence Theorem 1 [4]

Let D denote the region in $(n + 1)$ dimensional space, one dimension for t and n dimensions for the vector x)

$$|t-t_0|\le a,\ \|X-X_0\|\le b \tag{2.15}$$

And suppose that $f(x,t)$ satisfies the condition

$$\frac{\partial f_i}{\partial x_j},\ i,j=1,\ldots,n \tag{2.16}$$

are continuous in D.

Then there is a constant $\delta > 0$ such that there exists a unique continuous vector solution $X(t)$ of the system

$$\begin{aligned} X_1' &= f_1(X_1,\ldots..X_n,t), X_1(t_0)=X_{10}\\ X_2' &= f_2(X_1,\ldots..X_n,t), X_2(t_0)=X_{20}\\ X_n' &= f_n(X_1,\ldots...X_n,t), X_n(t_0)=X_{n0}\end{aligned} \tag{2.17}$$

In the interval $|t-t_0|\le\delta$.

***Theorem* 2.1**

Let $\frac{\partial u}{\partial t}=\lambda+\frac{\partial}{\partial y}\left(e^{\beta\left(1-y^{2}\right)}\frac{\partial u}{\partial y}+a\frac{\partial u}{\partial y}+\in\left(\frac{\partial u}{\partial y}\right)^{3}\right)$ with

$$\begin{aligned} &t\le 0:\ u=Sin(\pi y)\ \forall\ y\\ &t>0:\ u(-1)=0=u(1)\end{aligned}$$

Then there exists a unique solution of the problem

Proof

By using the above theorem. Neglecting small parameters by assuming that the second approximation

$$\left|\frac{1}{c}\frac{\partial u}{\partial y}\right|<1 \tag{2.18}$$

Then (2.13) becomes

$$\frac{\partial u}{\partial t}=\lambda+\frac{\partial}{\partial y}\left(e^{\beta\left(1-y^{2}\right)}\frac{\partial u}{\partial y}+a\frac{\partial u}{\partial y}\right) \tag{2.19}$$

Let us introduce a similarity variable

$$\eta = \frac{y}{2\sqrt{t}} \tag{2.20}$$

Let $\beta \to 0, \ \mu \to 1$. Then

$$\left(\frac{d^2u}{d\eta^2}\right) = -\frac{2}{(1+a)}\left(\lambda + \eta\frac{du}{d\eta}\right) \tag{2.21}$$

Now let

$$\begin{aligned} x_1 &= \eta, \\ x_2 &= u \\ x_3 &= \frac{du}{d\eta} \end{aligned} \tag{2.22}$$

to obtain

$$\begin{pmatrix} x_1' \\ x_2' \\ \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 \\ x_3 \\ \\ -\dfrac{2(\lambda + x_1x_2)}{1+a} \end{pmatrix} = \begin{pmatrix} f_1\ (x_1, x_{2,}\ x_3) \\ f_2(x_1,\ \ x_{2,} x_3) \\ \\ f_3(x_1\ \ x_{2,}\ \ x_3) \end{pmatrix} \tag{2.23}$$

Subject to the initial conditions

$$\begin{aligned} x_1(0) &= 0 \\ x_2(0) &= -1 \\ x_3(0) &= D \end{aligned} \tag{2.14}$$

Now we need to proof that $\frac{\partial f_i}{\partial x_j}$ are continuous in the domain *D*. Differentiating each term of f_i with respect to x_i, we obtain

$$\begin{aligned} &\frac{\partial f_1}{\partial x_1} = 0, \frac{\partial f_1}{\partial x_2} = 0, \frac{\partial f_1}{\partial x_3} = 0 \\ &\frac{\partial f_2}{\partial x_1} = 0, \frac{\partial f_2}{\partial x_2} = 0, \frac{\partial f_2}{\partial x_3} = 1 \\ &\frac{\partial f_3}{\partial x_1} = -\frac{2x_2}{(1+a)},\ \frac{\partial f_3}{\partial x_2} = -\frac{2x_1}{(1+a)}, \frac{\partial f_3}{\partial x_3} = 0 \end{aligned} \tag{2.15}$$

From above, since there is no singularities then $\frac{\partial f_i}{\partial x_j}$ exist and continuous, hence the proof.

3.0 Method of solution

Special case 1: Pulsatile pressure gradient i.e.

$$-\frac{\partial p}{\partial x} = P_0 + P_1 e^{i\omega t} \tag{3.1}$$

and

$$u = U_0 + U_1 e^{i\omega t} \tag{3.2}$$

Substituting (3.1) and (3.2) in (2.13), we obtain two ordinary differential equations

$$0 = P_0 + \frac{d}{dy}\left(\left(e^{\beta(1-y^2)}\right)\frac{dU_0}{dy} + a\frac{dU_0}{dy}\right) \tag{3.3}$$

and in terms of U_1, we obtain

$$0 = P_1 + \frac{d}{dy}\left(\left(e^{\beta(1-y^2)}\right)\frac{dU_1}{\partial y} + a\frac{dU_1}{dy}\right) - i\omega U_1 \tag{3.4}$$

To obtain the solutions of (3.3) and (3.4) we assume $0 < \beta << 1$

We then seek an asymptotic expansion about β in the form

$$U_0 = u_0 + \beta u_1 + \beta^2 u_2 \quad \text{(Neglecting other terms)} \tag{3.5}$$

and taking the Taylor's expansion of $e^{\beta(1-y^2)}$ about β

$$e^{\beta(1-y^2)} = 1 + \beta(1-y^2) + 0.5\beta^2(1-y^2)^2 \tag{3.6}$$

Substituting (3.5), (3.6) into (3.3) and (3.4) equating coefficients in favour of β, we have

$$\beta^0 : \ 0 = \lambda + (1+a)\frac{d^2u_0}{dy^2} \tag{3.7}$$

with

$$u_0(-1) = 0 = u_0(1) \tag{3.8}$$

$$\beta^1 : \ 0 = \frac{d}{dy}\left((1-y^2)\frac{du_0}{dy}\right) + (1+a)\frac{d^2u_1}{dy^2} \tag{3.9}$$

with

$$u_1(-1) = 0 = u_1(1) \tag{3.10}$$

$$\beta^2 : \ 0 = \frac{d}{dy}\left(\left(\frac{(1-y^2)^2}{2}\right)\frac{du_0}{dy}\right) + \frac{d}{dy}\left((1-y^2)\frac{du_1}{dy}\right) + (1+a)\frac{d^2u_2}{dy^2} \tag{3.11}$$

with

$$u_2(-1) = 0 = u_2(1) \tag{3.12}$$

For U_1, we have

$$\beta^0 : 0 = \lambda + (1+a)\frac{d^2u_0}{dy^2} - i\omega u_0 \tag{3.13}$$

with

$$u_0(-1) = 0 = u_0(1) \tag{3.14}$$

$$\beta^1 : 0 = \frac{d}{dy}\left((1-y^2)\frac{du_0}{dy}\right) + (1+a)\frac{d^2u_1}{dy^2} - i\omega u_1 \tag{3.15}$$

with

$$u_1(-1) = 0 = u_1(1) \tag{3.16}$$

$$\beta^2 : 0 = \frac{d}{dy}\left(\left(\frac{(1-y^2)^2}{2}\right)\frac{du_0}{dy}\right) + \frac{d}{dy}\left((1-y^2)\frac{du_1}{dy}\right) + (1+a)\frac{d^2u_2}{dy^2} - i\omega u_2 \quad (3.17)$$

with $$u_2(-1) = 0 = u_2(1) \quad (3.18)$$

Using mathematica version 6, equations (3.7) – (3.18) are solved analytically, due to the large size of the solution we present only the graphical solution of equation (3.2) in figures 3.1-3.3.

Special Case 2 : Constant pressure gradient

At the capillary level the flow is no more pulsatile but has attained a mean steady state. Now we then use the method of Adomian decomposition [3] to obtain our result. By using Adomian algorithm the solution of (2.19) is given as

$$u_0(y,t) = u(y,0) = Sin(\pi y) + L_t^{-1}G$$

$$u_{n+1}(y,t) = L_t^{-1}L_y\left\{\left(1 + \beta(1-y^2) + (1-y^2)^2\frac{\beta^2}{2}\right)L_y u_n(y,t) + aL_y u_n(y,t)\right\} \quad (3.19)$$

Using mathematica version 6 we iterate (3.19) and the numerical result is in figure 3.4 – 3.5

Special Case 3:- Steady flow

The initial condition independent model is given as,

$$0 = \lambda + \frac{d}{dy}\left(\exp\beta(1-y^2)\frac{du}{dy} + a\frac{du}{dy}\right) \quad (3.20)$$

$$u(-1) = 0 = u(1)$$

***Definition* 1**: [5] for a system of equation

$$x_i' = \sum_{j=1}^{n} a_{ij}(t)x_j \quad (i = 1,...,n) \quad (3.21)$$

where a_{ij} are continuous functions on some closed bounded t interval [a, b]. If f is the vector with components f_i defined by

$$f_i(t,x) = \sum_{j=1}^{n} a_{ij}(t)x_j \quad (i = 1,...,n) \quad (3.22)$$

then f satisfies the Lipschitz condition on the (n + 1)-dimensional region $D: a \le t \le b \quad |x| < \infty$.

Infact

$$|f(t,x_1) - f(t,x_2)| \le k|x_1 - x_2| \quad (3.23)$$

where

$$k = \sum_{i=1}^{n} |a_{ij}(t)| \quad (t\varepsilon[a,b];\ j = 1,...,n) \quad (3.24)$$

***Theorem* 3.1**: (Existence and Uniqueness Theorem)

For the non-linear system (3.19), where the functions $a_{ij}\ \varepsilon\ C$ on [a, b], there exists one and only one solution φ of (3.19) on [a,b] passing through any point $(\tau,\xi)\,\varepsilon\, D$ that is $\varphi(\tau) = \xi$

We shall use the above theorem to prove our result

Proof

We have to show that the vector f satisfies the Lipschiitz condition on D. then the existence and uniqueness is guaranteed at the initial point $x_1(0) = -1$. It suffices to show that the solution can be continued to a unique solution in the entire interval [a,b]

Now, let $x_1 = y,\ x_2 = u$ and $x_3 = u'$. (3.25)

Then, (3.20) becomes

$$\begin{pmatrix} x_1' \\ x_2' \\ \\ x_3' \end{pmatrix} = \begin{pmatrix} 1 \\ x_3 \\ \\ \dfrac{\lambda - 2\beta x_1 x_3 \exp \beta\left(1 - x_1^2\right)}{a + \exp \beta\left(1 - x_1^2\right)} \end{pmatrix} = \begin{pmatrix} f_1(x_1, x_2, x_3) \\ f_2(x_1, x_2, x_3) \\ \\ f_3(x_1, x_2, x_3) \end{pmatrix} \qquad (3.28)$$

subject to the initial conditions

$$x_1(0) = 0$$
$$x_2(0) = -1$$
$$x_3(0) = D$$

Again we have to show that $\dfrac{\partial f_i}{\partial x_j} = k$ for the problem to satisfy the Lipschitz condition in definition one. Differentiating each term of f_i with respect to x_i, we obtain

$$\frac{\partial f_1}{\partial x_1} = 0\ , \frac{\partial f_1}{\partial x_2} = 0,\ \frac{\partial f_1}{\partial x_3} = 0$$

$$\frac{\partial f_2}{\partial x_1} = 0,\ \frac{\partial f_2}{\partial x_2} = 0,\ \frac{\partial f_2}{\partial x_3} = 1$$

$$\frac{\partial f_3}{\partial x_1} = \frac{-2\beta x_3\left(a + e^{\beta\left(1-x_1^2\right)}\right)\left(1 + 2x_1\beta\right)e^{\beta\left(1-x_1^2\right)}}{\left(a + e^{\beta\left(1-y^2\right)}\right)^2},\ \ \frac{\partial f_3}{\partial x_2} = 0,\ \frac{\partial f_3}{\partial x_3} = \frac{-2\beta x_1 \exp \beta\left(1 - x_1^2\right)}{a + \exp \beta\left(1 - x_1^2\right)}$$

From above, it is now obvious that $\dfrac{\partial f_i}{\partial x_j}$ exist and continuous. Hence the proof, clearly, $\dfrac{\partial f_i}{\partial x_j} = k$ therefore the function satisfies the Lipschitz condition. It follows from (3.7) – (3.11) that (3.30) gives

$$u = \frac{\lambda\left(1 - y^2\right)}{2(1 + a)} + \frac{\beta\lambda\left(-1 + 2y^2 - y^4\right)}{4(1 + a)^2} + \frac{\beta^2\lambda(-1 + a)\left(-1 + y^2\right)^3}{12(1 + a)^3} \qquad (3.28)$$

As the asymptotic solution, while the graphical solutions are given in figures 3.1, 3.2, 3.3, 3.4, 3.5, 3.6, 3.7 and 3.8.

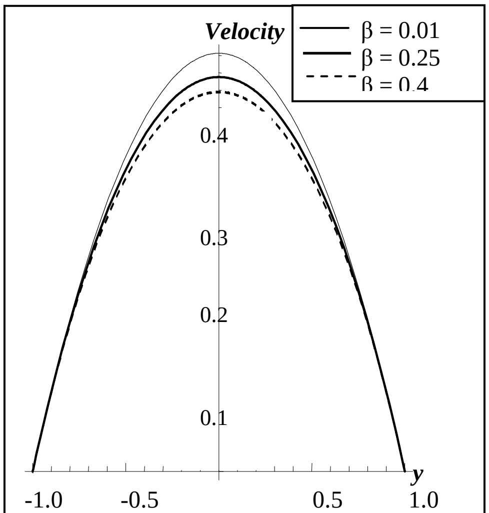

Figure: 3.1: Velocity against distance

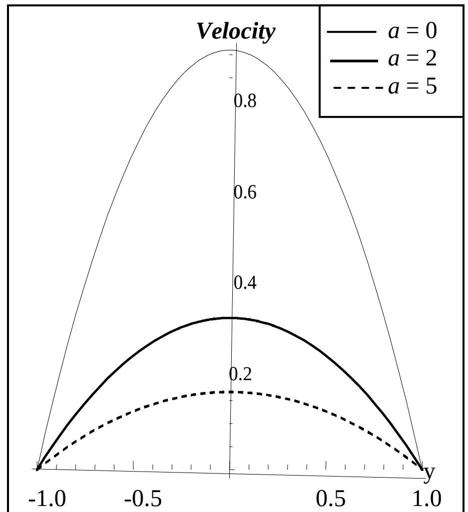

Figure 3:2: Velocity against distance

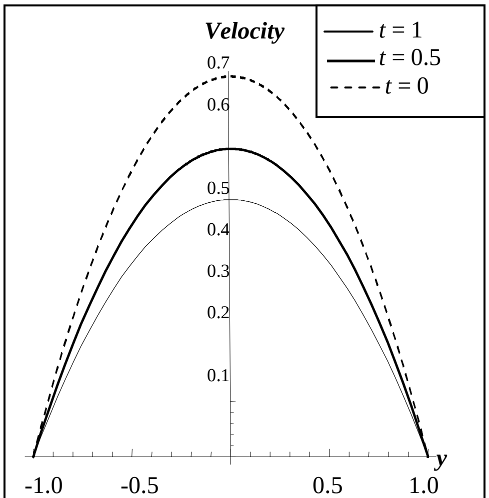

Figure: 3.3: Velocity against distance

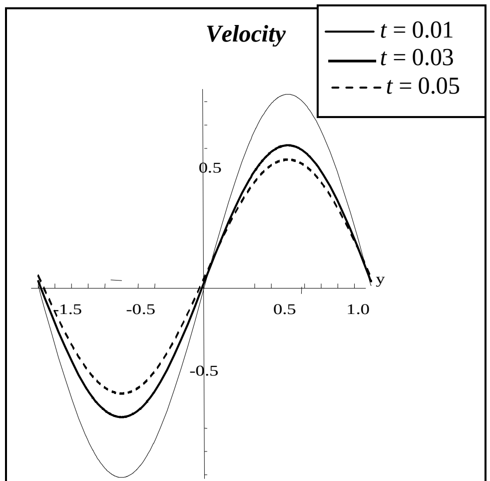

Figure: 3.4: Velocity against distance

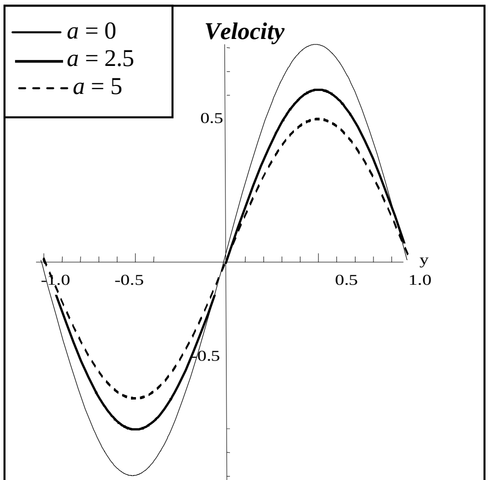

Figure: 3.5: Velocity against distance

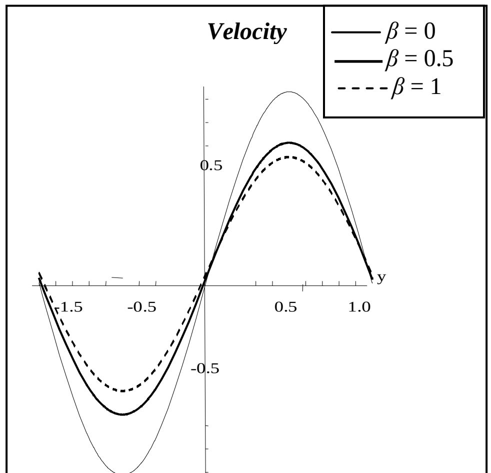

Figure: 3.6: Velocity against distance

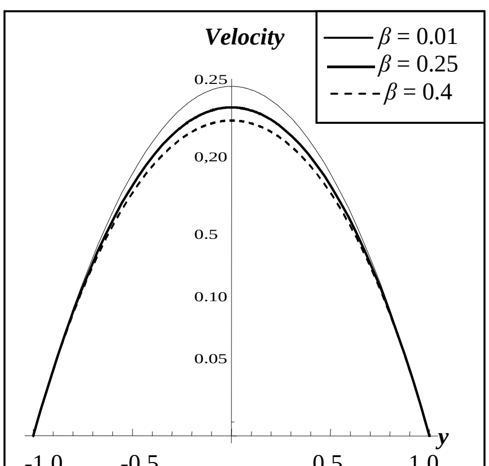

Figure 3.7: Velocity against distance

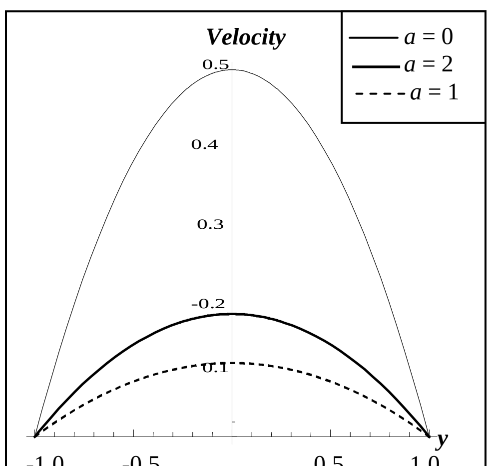

Figure 3.8: Velocity against distance

4.0 Discussion of results

In figures 3.1, 3.6 and 3.7 we observed that as the viscosity parameter increases the flow velocity reduces this is due largely to the fact that as viscosity of blood increases the non-Newtonian characteristic become more pronounced.

In figures 3.2, 3.5 and 3.8 variations of the elastic parameter shows that as the stenosis height increases the artery diameter obviously reduce and the elasticity of blood increases this has an end effect on the flow velocity causing reduction in flow velocity.

In figures 3.3 and 3.4 we observe that as time increases the velocity of flow in the artery increases while the flow velocity reduces with time in the capillary flow.

4.0 Conclusion

We have studied the hemodynamics of a stenosed artery; the possible application of this work is in the diagnosis and treatment of cardiovascular diseases. Our result shows that increase in the whole blood viscosity is a very dangerous situation which poses a major health risk.

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