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Batch arrival discrete time queue with gated vacation system

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Abstract

A class of single server vacation queues, which have batch arrivals and single server, is considered in discrete time. Here the server goes on vacation of random length as soon as the system becomes empty. On return from vacation, if he finds any customers waiting in the queue, the server starts serving the customers one by one until the queue size is zero (the queue discipline is FIFO); otherwise he takes another vacation and so on. The vacation model understudy here is the Gated systems: In a gated system, as soon as the server returns from vacation it places a gate behind the last waiting customer. It then begins to serve only the customers who are within the gate, based on some rules of how many or how long it could serve. It is shown here that the interarrival, service, vacation and server operation time can be cast with markov based representation then this class of vacation models can then be studied as matrix-product problem which belongs to a class of matrix analytic family - thereby allowing us to use result from* [2] *to solve the resulting matrix product problem. Most importantly it is shown that using discrete time modelling approach to study some vacation model is more appropriate and makes the model much more algorithmically tractable.

Keywords
Batch Arrival, Discrete time model, matrix product problem, Gated Vacation system.

1.0 Introduction

Vacation in queueing context mean the period the server is not attending to a particular targeted queue. The server may be under repair, attending to other queues or simply forced to stop serving customers in the particular queue. Vacation model have been used extensively to study various systems, such as polling and priority systems.

In a polling system, N queues are attended to by one server who attends to only one queue at a time. The server attends to one queue for a period of time based on some predefined rules and then proceeds to the next queue and so on.

Consider an arbitrary queue among the N queues. The customers in this particular queue view the server as being away on a vacation because they are not being attended to. This example of polling system is very common in computer systems where a processor has to attend to several queues of jobs.

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Another example is road intersection control by traffic signals. At any given time one section of the road receives the green signal for service while the other section receives the red signal to stop service. The sections which receive the red signal are not receiving service and to them the server is on a vacation**.**

Priority queues are also sometimes studied as vacation queues. Consider a single server system with at least two classes of customers in which there is a priority for service. A low priority group of customers may keep receiving service until a higher priority customer arrives, after which the server may switch, depending on the predefined rules, to serving this higher priority customer. While the higher priority customer is receiving service the low priority group of customers sees the server as having gone on a vacation.

Several types of method have been used to study vacation model, ranging from embedded Markov chain to the classical transform approach. It is my hope to use the matrix-geometric method, set up in discrete time [2] to investigate the batch arrival discrete time queue with server vacation. This includes single server vacation queues with batch arrival, provided that the service, vacation and operational times can be represented by a Markov based model and the system is set up in discrete time. Vacation models as classified by Alfa [1] are stated below:

1. **Gated systems**

In a gated system, as soon as the server returns from a vacation it places a gate behind the last waiting customer. It then begins to serve only the customers who are within the gate, based on some rules of how many or how long it could serve.

2. **Ungated systems**

In an ungated system the server only applies the rule of how many or how long it could serve.

Under each of the classification above, we have further features, such as: single or multiple vacations; time-limited service – preemptive and non-preemptive, random interruptions for vacation, and others.

The subject of vacation queues has appeared in different literature in the last fifty years. For detail study of previous work in vacation models [1].

Here we will mention some few papers and books that are in the direction of this research work.

Doshi [6] gave a survey of queuing systems where some vacation occurs. The queuing systems he study are those where the server works on primary and secondary customers such situation arise in many computer, communication, production and many other Stochastic systems. He formulated two models. First, the server keeps taking vacations until on return from vacation there is at least one customer present. Second, the server takes exactly one vacation at the end of each busy period. He also gave variety of techniques used in studying the system. He then show that with the help of the two model it is possible to analyze other model as vacation model for detail of his work see [5].

In their work [3] considers a discrete time gated vacation system. The numbers of customers arriving in the primary and secondary queues are model by means of independent and identically distributions (*i i d*) with common probability mass function and a corresponding probability generating function. The model combines both gated and exhaustive vacation system. They obtain various performance measures such as moments of the queue contents and moment of customer delay.

Their systems also take multiple vacations like those of [5]. In addition to the work above [4] investigate the gated multiple vacation queue in discrete time. The generalized multiple vacation queuing models he developed allows capture of performance amongst the multiple vacation, single vacation and limited multiple vacation gated queuing systems.

Choudhury [2] uses the compound Poisson arrival and generalized vacation to analyze Batch Arrival Poisson queue with vacation.

In another development [11] treat the batch Markovian arrival process whose vacation schedule and lengths of whose vacation times depend on the queue length of the system at the beginning of a vacation. Alfa [1] provides a generalized form for a class of discrete time vacation model. He provides a unified framework for analyzing vacation models in discrete time and present matrix-analytic method for analyzing them.

Jau-Chuan [7] studies the *N* policy $M^{[x]}/G/1$ queue with server vacations; startup and breakdowns, where the arrival form a compound poisson process and service times are generally distributed.

Ojobor and Omosigho [9] study the effect of two queue discipline (FIFO and LIFO) on some measure of performance of a single server queue system using simulation. The approach is to generate arrival times and service for 200 customers and the customer is served through a single server queueing system under each queue discipline.

Ojobor and Omosigho [10] study batch arrival discrete time queue with ungated server vacation. They consider class of single server vacation queues, which have batch arrivals and single server, is considered in discrete time. Here the server goes on vacation of random length as soon as the system becomes empty. On return from vacation, if he finds any customers waiting in the queue, the server starts serving the customers one by one until the queue size is zero (the queue discipline is FIFO); otherwise he takes another vacation and so on. The vacation model they study here is the Ungated systems**.**

Other class of vacation model of interest to researcher lately is the case of vacation models in retrial systems. This class is very important when studying mobile communication and some computer networks. For results see [12].

Modern telecommunication systems have become more digital systems than analog these days. It is therefore more appropriate to develop vacation models which are applicable to these systems using discrete time approach.

The aim of the current contribution is to investigate the Batch Arrival discrete time queue with server vacation. The goal is to model Batch Arrival Vacation models in discrete time and use the matrix analytic method to analyze the model. Here we restrict ourself to the **gated** vacation system.

2.0 Batch arrival discrete time queue model with gated vacation system

In this section we shall consider the extension of [2] model. The aim is to remodel [1] model to allow room for batch arrival with server vacation. A rephrase of the model is given as follows:

- i = the number of items in the system.
- i' = number of items inside a gate (when applicable).
- k = the phase of arrival: arrival is phase type with representation $(\boldsymbol{\alpha}, T)$. The arrival rate is λ. Here the arrival is in batches i.e. the batch arrival is phase type with representation $(\boldsymbol{\alpha}, T)$.
- j = the phase of service: service is phase type with representation $(\boldsymbol{\beta}, S)$.
- j' = the phase of service interruption. This has m+1 phases including phase 0, when there is no interrupted service.
- l = the phase of vacation: vacation is phase type with representation $(\boldsymbol{\delta}, L)$.

- u = the time clock of a server's visit (or the number served so far by a server during a visit) – the use depends on the model. This could also represent the phase of the operational time with representation *(***u***,U)*.

The following parameters are also define

- N = the limited time of a single visit by a server for a time-limited service.
- M = the limited number of customers to be served during a server visit for number-limited service.

For some models where services may be interrupted we need to define a matrix Q which represents the phase at which an interrupted service begins when it resumed, given the interruption phase. The elements $Q_{j_1' j_1}$ refer to the probability that a service interrupted in phase j'_1 resumes in phase j_1 when the service re-starts. For example, a preemptive resume service has $Q = I$ and a preemptive repeat service rule has $Q = \mathbf{1}\boldsymbol{\beta}$.

2.1 Gated

Here we consider the gated system. The state space for this system is described below:

- $\Delta_0 = (0,k,l), k = 1,2,\dots n, l = 1,2,\dots r$
- $\Delta_0^v = (i,k,l), i = 1,2,\dots,\ k = 1,2,\dots n, l = 1,2,\dots r,$
- $\Delta_i^s = (i,(i^1 k,j)), i = 1,2,\dots, k = 1,2,\dots,n, i^1 = 1,2,\dots i-1, j = 1,2,\dots,m$
- $\Delta_i^t = (i,((i^1 = i),k,j), i = 1,2,\dots, k = 1,2,\dots,n, j = 1,2,\dots,m.$

For $\Delta_0 = (0,k,l)$, the first 0 refers to an empty system, k refers to phase of arrival and l refers to the phase of vacation. By definition of the gated system we have more than one vacation.

Let $\Delta_i = \Delta_i^v U \Delta_i^s U \Delta_i^t$. The state space for the gated vacation system is given by Δ as

$$\Delta = \Delta_0 \prod_{i=1}^{\infty} \Delta_i$$

When the system is in states $\Delta_i, i \geq 1$, the chain can only have transitions to states $\Delta_{i-1}, \Delta_{i+1}$ or remain in state Δ_i. However, because we have a supplementary variable i^1 at level i which depend on, the transition probabilities are level dependent $\forall i \geq 0$. This Markov chain is thus a level dependent QBD. If we now label the states lexicographic order, and let the first index be know as level except for level i=1. When the system is in states Δ_0 the chain can only have transitions to states Δ_1 or remain in states Δ_0. This Markov chain is thus a level-independent QBD. If we now label the states in lexicographic order, and let the first index be level $i, i \geq 0$. The resulting transition matrix for this Markov chain can then be represented in equation (2.1) below. We then apply the matrix geometric result to solve it.

The interior block matrices for the system, i.e. the matrices $A_{i,i+1}$, $A_{i,i}$, and $A_{i,i-1}$ which represent the transition from Δ_i *to* Δ_{i+1}, Δ_i *to* Δ_i and Δ_i *to* Δ_{i-1}, $\forall i \geq 2$. Here we shall reblock Alfa model to allow room for batch arrival. The block matrices for the system are given below:

$$p = \begin{bmatrix} A_{0,0} & A_{0,1} & & & & \\ A_{1,0} & A_{1,1} & A_{1,2} & & & \\ & A_{2,1} & A_{2,2} & A_{2,3} & & \\ & & A_{3,2} & A_{3,3} & A_{3,4} & \\ & & & \vdots & \ddots & \vdots \\ & & & & \cdots & \end{bmatrix} \tag{2.1}$$

where

$$A_{i,i-1} = \begin{bmatrix} 0 & 0 \\ (e_1 \otimes e_1' (T \quad \otimes (S^0 \delta)) & \tilde{I}(i-1) \otimes T \otimes (S^0 \beta) \end{bmatrix}$$

$$A_{i,i+1} = \begin{bmatrix} (T^0 \alpha) \otimes L & e_1' \otimes (T^0 \alpha) \otimes (L^0 \beta) \\ 0 & I(i) \otimes (T^0 \alpha)\, S \end{bmatrix}$$

$$A_{i,i} = \begin{bmatrix} T \otimes L & e_1' \otimes T \otimes (L^0 \beta) \\ 0 & I(i) \otimes T \otimes S + \tilde{I}(i-1) \otimes (T^0 \alpha) \otimes (S^0 \beta) \end{bmatrix}$$

where

$$\tilde{I} = \begin{bmatrix} 0 & 0 \\ I(v) & 0 \end{bmatrix}$$

$I(v)$ is an identity matrix of size v and e_1 is of appropriate order as defined earlier. Here the term zero on the top left corner refers to transition during a vacation with batch arrival; that is no arrival. The term $zero$ on the top right corner refers to a transition from vacation to service commencement with batch arrival; that is no vacation yet. And the term $\tilde{I}(i-1) \otimes T \otimes (S^0 \beta)$ on the bottom right corner refers to a transition during service with batch arrival. The term on the bottom left corner $(e_1 \otimes e_1' (T \quad \otimes (S^0 \delta))$ refers to a transition from a service with batch arrival.

Next we apply the matrix -product solution by [12]. In order to obtain *x* we choose a level $i = k$, such that the probability of arrival during a vacation exceeding *k* is less than some very small value $\varepsilon = 10^{-q}$, usually $q \geq 12$. We then assume that the chances of the number in the gate exceeding *k* is negligible – hence we approximate *p* by *p*(*k*) with $A_{k+v,K+v-1} = A_{k+1}, A_{k+v,k+v} = A_k$ and $A_{k+v,k+v+1} = A_{k-1}\ \forall\ v \geq 0$. This now becomes a level independent system with transition matrix similar to that Ojobor (2008) but with a large boundary.

2.2 Matrix-Product solution for *P*

For the class of problems above results available in literature are for finite order ie. (n,m,r,s) < ∞.

The system describe above is a spatially inhomogenous QBD process in discrete time, sometimes referred to as level-dependent QBD. Because of this spatial inhomogeneity we cannot apply the standard matrix-geometric method as [10]. In order to calculate the equilibrium distribution we must first evaluate the matrices $R_k, k \geq 0$, which are minimal non-negative solutions to

$$R_k = A_{k,k+1} + R_k A_{k,k} + R_k \left[R_{k+1} A_{k,k-1}\right], \qquad \forall\, k \geq 0$$

where $(R_k)_{ij}$ is expected number of visits to state (k+1,j) before first visit to level k conditional on the process starting in state (k,i). It is easy to show here that the steady state vector x can be determined as follows

$$x_k = x_{k-1} R_{k-1} = x_0 \prod_{j=0}^{k-1} R_j, k \geq 1$$

where x_0 is obtained by solving the system of equations

$$x_0 \left[A_{0,0} + R_0 A_{1,0}\right] = x_0$$

and the normalization expression

$$[x_0 \textstyle\sum_{k=0}^{\infty}[\prod_{j=0}^{k-1} R_j]]\, \mathbf{1} = 1, where\, R_{-1} = 1$$

is then applied. Note that provided the system is stable the solution to

$$x_0 \left[A_{0,0} + R_0 A_{1,0}\right] = x_0$$

satisfies

$$x_0 [\textstyle\sum_{k=0}^{\infty}[\prod_{j=0}^{k-1} R_j]]\, \mathbf{1} < \infty.$$

Ramaswami and Taylor obtained explicit solution for Ri, i = 0, 1, 2, 3, … see [6].
Note that the matrices Gk are the minimal non-negative solutions to the matrix Quadratic equations

$$G_k = A_{k,k-1} + A_{k,k} G_k + A_{k,k+1} G_{k+1} G_k, \qquad k \geq 0.$$

All gated systems are of this type.

3.0 Performance measures

We shall look at the performance of the system understudy. First we define f_i as a column vectors whose element is either 0 or 1. We also define $f_i^{\,1}$ as row vector which is of the same order as x_i. The element of x_i have a relationship which corresponds to the element of $f_i^{\,1}$.

Let Γ_i represent the set of all location in $f_i^{\,1}$ for which there are 1's, hence all the other location have 0's. Example, $\Gamma_i(i = \overline{\iota_1^1 \quad \iota_2^1})$ which implies the values of 1's in all location where the number in the gate is between $i_1^1\, and\, i_2^1$ inclusive. Hence we write the corresponding $f_i^{\,1}$ vector as $f_i^{\,1}(\Gamma_i) = f_i^{\,1}(i^1 = \overline{\iota_1^1 \quad \iota_2^1})$.

3.1 Queue length Distribution (Gated Case)

Based on the result in section 2.2, the vector x can be partitioned as

$$x = [x_0, x_1, x_2 \ldots],$$

and further as

$$\boldsymbol{x}_i = [x_{i,1}, x_{i,2}, \ldots, x_{i,i'-1}, x_{i\ i'}, x_{i\ i'+1}, \ldots x_{i,i}].$$

The marginal probability density, y_i , of finding I customers in the system at an arbitrary time is given as

$$y_i = x_i \mathbf{1}.$$

Let y_i be the marginal probability of finding i (Batch) customers in the system at an arbitrary time, then

$$y_i = x_i 1$$

We denote the mean number of customers waiting in the system at arbitrary time by μ_L.

$$\mu_L = x_0 \left[\sum_{\xi=1}^{\infty} \xi \prod_{i=0}^{\xi-1} R_i \right] \mathbf{1}$$

The conditional probability y_i^v, of finding I customers waiting in the system during a vacation, is obtained as follows. Let v_o be the probability that the server is on vacation then

$$v_o = x_o \mathbf{1} + x_o \sum_{i=1}^{\infty} [\prod_{\xi=0}^{i-1} R_\xi] f_i(l = \overline{1,r}), \text{ and } y_i^v = \frac{x_i f_i(l=\overline{1,r})}{v_0}, i \geq 1$$

$$y_0^v = \frac{x_0 \mathbf{1}}{v_0}$$

Also let denote the conditional expectation of the number of customers waiting when the server is on vacation by μ_v,

$$\mu_v = x_0 \left[\sum_{\xi=1}^{\infty} \xi \prod_{i=0}^{\xi-1} R_i \right] \frac{f_i(l=\overline{1,r})}{v_0},$$

The conditional probability y_i^s, of finding I customers waiting in the system during service is obtained as follows. Let s_0 be the probability that service is ongoing at an arbitrary time. Then $s_0 = 1 - v_0$. Therefore we have

$$y_i^s = \frac{x_i [\mathbf{1} - f_i(l=\overline{1,r})]}{s_0}.$$

The conditional expectation of the number of customers waiting in the system when the server is busy serving, μ_s , is given by

$$\mu_s = x_0 \left[\sum_{\xi=1}^{\infty} \xi \prod_{i=0}^{\xi-1} R_i \right] \frac{\mathbf{1} - f_i(l=\overline{1,r})}{s_0},$$

3.2 Distribution of work in the system

To write the equation for the work in the system for the level-dependent case, we need to define the following $S_1 = S^0\beta$, $G^{(\psi)}(\psi) = S^{\psi-1}S_1$, $\psi \geq 1$, $G^{\xi}(\xi) = S_1^{\xi}, \xi \geq 1$ and

$G^{\xi}(\psi) = S_1 G^{(\xi-1)}(\psi - 1) + SG^{(\xi)}(\psi - 1), \psi \geq \xi + 1,\ \xi \geq 2$. The matrix $G^{(\xi)}(\psi)$ is of dimension $m \times n$. Note that $G^0(\psi) = 0$, for all $\psi \geq 1$. $G^{(v)}(\psi) = 0$, for all $\psi < v$ and $G^{(0)}(0) = 1$ by definition. Also let $D_0 = T$ and $D_1 = T^0\alpha$.

For the level-dependent case let

$$x_{i,i} = [x^v_{i,i}, x^s_{i,i}],$$

where the superscript v represents vacation and s represents service, and $x_{i,i}$ implies $x_{i,\ i'=i}$. Note that for the level-dependent case we have

$$x^s_{i,i'} = [x^s_{i,i',1}, x^s_{i,i',2}, \dots, x^s_{i,i',N}].$$

The probability V^p_a that the amount of work in the system at instant of return from a vacation is 'a' is given by

$$v^p_a = c^{-1}[\sum_{i=1}^{a}\sum_{d=0}^{1} x^v_{i,i}\left((D_d \otimes L^o)\otimes\left(Q^* G^{(i+d)}(a)\right)\mathbf{1}\right) + x^v_o(D_1 \otimes L^o)\otimes G^{(1)}_a \mathbf{1}]$$

$a \geq 1$

where

$$c = \sum_{a=1}^{\infty}[x^v_o((D_1 \otimes L^o)\otimes G^{(1)}_a)\mathbf{1} + \sum\sum x^v_{i,i}\left((D_d \otimes L^o)\otimes\left(Q^* G^{(i+d)}(a)\right)\right)\mathbf{1}]$$

The probability v^v_a that the amount of work left behind just after vacation start 'a' is given by

$$v^v_o = b^{-1}\sum_{u=1}^{N} x^s_{1,1,u}(D_o \otimes s^o)\mathbf{1}\ and$$

$$v^v_a = b^{-1}\sum_{i=1}^{a}\sum_{i'=1}^{i}\sum_{w=1}^{a}\sum_{d=0}^{1} x^s_{i,i',N}(D_d \otimes((s^{w-1}s^o)G^{(i'+d-1)}(a-w+1)))\mathbf{1}$$

where

$$b = \sum_{u=1}^{N} x^s_{1,1,u}(D_o \otimes s^o)\mathbf{1}$$

$$+\sum_{a=1}^{\infty}\sum_{i=1}^{a}\sum_{i'=1}^{i}\sum_{w=1}^{a}\sum_{d=0}^{1} x^s_{i,i',N}(D_d \otimes((s^{w-1}s^o)G^{(i'+d-1)}(a-w+1)))\mathbf{1}.$$

3.3 Waiting time distribution

Let $x_i, i \geq 0$ be partition as $x_i = [x^v_i, x^s_{i,1}, \dots, x^s_{i,N}]$. Define the vectors $Z_i, i \geq 0$ as the corresponding vectors to x_i, where z_i corresponds to the steady state vector of the system as seen by a (Batch) arriving customers. Then

$$z_0^v = \lambda^{-1} x_o^v\big((T^o\alpha)\otimes(L + L^o\delta)\big) + \lambda^{-1}\sum_{u=1}^{N} x_{1,u}^s((T^o\alpha)\otimes(s^o\delta))$$

$$z_i^v = \lambda^{-1}\big[x_i^v\big((T^o\alpha)\otimes L\otimes I(m+1)\big) + x_{i,N}^s\big((T^o\alpha)\otimes\delta\otimes s^*\big)$$
$$+ x_{i+1,N}^s\big((T^o\alpha)\otimes\delta\otimes s^o\big)\big] \quad i \geq 1$$

$$z_{i,1}^s = \lambda^{-1}\big[x_i^v\big((T^o\alpha)\otimes(L^o\otimes Q^*)\big)\big]$$

$$z_{i,u}^s = \lambda^{-1}\big[x_{i,u-1}^s\big((T^o\alpha)\otimes s\big) + x_{i+1,u-1}^s\big((T^o\alpha)\otimes(s^o\beta)\big)\big]$$
$$i \geq 1, 2 \leq u \leq N$$

We define

$$\tilde{z}_o^o = z_o^v(1\otimes I) \, and \text{ for } i \geq 1 \text{ define } \tilde{z}_{i,o}^{v,o} = z_i^v(1\otimes I)$$
$$\tilde{z}_{i,u}^{s,o} = z_{i,u}^s(1\otimes I)$$

Let

$$\tilde{z}_i^o = [\tilde{z}_{i,o}^{v,o}, \tilde{z}_{i,1}^{s,o}, \tilde{z}_{i,2}^{s,o}] \qquad i \geq 1.$$

Finally, let

$$\tilde{z}^o = \big[\tilde{z}_{o,}^o\ \tilde{z}_{1,}^o\ \tilde{z}_{3,}^o \ldots\big]\ also \text{ let } \tilde{z}^{n+1} = \tilde{z}^n\tilde{p},\ \ n \geq 0,\ \tilde{z}^n$$

is partition the same way as $\tilde{z}^o$.

Let w_a be the probability that a customer waiting time is less than or equal to a unit of time. Then

$$w_a = \tilde{z}_o^a 1 \quad a \geq 0.$$

4.0 Conclusion

We have showed here that the matrices Presented by [2] can be re-blocked to allow batch arrival with gated vacation and the method of matrix product is then used to solve the resulting matrix problem.

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