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Control derivative to optimal control analysis

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Abstract

Optimal control theory, generally, is to determine the control signals which will cause a process to satisfy the physical constraints and at the same time optimize some performance criterion. In this work, a numerical method for finding solution to linear optimal control problems with bounded state constraints is examined. The method applied is based on Legendre series by parameterization of both the state and the control variables involving a derivative.

1.0 Introduction

Human beings are constantly confronted with the problem of controlling a system and faced with many alternative ways of doing certain things.

Constrained optimal control problem arises from the basic control problem $u \in U$ and is to choose the control vector such that the state vector *x* is transferred from x_0 to a terminal point at a particular time *T*, where some (or all or none) of the state variables are specified.

The region *U* of the control problem is called the admissible control region and it is the set from where the control and control region can be obtained.

If the transfer can be accomplished, the problem in optimal control is to effect the transfer so that functional $J = \int_0^T f_0(x,u,t)$ is optimized, f_0 is a function depending on $x_1, x_2, x_3, \cdots, x_n, u_1, u_{2.}, \cdots, u_m$ and t which is continuous with continuous partial derivatives.

The formulation of the problem requires: A mathematical model of the process to be constructed, a statement of the physical constraints and specification of a performance criterion.

A non trivial part of this problem is to model the process and the objective is to obtain the simplest mathematical description that adequately predicts the response of the physical system to all anticipated inputs.

For example, if $x_1(t), x_2(t), ..., x_n(t)$ are the state variables of the process at time *t* and $u_1(t), u_2(t), ..., u_n(t)$ are the control input into the process at time *t*, then the system may be described by *n*-first-order differential equations

$$x_1(t) = a_1(x_1(t), x_2(t), ..., x_n(t), u_1(t), u_2(t)), ..., u_m(t), t)$$
$$x_2(t) = a_2(x_1(t), x_2(t), .., x_n(t), u_1(t), u_2(t)), ..., u_m(t), t)$$

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$$x_n(t) = a_n(x_1(t), x_2(t), \ldots, x_n(t), u_1(t), u_2(t)), \ldots, u_n(t), t)$$

with $x(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{bmatrix}$ as the state vector of the system and $u(t) = \begin{bmatrix} u_1(t) \\ u_2(t) \\ \vdots \\ u_n(t) \end{bmatrix}$ as the control vector.

The state equation can be written as: $\dot{x}(t) = a(x(t), u(t), t)$

A typical optimal control problem is of the form:

Minimize the integral: $\int_{t_0}^{t_f} f[x(t), u(t), t]$ (1.1)

Subject to $x(t) = a(t, x(t), u(t))$ (1.2)

With known initial condition of the state variables as: $x(t_0) = x_0$ and $t_0 \le t \ge t_f$, where, $J(x,u)$ is the objective function, $u(t)$ is the control applied to the system at time t, t_0 is the given initial time, t_f is the given final time, $x(t) = (x_1, x_2, x_3, \ldots, x_n)^T$ is an *n*-vector called the state vector, $u(t) = (u_1, u_2, u_3, \ldots, u_n)^T$ is an *m*-vector called the control vector, $u \in \Re^m$ is referred to as the set of controls, $u(t) \in$ U, $\forall$ $t \in [t_0, t_f]$ and a corresponding trajectory x; define on $[t_0, t_f]$ which satisfy equation (1.2) is called a dynamic system.

It is important to note that:

(i) The process of adjusting the objective functional to achieve a specific goal is called the control process and the controlling mechanism is called the control vector.

(ii) The final time t_f would be needed to be specified because it is the time which minimizes the objective functional (1.1).

(iii) Maximizing a problem $J(x,u)$ may be re-stated as minimization of the problem $-J(x,u)$.

The solution of optimal control problems could be determined by either computational or analytical method.

In analytical methods, given a problem of the form:

Minimize $J(x,u) = \int_{t_0}^{t_f} f_0(x,u,t)dt$, such that $\dot{x}(t) = f(x,u,t)$, $x(t_0) = a$, and $x(t_f) \in S$. By introducing the Lagrange multiplier $\lambda(t)$ we form the augmented functional:

$$J^* = \int_{t_0}^{t_f} [f_0(x,u,t)dt + \lambda f(x,u,t) - \dot{x}]dt.$$

Assuming that *f* and *g* are smooth sufficiently, solution of the form $x(t)$ and λ(t) which are piecewise smooth and *u*(*t*) which is piecewise continuous are obtained.

According to [1] and [3] the integral $F = f_0 + \lambda(f - x)$ is a function of two variables *x* and *u* arriving at the two Euler equations

$$\frac{\partial f}{\partial x} - \frac{d}{dt}\left(\frac{\partial f}{\partial u}\right) = 0 \text{ and } \frac{\partial F}{\partial u} - \frac{d}{dt}\left(\frac{\partial f}{\partial u}\right) = 0 \quad i.e \quad \frac{\partial f_0}{\partial u} + \lambda\frac{d}{dt}\left(\frac{\partial f}{\partial u}\right) = 0.$$

This result is the Hamiltonian $H = \lambda f_0 + \lambda f$, and can be expressed as:

$$\lambda = -\frac{\partial H}{\partial x} \tag{1.3}$$

and $$\frac{\partial H}{\partial u} = 0 \tag{1.4}$$

Hence, (1.3) and (1.4) together with $\dot{x} + x = u$ govern the optional paths.

Two conditions are needed to complete the solution on integrating, since we have two first order differential equations, these are given by (i) $x(t_0) = a$ (ii) $x(t_f) = b$.

In computational method, [4] observed that "obtaining an exact or analytical solutions for linear optimal control problems are available only for relatively simple problems while non-linear analytical solution of optimal control problems are out of reach".

Teo, [13] worked on approximate analytical solution based on power series expansion while [8], [9] and [12] shed more light on parameterization method.

In Parameterization method, the optimal control problem is transformed into parameter optimization problem which is believe to be simpler than the original control problem.

This work, therefore, consider parameterization of optimal control problem with derivative control via Legendre polynomial.

2.0 Main result

Consider the time-invariant linear quadratic optimal control $u^*(t)$ that minimizes the quadratic performance index

$$J = \int_{to}^{tf} [Px^2(t) + Qu^2(t)]\,dt \tag{2.1}$$

Subject to the system state equations and the initial conditions given by

$$\dot{x}(t) - \dot{u}(t) = Cx(t) + Du(t) \text{ and } x(t_0) = t_o$$

where, $X \in \Re^n, U \in \Re^m, m \le n$. P and Q are positive, semi definite and positive definite symmetric matrices respectively.

The time variable $t \in [t_0, t_f]$ in the optimal control problem is transformed into variable $\tau \in [-1,1]$ of the Legendry polynomial using the relation:

$$t = \frac{1}{2}\left\{(t_f - t_0)\tau + (t_0 + t_f)\right\}$$

(Since Legendre functions are valid in this plane) and $\frac{dt}{d\tau} = \frac{1}{2}(t_f - t_0)$. Hence, (2.1) becomes

$$\text{Minimize } J = \frac{t_f}{2}\int_{t_0}^{t_f} [Px^2(\tau) + Qu^2(\tau)d\tau] \text{ subject to } \dot{x}(\tau) - \dot{u}(\tau) = \frac{1}{2}\left(Ax(\tau) + Bu(\tau)\right)$$

and

$$x(t_0) = t_o \tag{2.2}$$

where, A and B are constant, $J = \int_{t0}^{tf}[ax^2(t) + bu^2(t)]\,dt$ is the cost function, $\dot{x}(t)$ is the state variable at time t and $\dot{u}(t)$ is the control variable at time t See [5, 6, 7, 10, 11 and 14].

Assume a Legendre series of the form:

$$x(\tau) = \sum_{i=0}^{n} a_i p_i = a_0 p_0 + a_1 p_1 + a_2 p_2 + \ldots + a_N p_N$$

where p_i's are polynomials, such that

$$x^2(\tau) = \sum_{i=0}^{n} a_i^{\,2}\, p_i^{\,2} \tag{2.3}$$

$$= a^2{}_0\, p^2{}_0 + a^2{}_1\, p^2{}_1 + \cdots + a^2{}_N\, p^2{}_N$$

and $\dot{x}(\tau) = \sum_{i=0}^{N/2} (4i-3) P_{2i-2} \sum_{j=i}^{N/2} a_{2j-1} + \sum_{i=1}^{N/2} (4i-3) P_{2i-1} \sum_{j=i}^{N/2} a_{2j}$. From (2.2), we have

$\dot{x}(\tau) - \dot{u}(\tau) = \frac{1}{2}\left(Ax(\tau) + Bu(\tau)\right)$

$$\left[\frac{du}{dt} + Au(t) = \frac{1}{2}\left(\frac{dx}{dt} - Bx(t)\right)\right] \tag{2.4}$$

Substituting (2.2) and (2.3) into (2.4), we have;

$$\dot{u}(t) + Au(t) = \frac{1}{2}\left(\sum_{i=1}^{N/2} (4i-3)\; P_{2i-2} \sum_{j=i}^{N/2} a_{2j-1} + \sum_{i=0}^{N/2} (4i-3)\; P_{2i-1} \sum_{j=1}^{N/2} a_{2j} - B\sum_{i=1}^{n} a_i\, p_i\right).$$

Let $Z = \frac{1}{2}\left(\sum_{i=1}^{N/2} (4i-3)\; P_{2i-2} \sum_{j=i}^{N/2} a_{2j-1} + \sum_{i=0}^{N/2} (4i-3)\; P_{2i-1} \sum_{j=1}^{N/2} a_{2j} - B\sum_{i=1}^{n} a_i\, p_i\right)$, then

$$\dot{u}(t) + Au(t) = Z \tag{2.5}$$

i.e $\frac{du}{dt} + Bu(t) = Z$. This is a first order differential equation and by solving (2.5), we have

$$\mathrm{u(t)} = \mathrm{Ce}^{-\mathrm{B}\tau} + \frac{1}{B}\, Z \tag{2.6}$$

Substituting for *Z* in (2.6) becomes

$$u(\tau) = Ce^{-Bt} + \frac{1}{B}\left\{\left(\sum_{i=0}^{N/2} (4i-3)\; P_{2i-2}\right)\sum_{j=i}^{N/2} a_{2j-1} + \left(\sum_{i=0}^{N/2} (4i-3)\; P_{2i-1}\right)\sum_{j=i}^{N/2} a_{2j}\right\},$$

$$u(\tau) = Ce^{-Bt} + \frac{1}{B}\left(\sum_{i=0}^{N/2} (4i-3)\; P_{2i-2} - C\,P_{2j-1}\right)\sum_{j=i}^{N/2} a_{2j-1} + \left(\sum_{i=0}^{N/2} (4i-3)\; P_{2i-1} - CP_{2j}\right)\sum_{j=i}^{N/2} a_{2j}$$

$$u^2(t) = Ce^{-2Bt} + \frac{1}{B}\left(\left(\sum_{i=0}^{N/2} (4i-3)\, P_{2i-2} - C\,P_{2j-1}\right)\sum_{j=i}^{N/2} a_{2j-1} + \left(\sum_{i=0}^{N/2} (4i-3)\, P_{2i-1} - CP_{2j}\right)\sum_{j=i}^{N/2} a_{2j}\right)^2 \tag{2.7}$$

When substituting (2.3) and (2.7) into (2.2) with series of integrations using

$$\int_a^b P_i\; P_j\; d\tau = \begin{cases} \dfrac{2}{2i+1}\,. & i = j \\[2ex] 0\,, & i \neq j \end{cases}$$

Then, the state and control variables are approximated. $J(x)$ is reduced by applying the following theorem:

***Theorem* 2.1**

If $J = \sum_{i=1}^{N+1} c\left(\frac{1}{2i+1}\right)$ which is obtained from the Hessian Matrix of $H = \frac{\partial^2 J}{\partial a_i \partial a_j}$ where $i, j = 0,1,\ldots,N$. Then, the new optimal control problem is a transformed parameterized optimization problem.

This parameterized optimization problem is likely to be quadratic in the unknown parameters.

Proof

See, [9].

***Theorem* 2.2**

If H is positive definite, then, $a^* = H^{-1}F^T(FH^{-1}F^T)^{-1}b$ and the value of J converges to the solution of the problem as N blows up from 2, 4, 6, 8 to infinity.

Proof

See, [13].

3.0 Numerical examples

The above method is implemented to solve the following examples. MatLab package was used to obtain results for the problems [2].

***Example* 3.1**

Minimize the functional

$$J = \int_0^1 (0.5x^2 + 0.2u^2)dt \qquad (3.1)$$

Subject to $\dot{x} - 0.5\dot{u} = x + 0.2u, \quad x(0) = 0.16$

which can also be expressed as Min J = $\int_0^1 (\frac{1}{2}x^2 + \frac{1}{5})dt$ subject to

$$\dot{x} - 0.5\dot{u} = x + 0.2u, \quad x(0) = 0.16 \qquad (3.2)$$

The numerical solution is shown in Table 3.1.

***Example* 3.2**

Consider problem of the form: $J = \int_0^1 (x^2 + u^2)dt$ subject to

$$\dot{x} - 0.05\dot{u} = x + 0.5u, \quad x(-1) = 0.124 \qquad (3.3)$$

The numerical solution is shown in Table 2.2.

Table 3.1: Results for Numerical Example 3.1

Value of *N*	**Numerical value of *J***	**Exact solution of *J***
2	0.01729678078504	0.01685546779576
4	0.01685488458218	0.01685546779576
6	0.01685456779576	0.01685546779576
8	0.01684546374862	0.01685546779576
10	0.01684546367321	0.01685546779576

Table 3.2: Result of Numerical Example 3.2

Value of N	Numerical value of J	Exact solution of J
2	0.12647262524285	0.12647262524285
4	0.12633548972548	0.12647262524285
6	0.12633445678099	0.12647262524285
8	0.12633445678078	0.12647262524285
10	0.12633445678077	0.12647262524285

4.0 Discussion and conclusion

From the results presented in Tables 3.1, and 3.2, it can be seen that our method gives results that are favourably comparable to the exact solutions using MatLab.

This is another attempt to solve continuous optimal problem with the control variable being a derivative. It has, generally, been established, by Cauchy, that if N grows, in a particular system and $|y_n - y_{n+1}| < \epsilon$ (ϵ- being small), then the system converges. From Table 3.1 and 3.2, we note that the value of J converges as N increases. We intend to use Chebyshev polynomial to linearise our problem and compare the solution with the present work and existing results.

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