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Marshall-Olkin multivariate semi-logistic distribution and minification processes

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Abstract

Marshall-Olkin Multivariate semi-logistic distribution (MO-MSL) and Marshall-Olkin multivariate logistic distribution (MO-ML) are introduced and studied. Various characterizations properties of Marshall-Olkin multivariate semi-logistic distribution are investigated and studied. First order autoregressive minification processes model and its extension to k^{th} order autoregressive minification processes model both with Marshall-Olkin multivariate semi-logistic distribution as marginal distribution are constructed and studied.

Keywords
Autoregressive minification processes of order 1 and *k*; Chacterizations; Marshall-Olkin multivariate logistic distribution; Marshall-Olkin multivariate Semi-Logistic distribution; Stationarity; Survival function.

1.0 Introduction

Researchers focus attentions to logistic distribution, semi-logistic distribution and even its extension to Marshall-Olkin logistic and semi-logistic distributions. This is so due to the wide applications of these distributions in various fields of human endeavours.

Various methods can be used to introduce new parameters in order to expand families of distributions. Introduction of a scale parameter leads to accelerate life model and taking powers of a survival function introduces a parameter that leads to proportional hazards model.

Minification process can be of several orders and its studies began with the work of [17]. In his work, he defined the process as:

$$X_n = K \min\left(X_{n-1}, \varepsilon_n\right), \quad n \geq 1 \tag{1.1}$$

where $k > 1$ is a constant and $\{\varepsilon_n\}$ is an innovation process of independent and identically distributed random variables chosen to ensure that $\{X_n\}$ is a stationary Markov process with a given marginal distribution. Due to the structure of equation (1.1), the process $\{X_n\}$ is called minification process.

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Several authors have introduced minification processes with given marginal distribution. Sim [16] developed a first order autoregressive Weibull process and studied its properties. Arnold [1] developed a logistic process involving Markovian minimization. Giving slight modifications to (1.1), several authors constructed other minification processes models. Yeh [21] considered a first order autoregressive minification process having Pareto marginal distribution. Arnold and Robertson [2] developed a minification process having logistic marginal distribution. Pillai, Jose and Jayakumar [14] discussed the autoregressive minification processes and the class of distribution of universal geometric minima. Ristic [15] introduced stationary bivariate minification processes.

Univariate Marshall-Olkin logistic distribution is given by:

$$\overline{F}\left(x\right)=\frac{1}{1+\frac{1}{\alpha}\exp\left\{x\right\}},-\infty\leq x\leq\infty,0<\alpha<1 \tag{1.2}$$

and $f\left(x\right)=\dfrac{\exp\left\{x\right\}}{\alpha\left(1+\frac{1}{\alpha}\exp\left\{x\right\}\right)^{2}},-\infty\leq x\leq\infty,\ 0<\alpha<1.$

This distribution is symmetric about zero and it is closely resembles the normal distribution. Although multivariate data sets with logistic like marginals have always been around, it was not until 1961 that a bivariate logistic model was proposed. Gumbel [6] actually provided three bivariate logistic models, one of which has cumulative distribution function

$$F\left(x,y\right)=\frac{1}{1+\exp\left\{x\right\}+\exp\left\{y\right\}},-\infty\leq x,y\leq\infty. \tag{1.3}$$

Location and scale parameters can be introduced to generalize this expression. Gumbel [6] in his paper studied the regression properties and verified that the correlation coefficient is $\frac{1}{2}$. A multivariate extension of the Gumbel` s bivariate logistic is proposed by [10]. More applications of Marshall-Olkin logistic distribution can be obtained in [7].

Lewis and McKenzie [9] developed minification processes and their transformations. In his work, he defined a first order autoregressive minification process as a sequence having the general structure $X_n=\begin{cases} kX_{n-1} & \text{with probability } P \\ k\min\left(X_{n-1},\in_n\right) & \text{with probability } 1-P \end{cases}$ where $\left\{\in_n\right\}$ is an innovation process of independent and identically distributed random variables chosen to ensure that $\left\{X_n\right\}$ is a stationary Markov process with a specified marginal distribution function $F_X\left(X\right)$. Another form of minification process is the one with structure

$$X_n=\begin{cases} k\in_{n-1} & \text{with probability } P \\ k\min\left(X_{n-1},\in_n\right) & \text{with probability } 1-P. \end{cases}$$

Thomas and Jose [19] defined first order autoregressive minification process model of random vectors $\left\{\left(X_n,Y_n\right)\right\}$ with Marshall and Olkin bivariate semi-Pareto distribution as

$$X_n = \begin{cases} U_n & \text{with probability } p \\ \min\left(X_{n-1}, U_n\right) & \text{with probability } 1-p \end{cases}$$

$$Y_n = \begin{cases} V_n & \text{with probability } p \\ \min\left(Y_{n-1}, V_n\right) & \text{with probability } 1-p \end{cases}$$

where $\left\{\left(U_n, V_n\right)\right\}$ are innovations, which are independent of $\left\{\left(X_{n-k}, Y_{n-k}\right)\right\}$ for $k = 1,2,\ldots,n$.

Miroslav, Biljana, Aleksandar and Miodrag [13] introduced and studied a stationary bivariate minification process. They defined the process as

$$X_n = K \min\left(X_{n-1}, Y_{n-1}, \varepsilon_{n1}\right)$$
$$Y_n = K \min\left(X_{n-1}, Y_{n-1}, \varepsilon_{n2}\right).$$

Thomas and Jayakumar [20] defined first order autoregressive bivariate semi-logistic process as

$$X_n = \min\left(X_{n-1}, \tfrac{1}{\alpha_1}\ln p, \varepsilon_n\right)$$
$$Y_n = \min\left(Y_{n-1}, \tfrac{1}{\alpha_2}\ln p, v_n\right), n \geq 1,\ 0 \leq p \leq 1,\ \ \alpha_1, \alpha_2 > 0,$$

Thomas and Jose [18] introduced and studied the univariate Marshall-Olkin Pareto processes. Thomas and Jose [19] developed a new family of distributions that were earlier studied by [12] which is similar to those of [14]. In [19], also, AR(1) and AR(k) times series models useful in generating first order and k^{th} order autoregressive minification processes having a specified stationary bivariate marginal distribution are presented and studied. In that paper, they developed the Marshall-Olkin bivariate semi-Pareto distribution as a generalization of the bivariate semi-Pareto distribution of [3]. They also presented bivariate semi-Pareto AR(1) model and its generalization to AR(k) model with MO-BSP stationary distribution were constructed. These models developed are analogous to the model studied by [8] where the role of addition is replaced by minimization. Recently, [20] developed bivariate semi-logistic distribution and processes. Miroslav, Biljana, Aleksandar and Miodrag [13] introduced and studied a stationary bivariate minification process with Marshall and Olkin exponential distribution. The innovation and joint distributions of random vectors $\left(X_n, Y_n\right)$ are also presented by [13]. In that paper, autocovariance and autocorrelation matrices are also developed, unknown parameters in the model are estimated and asymptotic properties of the estimated parameters are also investigated. Balakrishnan [5] discussed the analysis of bioavailability data when successive samples are from logistic distribution.

Some bivariate and multivariate minification processes are introduced and studied by [3], [4], [18] and [19] as well as [15]. Marshall and Olkin [11] introduced the bivariate exponential distribution with survival function $P\left(X > x, Y > y\right) = e^{-\lambda_1 x - \lambda_2 y - \lambda_{12}\max(x,y)}, x, y > 0,.$

with $\lambda_1 > 0, \lambda_2 > 0$ and $\lambda_{12} > 0$. The random variables are constructed such that X and Y are dependent exponentially distributed random variables with parameters $\lambda_1 + \lambda_{12}$ and $\lambda_2 + \lambda_{12}$ respectively.

In this paper**,** we present the Marshall-Olkin multivariate logistic and semi logistic distributions as a generalization of Marshall-Olkin bivariate logistic and semi-logistic distribution presented by [20]. The characterizations properties of Marshall-Olkin multivariate semi-logistic distribution are investigated and studied. First order autoregressive minification process and its extension to k^{th} order autoregressive minification models are developed with Marshall-Olkin multivariate semi-logistic distribution as a stationary marginal distribution.

Section 2 of this paper, dealt with the introduction of Marshall-Olkin multivariate logistic and semi-logistic distributions and the properties of the Marshall-Olkin multivariate semi-logistic distribution. Section 3, we give characterizations of Marshall-Olkin multivariate semi-logistic distribution. First order autoregressive minification process model with Marshall- Olkin multivariate semi-logistic distribution as its marginal distribution is constructed in Sect-

ion 4. We generalized the model to k^{th} order autoregressive minification model with Marshall-Olkin multivariate semi-logistic distribution as a stationary marginal distribution in section 5. Section 6, gives the conclusion of the entire paper.

2.0 Marshall-Olkin multivariate logistic and semi-logistic distributions

In this section, we consider the multivariate logistic and semi-logistic distributions and distributions related to it, which are generally used for modeling population growth, medical diagnosis and public health related data.

***Definition* 2.1**

The random vector $\underline{X} = (X_1, X_2 ..., X_n)$ is said to have a *n*-variate semi-logistic distribution $\underline{X} : MO - MSL,$ if its survival function is of the form

$$\bar{F}(x_1, x_2, ..., x_n) = P(X_1 > x_1, X_2 > x_2, ..., X_n > x_n)$$

$$= \frac{1}{1 + \eta(x_1, x_2, ..., x_n)} \tag{2.1}$$

such that $\eta(x_1, x_2, ..., x_n)$ satisfies the functional equation

$$\eta(x_1, x_2, ..., x_n) = \frac{1}{p}\eta\left(x_1 + \tfrac{1}{\alpha_1}\ln p, x_2 + \tfrac{1}{\alpha_2}\ln p, ..., x_n + \tfrac{1}{\alpha_n}\ln p\right),$$

$$0 < p < 1,\ \alpha_1, \alpha_2, ..., \alpha_n > 0,\ -\infty < x_1, x_2, ..., x_n < \infty \tag{2.2}$$

$\eta(x_1, x_2, ..., x_n)$ is a monotonically increasing function in $\underline{X}$ satisfying

$$\lim_{x_1 \to 0} \lim_{x_2 \to 0} ... \lim_{x_n \to 0} \eta(x_1, x_2, ..., x_n) = 0 \text{ and } \lim_{x_1 \to \infty} \lim_{x_2 \to \infty} ... \lim_{x_n \to \infty} \eta(x_1, x_2, ..., x_n) = \infty.$$

***Lemma* 2.1**

The solution of the functional equation (2.2) is given by

$$\eta(x_1, x_2, ..., x_n) = \exp\{\alpha_1 x_1\} h_1(x_1) + \exp\{\alpha_2 x_2\} h_2(x_2) + ... + \exp\{\alpha_n x_n\} h_n(x_n) \tag{2.3}$$

where $h_i(x_i), i = 1, 2, ..., n,$ are periodic functions in $x_i,\ i = 1, 2, ..., n$ with period $\frac{1}{\alpha_i}\ln p, i = 1, 2, ..., n,$ respectively.

Proof

Considering equations (2.2) and (2.3), we have

$$\eta(x_1, x_2, ..., x_n) = \frac{1}{p}\eta\left(x_1 + \tfrac{1}{\alpha_1}\ln p, x_2 + \tfrac{1}{\alpha_2}\ln p, ..., x_n + \tfrac{1}{\alpha_n}\ln p\right)$$
$$= \frac{1}{p}\left[\exp\left\{\alpha_1\left(x_1 + \tfrac{1}{\alpha_1}\ln p\right)\right\}h_1(x_1) + ... + \exp\left\{\alpha_n\left(x_n + \tfrac{1}{\alpha_n}\ln p\right)\right\}h_n(x_n)\right]$$
$$= \frac{1}{p}\left[p.\exp\{\alpha_1 x_1\}h_1(x_1) + ... + p.\exp\{\alpha_n x_n\}h_n(x_n)\right]$$
$$= \exp\{\alpha_1 x_1\}h_1(x_1) + \exp\{\alpha_2 x_2\}h_2(x_2) + ... + \exp\{\alpha_n x_n\}h_n(x_n)$$
$$= \eta(x_1, x_2, ..., x_n).$$

Hence the proof is complete. ■

In particular, if we choose $h_i(x_i) = 1,\ i = 1, 2, ..., n,$ the $MSL(\alpha_1, \alpha_2, ..., \alpha_n; p)$ reduces to a multivariate logistic distribution with survival function

$$\overline{F}(x_1, x_2, ..., x_n) = \frac{1}{1 + \exp\{\alpha_1 x_1\} + \exp\{\alpha_2 x_2\} + ... + \exp\{\alpha_n x_n\}}, \alpha_i > 0,\ \forall i \qquad (2.4)$$

Gumbel [6] proved that the bivariate logistic distribution having distribution function (1.2) is asymmetric. Similarly, it can be shown that the survival function (2.4) is also asymmetric.

Let $\underline{X} = (X_1, X_2 ..., X_n)$ be a random vector with joint survival function as in equation (2.1). Then the modified multivariate survival function is in the form

$$\overline{G}_{\underline{X}}(\underline{X}) = \frac{\beta \overline{F}_{\underline{X}}(\underline{X})}{1 - (1-\beta)\overline{F}_{\underline{X}}(\underline{X})} \quad \underline{X} = (x_1, x_2, ..., x_n) \geq 0,\ 0 < \beta < 1, \qquad (2.5)$$

The family of distributions of the form (2.5) shall be called Marshall-Olkin multivariate family of distributions. From (2.5), we can see that the new family of multivariate semi-logistic survival function is

$$\overline{G}(x_1, x_2, ..., x_n; \beta) = \frac{1}{1 + \frac{1}{\beta}\eta(x_1, x_2, ..., x_n)} \quad \underline{X} = (x_1, x_2, ..., x_n) \geq 0,\ 0 < \beta < 1, \qquad (2.6)$$

which we shall refer to as Marshall-Olkin multivariate semi-logistic distribution denoted by MO-MSL. Similar Marshall-Olkin multivariate distribution can be developed by considering multivariate Weibull survival function. For instance, a multivariate semi-Weibull distribution has a survival function of the form $\overline{F}_{\underline{X}}(\underline{X}) = EXP(-\eta(x_1, x_2, ..., x_n))$ where $\eta(x_1, x_2, ..., x_n)$ satisfied the functional equation (2.2). Then the Marshall-Olkin multivariate semi-Weibull distribution has the survival function given by

$$\overline{G}(x_1, x_2, ..., x_n; \beta) = \frac{\beta e^{-\eta(x_1, x_2, ..., x_n)}}{1 - (1-\beta)e^{-\eta(x_1, x_2, ..., x_n)}}. \qquad (2.7)$$

Similarly, from equations (2.4) and (2.5) the Marshall-Olkin multivariate logistic distribution has the survival function given as:

$$\overline{F}(x_1, x_2, ..., x_n) = \frac{1}{1 + \frac{1}{\beta}\left[\exp\{\alpha_1 x_1\} + \exp\{\alpha_2 x_2\} + ... + \exp\{\alpha_n x_n\}\right]}, \alpha_i > 0,\ \forall i\ and\ 0 < \beta < 1. \quad (2.8)$$

The Marshall-Olkin multivariate logistic density function is:

$$f\left(x_1,x_2,...,x_n\right)=\frac{2\prod_{i=1}^{n}\alpha_i\prod_{i=1}^{n}\exp\{\alpha_i x_i\}}{\beta^2\left(1+\frac{1}{\beta}\left[\sum_{i=1}^{n}\exp\{\alpha_i x_i\}\right]\right)^3},\ -\infty<x_i<\infty,\ \ \alpha_i>0,\ \forall i\ and\ 0<\beta<1.$$

3.0 Characterizations

This section is devoted to characterizations properties of Marshall-Olkin multivariate semi-logistic distribution. The following theorems give the characterizations properties of this distribution. Proofs of these theorems are also given in this section.

***Theorem* 3.1**

Let N be a geometric random variable with parameter p such that $P\{N = n\} = pq^{n-1}$, $n = 1,2,3, \ldots, 0 < p < 1, q = 1 - p$. Consider a sequence $\left\{X_i^{(1)}, X_i^{(2)}, X_i^{(3)}, \ldots, X_i^{(n)}, i \geq 1\right\}$ of

independent and identically distributed random vectors with common survival function $\overline{F}\left(x_1, x_2, \ldots, x_n\right)$. Assume that N and $\left\{X_i^{(1)}, X_i^{(2)}, X_i^{(3)}, \ldots, X_i^{(n)}, i \geq 1\right\}$ are independent for all $i \geq 1$. Let $U_1 = \min_{1\leq i\leq N} X_i^{(1)}, U_2 = \min_{1\leq i\leq N} X_i^{(2)}, \ldots, U_n = \min_{1\leq i\leq N} X_i^{(n)}$. Then the random vectors $\left(U_1,U_2,...,U_n\right)$ are distributed as Marshall-Olkin multivariate semi-logistic if and only if $\left\{X_i^{(1)}, X_i^{(2)}, X_i^{(3)}, \ldots, X_i^{(n)}, i \geq 1\right\}$ have the multivariate semi-logistic distribution.

Proof

Suppose

$$\begin{aligned}\overline{G}\left(x_1, x_2, \ldots, x_n\right) &= P\left(U_1 > x_1, U_2 > x_2, \ldots, U_n > x_n\right)\\ &=\sum_{n=1}^{\infty}\left[\overline{F}\left(x_1,x_2,...,x_n\right)\right]^n pq^{n-1}\\ &= p\overline{F}\left(x_1,x_2,...,x_n\right)\sum_{n=1}^{\infty}\left[\overline{F}\left(x_1,x_2,...,x_n\right)\right]^{n-1}q^{n-1}\\ &=\frac{p\overline{F}\left(x_1,x_2,...,x_n\right)}{1-\left(1-P\right)\overline{F}\left(x_1,x_2,...,x_n\right)}.\end{aligned}$$

Let $\overline{F}\left(x_1,x_2,...,x_n\right)=\frac{1}{1+\eta\left(x_1,x_2,...,x_n\right)}$, which is the survival function of multivariate semi-logistic. Substituting this into the equation above, we have

$$\overline{G}\left(x_1, x_2, \ldots, x_n\right) = \frac{p}{p+\eta\left(x_1, x_2, \ldots, x_n\right)} \tag{3.1}$$

which is the survival function of Marshall-Olkin multivariate semi-logistic distribution with $\beta = p$. Conversely, suppose that $\overline{G}\left(x_1, x_2, \ldots, x_n\right) = \frac{p}{p+\eta\left(x_1, x_2, \ldots, x_n\right)}$.

Then $\dfrac{p\overline{F}(x_1, x_2, \ldots, x_n)}{1-(1-p)\overline{F}(x_1, x_2, \ldots, x_n)} = \dfrac{p}{p+\eta(x_1, x_2, \ldots, x_n)}$. Therefore, by simplification, we have $\overline{F}(x_1, x_2, ..., x_n) = \dfrac{1}{1+\eta(x_1, x_2, ..., x_n)}$, which is survival function of multivariate semi-logistic distribution. This gives the complete proof. ■

The next theorem gives additional characterization property of MO-MSL distribution. Before the theorem let us consider the following.

Let $\{N_k : k \geq 1\}$ be a sequence of geometric random variables with parameters p_k, $0 \leq p_k \leq 1$. Define

$$\overline{F}_k(x_1, x_2, ..., x_n) = p\left(U_{N_{k-1}}^{(1)} > x_1, U_{N_{k-1}}^{(2)} > x_2, ..., U_{N_{k-1}}^{(n)} > x_n\right), \ k = 2, 3, ...$$

$$= \frac{p_{k-1}\overline{F}_{k-1}(x_1, x_2, ..., x_n)}{1-(1-p_{k-1})\overline{F}_{k-1}(x_1, x_2, ..., x_n)}. \tag{3.2}$$

Here, we refer $\overline{F}_k$ as the survival function of the geometric p_{k-1} minimum of independent and identically distributed random vectors with $\overline{F}_{k-1}$ as the common survival function.

***Theorem* 3.2**

Let $\left\{X_i^{(1)}, X_i^{(2)}, X_i^{(3)}, \ldots, X_i^{(n)}, i \geq 1\right\}$ be a sequence of independent and identically distributed non-negative random vectors with common survival function $\overline{R}(x_1, x_2, ..., x_n)$. Define $\overline{R}_1 = \overline{R}$ and $\overline{F}_k$ as the survival function of the geometric p_{k-1} minimum of independent and identically distributed random vectors with $\overline{F}_{k-1}$, $k = 2, 3, \ldots$ as the common survival function. Then

$$\overline{F}_k(x_1, x_2, ..., x_n) = \overline{R}(x_1, x_2, ..., x_n) \tag{3.3}$$

if and only if $\left\{X_i^{(1)}, X_i^{(2)}, X_i^{(3)}, \ldots, X_i^{(n)}, i \geq 1\right\}$ has MO-MSL distribution.

Proof

Considering the definition, the survival function $\overline{F}_k$ satisfies the equation (3.2). Hence, we have $\overline{R}(x_1, x_2, ..., x_n) = \dfrac{1}{1+\frac{1}{p}\eta(x_1, x_2, ..., x_n)} = \dfrac{1}{1+\Psi(x_1, x_2, ..., x_n)}$, where $\Psi(x_1, x_2, ..., x_n)$ is a monotonically increasing function in $x_i(i=1,2,3,...n)$, $x_i \geq 0, (i=1,2,3,...n)$ as well as $\lim_{x_1\to 0}\lim_{x_2\to 0}...\lim_{x_n\to 0} \Psi(x_1, x_2, ..., x_n) = 0$ and $\lim_{x_1\to\infty}\lim_{x_2\to\infty}...\lim_{x_n\to\infty} \Psi(x_1, x_2, ..., x_n) = \infty$. Therefore, we can write $\overline{R}_k(x_1, x_2, ..., x_n) = \dfrac{1}{1+\Psi_k(x_1, x_2, ..., x_n)}$; $k = 1, 2, 3, \ldots$ Putting this in (3.2), we get $\Psi_k(x_1, x_2, ..., x_n) = \dfrac{\Psi_{k-1}(x_1, x_2, ..., x_n)}{p_{k-1}}$, $k = 2, 3, \ldots$. Using this relation recursively, we have $\Psi_k(x_1, x_2, ..., x_n) = \dfrac{\Psi_1(x_1, x_2, ..., x_n)}{p_1 p_2 \cdots p_{k-1}}$, as $\overline{R}_1 = \overline{R}$ that means $\Psi_1 = \Psi$.

This implies that $$\Psi_k(x_1,x_2,...,x_n)=\frac{\Psi_1(x_1,x_2,...,x_n)}{p_1p_2\cdots p_{k-1}}. \qquad (3.4)$$

Hence, $\overline{F}_K(x_1,x_2,...,x_n)=\overline{R}(x_1,x_2,...,x_n)$. On the other hand, assume equation (3.3) holds. By the hypothesis of the theorem equation (3.4) follows. Therefore equations (3.3) and (3.4) together lead to the equation

$$\left\{1+\frac{1}{p_1p_2\ldots p_{k-1}}\Psi_1(x_1,x_2,...,x_n)\right\}^{-1}=\overline{R}(x_1,x_2,...,x_n)=\frac{1}{1+\Psi(x_1,x_2,...,x_n)}.$$

This implies that $\Psi(x_1,x_2,...,x_n)=\dfrac{\Psi_1(x_1,x_2,...,x_n)}{p_1p_2\cdots p_{k-1}}$. Hence the proof is complete. ■

4.0 Marshall-Olkin multivariate semi-logistic AR(1) ,model

First order autoregressive minification process model with MO-MSL distribution as stationary marginal distribution is developed in this section. The following theorem gives the model.

***Theorem* 4.1**

Consider the multivariate autoregressive minification process $\left\{\left(X_n^{(1)},X_n^{(2)},...,X_n^{(k)}\right)\right\}$ having the structure:

$$X_n^{(1)}=\begin{cases}\mu_n^{(1)} & \text{with probability } p\\ \min\left(X_{n-1}^{(1)},\mu_n^{(1)}\right) & \text{with probability } 1-p\end{cases}$$

$$X_n^{(2)}=\begin{cases}\mu_n^{(2)} & \text{with probability } p\\ \min\left(X_{n-1}^{(2)},\mu_n^{(2)}\right) & \text{with probability } 1-p\end{cases} \qquad (4.1)$$

$$\vdots \qquad \vdots \qquad \qquad \vdots$$

$$X_n^{(k)}=\begin{cases}\mu_n^{(k)} & \text{with probability } p\\ \min\left(X_{n-1}^{(k)},\mu_n^{(k)}\right) & \text{with probability } 1-p\end{cases}$$

where $\left\{\left(\mu_n^{(1)},\mu_n^{(2)},...,\mu_n^{(k)}\right)\right\}$ are the sequence of independent and identically distributed innovations random variables. Then $\left\{\left(X_n^{(1)},X_n^{(2)},...,X_n^{(k)}\right)\right\}$ is stationary with marginal distribution of Marshall-Olkin semi-logistic (MO-MSL) if and only if $\left\{\left(\mu_n^{(1)},\mu_n^{(2)},...,\mu_n^{(k)}\right)\right\}$ is jointly distributed as multivariate semi-logistic distribution (MSL).

Proof

Considering equation (4.1), we have

$$\overline{F}_{X_n^{(1)},...,X_n^{(k)}}(x_1,x_2,...,x_n)$$
$$=p\overline{R}_{\mu_n^{(1)},...,\mu_n^{(k)}}(x_1,x_2,...,x_n)+(1-P)\overline{F}_{X_{n-1}^{(1)},...,X_{n-1}^{(k)}}(x_1,x_2,...,x_n)\overline{R}_{\mu_n^{(1)},...,\mu_n^{(k)}}(x_1,x_2,...,x_n) \qquad (4.2)$$

Under stationary condition, we have

$$\overline{F}_{X_n^{(1)},\dots,X_n^{(k)}}\left(x_1,x_2,\dots,x_n\right)=\frac{p\overline{R}_{\mu^{(1)},\dots,\mu^{(k)}}\left(x_1,x_2,\dots,x_n\right)}{\left\{1-\left(1-p\right)\overline{R}_{\mu^{(1)},\dots,\mu^{(k)}}\left(x_1,x_2,\dots,x_n\right)\right\}}.$$

If we take $\overline{R}_{\mu_n^{(1)},\dots,\mu_n^{(k)}}\left(x_1,x_2,\dots,x_n\right)=\dfrac{1}{1+\eta\left(x_1,x_2,\dots,x_n\right)}$, then

$$\overline{F}_{X^{(1)},\dots,X^{(k)}}\left(x_1,x_2,\dots,x_n\right)=\frac{p}{p+\eta\left(x_1,x_2,\dots,x_n\right)},$$

which is the survival function of MO-MSL. On the other hand, if we consider $\overline{F}_{X^{(1)},\dots,X^{(k)}}\left(x_1,x_2,\dots,x_n\right)=\dfrac{p}{p+\eta\left(x_1,x_2,\dots,x_n\right)}$, which is the survival function of MO-MSL, it can be shown that $\overline{R}_{\mu^{(1)},\dots,\mu^{(k)}}\left(x_1,x_2,\dots,x_n\right)=\dfrac{1}{1+\eta\left(x_1,x_2,\dots,x_n\right)}$ is distributed as multivariate

semi-logistic distribution and the process is stationary. Stationarity can be established as follows. Assume $\left\{\left(X_{n-1}^{(1)},X_{n-1}^{(2)},\dots,X_{n-1}^{(k)}\right)\right\}\underline{\underline{d}}\ MO-MSL$ and $\left\{\left(\mu_n^{(1)},\mu_n^{(2)},\dots,\mu_n^{(k)}\right)\right\}\underline{\underline{d}}\ MSL.$

Then from (4.2) $\overline{F}_{X_n^{(1)},\dots,X_n^{(k)}}\left(x_1,x_2,\dots,x_n\right)=\dfrac{p}{p+\eta\left(x_1,x_2,\dots,x_n\right)}$. This established that $\left\{\left(X_n^{(1)},X_n^{(2)},\dots,X_n^{(k)}\right)\right\}$ is distributed as MO-MSL. It is also possible to show that $\left\{\left(X_n^{(1)},X_n^{(2)},\dots,X_n^{(k)}\right)\right\}$ is stationary and is asymptotically marginally distributed as MO-MSL.

5.0 Generalization to the k^{th} order autoregressive model

In this section, we extended AR(1) model to the k^{th} order autoregressive time series model. The structure of the k^{th} order autoregressive time series model is given in the following theorem.

***Theorem* 5.1**

Consider an autoregressive time series (AR(k)) model with structure

$$X_n^{(1)}=\begin{cases}\mu_n^{(1)} & with\ probability\ \ p_0\\ \min\left(X_{n-1}^{(1)},\mu_n^{(1)}\right) & with\ probability\ \ p_1\\ \min\left(X_{n-2}^{(1)},\mu_n^{(1)}\right) & with\ probability\ \ p_2\\ \dots & \dots\qquad\dots\\ \min\left(X_{n-k}^{(1)},\mu_n^{(1)}\right) & with\ probability\ \ p_k\end{cases}$$

$$X_n^{(k)} = \begin{cases} \mu_n^{(k)} & \text{with probability } p_0 \\ \min\left(X_{n-1}^{(k)}, \mu_n^{(k)}\right) & \text{with probability } p_1 \\ \min\left(X_{n-2}^{(k)}, \mu_n^{(k)}\right) & \text{with probability } p_2 \\ \ldots & \ldots \quad \ldots \\ \min\left(X_{n-k}^{(k)}, \mu_n^{(k)}\right) & \text{with probability } p_k \end{cases} \quad (5.1)$$

where $0 < p_i < 1, \sum_{i=0}^{k} p_i = 1$. Then $\left\{\left(X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(k)}\right)\right\}$ has stationary marginal distribution as MO-MSL if and only if $\left\{\left(\mu_n^{(1)}, \mu_n^{(2)}, \ldots, \mu_n^{(k)}\right)\right\}$ is jointly distributed as multivariate semi-logistic (MSL) distribution.

Note that the proof of the theorem 5.1 is similar to the one given in theorem 4.1 above.

6.0 Conclusion

At the end of this paper, we can conclude that a new family of multivariate logistic and another family of multivariate semi-logistic distributions are introduced and presented. These new families are called Marshall-Olkin multivariate logistic and semi-logistic distributions.

The characterizations properties of these new families of multivariate logistic and semi-logistic distributions are presented as an extension of characterizations properties multivariate logistic and semi-logistic distributions.

Marshall-Olkin multivariate semi-logistic autoregression model of first order and its structure are developed with Marshall-Olkin multivariate semi-logistic distribution as stationary marginal distribution. The model and its structure were generalized and extended to k^{th} order autoregression model also using the same distribution as its stationary marginal distribution.

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