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Oscillatory behaviour of solutions of linear neutral differential equations with several time lags driven by space-time noise

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Abstract

The paper considers the contribution of space-time noise to the oscillatory behaviour of solutions of a linear neutral stochastic delay differential equation. It was established that under certain conditions on the time lags and their speed of adjustments, the presence of noise generates oscillation in the solution of the equation irrespective of the magnitude of the time lags. This is contrary to the comparable classical neutral differential equation which can permit a non-oscillatory solution.

1.0 Introduction

Neutral stochastic delay differential equations form a special class of stochastic functional differential equations. In recent years, there has been much activities concerning the oscillation theory of classical delay differential equations as well as neutral differential equations (see for example[3, 2, 7], [5], [11], [12], [13]. For instance, [9] considered the first order neutral differential equation

$$\frac{d}{dt}(x(t)-R(t)x(t-\Gamma))+P(t)x(t-r)-Q(t)x(t-\sigma)=0 \qquad (1.1)$$

It has been established by the authors that the new sharp conditions for the oscillation of all solutions of (1.1) are as follows:

Suppose that

(i) $P,Q,R\in C([t_0,\infty),\Re^+)$, $r\in(0,\infty)$ *and* $\Gamma,\sigma\in\Re^+$

(ii) $r>\sigma$, $\overline{P}(t)=P(t)-Q(t-r+\sigma)\geq 0$ *and not identically zero.*

(iii) $R(t)+\int_{t-r+\sigma}^{t}Q(s)ds\leq 1,\ t\leq t_1\leq t_0$

(iv) $\underset{t\to\infty}{Lim}\inf\int_{t-r}^{t}\overline{P}(s)ds>0$

(v) *There exists a positive continuous function* $H(t)$ *such that* $\underset{t\to\infty}{Lim}\inf\int_{t-r}^{t}H(s)ds>0$

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$$(vi)\ \inf_{\lambda>0,t\geq\Gamma}\left\{\frac{R(t-r)P(t)H(t-r)}{\overline{P}(t-r)H(t)}\exp\left(\lambda\int_{t-r}^{t}H(s)ds\right)+\frac{1}{H(t)\lambda}\overline{P}(t)\exp\left(\lambda\int_{t-r}^{t}H(s)ds\right)\right.$$
$$\left.+\frac{P(t)}{H(t)}\int_{t-r+\sigma}^{t}\frac{Q(s-r)H(s+\sigma)}{\overline{P}(s-\sigma)}\exp\left(\lambda\int_{s-\sigma}^{t}H(u)du\right)ds>1\right.$$

for $\overline{P}(t)>0, t\geq T$

Then every solution of (1.1) is oscillatory.

Also [8] obtained sufficient conditions for oscillation of all solutions of the neutral system

$$\frac{d}{dt}\left[x(t)-p_i x_i(t-r_i)\right]=-\sum_{k=1}^{m}\left[\sum_{j=1}^{n}q_{ij}^{(k)}x_j(t-\sigma_k)\right] \tag{1.2}$$

In spite of all these efforts, it appears however, that not much work has been done in respect of the effects of space-time noise of Ito type on the oscillatory behaviour of solutions of neutral stochastic delay differential equations (NSDDEs).

In the present paper, we study the oscillatory properties of solutions of a first order linear neutral stochastic delay differential equation of the form

$$\left.\begin{aligned} d\left[X_i(t)-\sum_{j=1}^{n}a_{ij}(t)X_j(t-r)\right]&=-\sum_{j=1}^{n}b_{ij}(t)X_j(t-\sigma_j)dt+\sum_{j=1}^{n}\mu_{11}^{j}X_i(t)dB_k(t)\\ X_i(t)&=\phi(t),\ \ for\ all\ t\in[-r,0],\ i=1,2,...,n\end{aligned}\right\} \tag{1.3}$$

where $\Gamma=\max_{1\leq j\leq n}\{r,\sigma_j\}$, $a_{ij}>0$, $b_{ij}>0$ are continuous functions, r is a positive real number, $\sigma_j=\sigma_1,\ \sigma_2,....,\sigma_n$ are non-negative constants and $B(t)=(B_1(t),.......,B_n(t))^*$ is a standard Brownian motion on a given probability space $(\Omega,\ F,\ P)$.

A solution of (1.3) is a random process $X_i(t)=X_1,X_2,\Lambda\ ,X_n$ which is continuous with probability 1 and satisfies (1.3) for all *t*. The solution *X*(*t*) is adapted to a family of $\sigma-a\lg ebras$ generated by the process *B*(*t*). It is well known that oscillation in first order linear classical delay differential equations is caused by the presence of time lags or delays. In our main result, it is shown that under certain conditions on the time lags and their speed of adjustments, the presence of space-time noise generates oscillation in the solution in the NSDDE (1.3). This happens even if the comparable classical neutral differential equation permits a non-oscillatory solution.

2.0 Preliminary notes

Throughout this article, we will always compare the oscillatory results of the solution of the NSDDE (1.3) with the oscillatory results of the corresponding classical differential equation of neutral type of the form

$$\frac{d}{dt}\left[x_i(t)-\sum_{j=1}^{n}a_{ij}x_i(t-r)\right]=-\sum_{j=1}^{n}b_{ij}(t)x_j(t-\sigma_j),\ i=1,2,...,n \tag{2.1}$$

which satisfies the same initial function as (1.3). We consider the coefficient of the neutral term to be a general matrix but not a diagonal one. Also our main result is reduced to the oscillation of scalar neutral differential equation so that the effects of the adjustment speeds are preserved.

***Definition* 2.1**

A solution $x=(x_1,x_2,\Lambda\ ,x_n)$ of (2.1) is said to be oscillatory if at least one of the components of the solution is oscillatory. This holds if it contains arbitrarIily large zeros; i.e. if there exists a sequence $\{t_n : x(t_n)=0\}$ *of* $x(t)$ *such that* $\underset{n\to\infty}{Lim}\, t_n=+\infty$, otherwise, it is said to be non-oscillatory. Hence *x*(*t*) is non-oscillatory if there exists a $t_1 > 0$ such that $|x_i(t)| > 0$ for all $t\geq t_1,\ i=1,2,\Lambda\ ,n.$ If $x_i(t) > 0$ for all $t\geq t_1$, then the solution is said to be eventually positive. If $x_i(t) < 0$ for all $t\geq t_1$, then the solution is said to be eventually negative. A solution which is either eventually positive or eventually negative is said to be non-oscillatory. For random processes, we have the following:

***Definition* 2.2** [1]

A non-trivial continuous function $f:[t_0,\infty)\to\Re$ is said to be oscillatory if the collection $W_f=\{t\geq t_0 : f(t)=0\}$ *satisfies* $SupW_f=\infty$. A function which is not oscillatory is said to be non-oscillatory. This is extended to random processes as follows:

A stochastic process $\{X(t,w)\}_{t\geq 0}$ *defined on the probabilty triple* $(\Omega,F,\ P)$ and with continuous sample paths is said to be oscillatory almost surely if there exists a subset $\Omega^*\subseteq\Omega$ *with* $P[\Omega^*]=1$ such that for all $w\in\Omega^*$ the path *X*(.,*w*) is oscillatory. Otherwise it is said to be non-oscillatory.

The first step towards achieving our result is to write the solution *X*(*t*) of the NSDDE (1.3), a random process (which though continuous, is nowhere differentiable) in terms of the differentiable solution *Z*(*t*) of a random neutral differential equation

$$Z^{|}(t)=-\sum_{j=1}^{n}P_j(t)Z_j(t-\sigma_j)+cZ^{|}(t-r),\ t>0 \tag{2.2}$$

where $P_j(t)$ is a positive continuous function defined on some subset $\Omega^*\subseteq\Omega$ by

$$P(t)(w)=\begin{cases}-b\exp\left(-\left(I-\dfrac{\mu^2}{2}\right)\Gamma\right)\exp(-\mu(B(t)(w)-B(t-\sigma)(w))),\ t>\bar{t}\\[2ex] -b\exp\left(-\left(I-\dfrac{\mu^2}{2}\right)\right)t-\mu B(t), \qquad t\leq\bar{t}\end{cases} \tag{2.3}$$

$where\ \bar{t}=\inf\{t\geq 0 : t-\sigma\},\ I\ is\ the\ identity\ matrix\ of\ appropriate\ order.$

We note that P as in (2.3) depends on the increments of a standard Brownian motion. The large deviations in these increments ensure that *P* is sufficiently large to stimulate oscillation in (2.2).

We now tap from many existing and extensive oscillatory and non-oscillatory results in the deterministic theory of oscillation (covering differential equations of neutral types) for use on a path-wise basis, that is, for all $w\in\Omega$, which apply directly to (2.2). The following concerning oscillation of solutions is a special case of the result found in [6].

Proposition 2.1

Assume that

$$G_1 : c, r, \sigma \text{ are positive numbers } 0 < c < 1, \sigma \geq r \geq 0$$
$$G_2 : P_j \in (\Re, \Re_+), P_J(t+r) = P_j(t), t \in \Re, j = 1,2,\Lambda, n$$
$$G_3 : P_0 > \frac{1-c}{e}, \int_{t-r}^{t} \sum_{j=1}^{n} P_j(s) ds = P_0$$

Then all non-trivial solutions of

$$x^{|}(t) = -\sum_{j=1}^{n} P_j(t) x_j(t - \sigma_j) + c x^{|}(t-r) \tag{2.4}$$

are oscillatory.

We can also have results pertaining to non-oscillatory solutions. The following is a special case of the result found in Gopalsamy [4] (Theorem 5.2.12).

Proposition 2.2

Let c, r, σ be non-negative numbers, 0< c < 1, $r \geq 0, \sigma > 0. Let\, P_j \in C(\Re, \Re_+)$ and $\sum_{j=1}^{n} P_j(t) \to P_0 > 0 \text{ as } t \to \infty$. If there exists a positive number μ satisfying

$$c e^{\mu\sigma} + \frac{P_0 e^{\mu\sigma}}{\mu} \leq 1 \tag{2.5}$$

Then (2.4) has a non-oscillatory solution.

In the remaining part of the paper, we apply a method of proof used in Appleby and Buckwar [1], [11] and a result concerning solution transformation (See [2], [10]) to create a conjugation relation which enables us to obtain oscillatory information about the solution of the NSDDE (1.3). This is done by analyzing the oscillatory behaviour of the solution of (2.2) through appropriate choice of deterministic results as in proposition 2.1 and proposition 2.2, which apply directly to (2.2).

3.0 Solution transformations:

In this section, we introduce a stationary random bijective coordinate change $\{\Lambda(t,.)\}_{t\geq 0}$ which satisfies the following properties:

$H_1 : \{\Lambda(t,.)\}_{t\in\Re}$ *is a continuous* $C^{k+1,\in}$ *semi – martingale with* $(\text{with } 0 < \in < \sigma)$ *such that for all* $w \in \Omega, \Re^d \ni v \to \Lambda(t,v) \in \Re^d$ *is a* C^{k+1} *diffeomorphism of* $\Re^d$

H$_2$: There exists a continuous semi-martingale $\{\Gamma(t,.)\}_{t\in\Re}$ *with* $(0 < \in < \sigma)$ *such that* the process $\{\Lambda(t,.)\}_{t\in\Re}$ *and* $\{\Gamma(t,.)\}_{t\in\Re}$ are perfectly stationary, that is

$$\Lambda(t,v,w) = \Lambda(0,v,\theta(t,w)) \text{ and } \Gamma(t,v,w) = \Gamma(0,v,\theta(t,w)) \text{ for all } t \in \Re^d, w \in \Omega$$

H_3 : *For all* $t \geq s, v \in \Re^d$ *and a.e.* $w \in \Omega$

$$\Lambda(t,v) = \Lambda(s,v) + \int_s^t \mu(du, \Lambda(u,v)) du + \int_s^t \Gamma(u,v) du$$

For details of the properties of the bijective random coordinate change, $\{\Lambda(t,.)\}_{t\in\Re}$ we refer to [10] and the references therein. Now as before, we let $\{X(t)\}_{t\geq 0}$ be the solution of the NSDDE (1.3), $\{Z(t)\}_{t\geq 0}$ be the solution of the random neutral differential equation (2.2) and let $\{\Lambda(t,.)\}_{t\in\Re}$ be the stationary bijective coordinate change with the properties as above. We define for all $t \geq 0, w \in \Omega$, the following conjugation relation

$$X(t,w)=\Lambda(0,\theta(t,w)\circ Z(t,w))\circ\Lambda^{-1}(0,w) \tag{3.1}$$

Hence the solution of the NSDDE, $X(t)$ is expressed as the conjugation relation of the random

bijective coordinate change or process $\Lambda(t,.)$ and the solution $Z(t)$ of the random neutral differential equation (2.2). By this conjugation relation, the zeros of the process Z correspond *w.p.*1 to the zeros of the solution X of the SNDDE (1.3). We can now obtain information about the oscillatory behaviour of the process X on a path-wise basis by analyzing the oscillatory properties of Z.

4.0 The main result:

4.1 Assumptions

We need the following assumptions on the time lags and their speed of adjustments [5]:

G_1: $\sigma_1,\sigma_2,\Lambda\ ,\sigma_n$ are positive numbers and r is a non-negative number.

G_2: b_{ij}, $(i=1,2,\Lambda\ ,n)$ are bounded continuous functions defined for $t\geq 0$.

G_3: a_{ij}, $(i=1,2,\Lambda\ ,n)$ are bounded continuous functions with bounded derivatives such that

$$a=\max_{1\leq i\leq n}\ \underset{t\geq 0}{Sup}\sum_{j=1}^{n}\left|a_{ji}(t)\right|<1 \tag{4.1}$$

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$$A=\min_{1\leq i\leq n,t\geq 0}\inf a_{ii}(t)=\max_{1\leq i\leq n,\ t\geq 0}Sup\left|a_{ji}(t)\right|>0 \tag{4.2}$$

$_4$: All non-trivial solutions of the scalar neutral differential equation

$$\frac{d}{dt}\left[u(t)-Cu(t-r)\right]+\mu u(t-\sigma_0)=0 \tag{4.3}$$

are oscillatory, where

$$\sigma_0=\min_{1\leq i\leq n}\{\sigma_i\},\qquad \mu=\min_{1\leq i\leq n}\left[\alpha_{ii}-\sum_{j=1}^{n}\beta_{ji}\right]$$

$$\alpha_{ii}=\inf_{t\geq 0}a_{ii}(t),\qquad \beta_{ji}=\underset{t\geq 0}{Sup}\left|a_{ji}(t)\right|\ \ j=i$$

In the main result below, we establish that whenever $h(t)$ = t-r satisfies the hypothesis of proposition 2.1 and assumptions G_1 – G_4 hold, then the solution of the SNDDE (1.3), where noise is present, oscillates with probability 1 for any given initial datum.

***Theorem* 4.1**

Assume that the time lags and their speed of adjustments σ_j, r, c_{ij}, a_{ij} satisfy the assumptions G_1 – G_2 and h(t) = t – r is non-decreasing. Then the NSDDE (1.3) has an oscillatory solution *w. p.* 1 on $[0,\infty)$ for every given initial datum ϕ.

Proof

By the conjugation relation $X(t,w)=\Lambda(0,\theta(t,w)\circ Z(t,w))\circ\Lambda^{-1}(0,w)$, the class $W=\{t\geq 0:X(t)=0\}$ satisfies the condition $SupW=\infty$ if and only if the collection $\overline{W}=\{t\geq 0:Z(t)=0\}$ *satisfies* $Sup\overline{W}=0,\quad w.p.1$

Now for all $t>0$ and $w\in\Omega$, we define

$$P(t,w)=-a\Lambda(t-r,w)\circ\Lambda^{-1}(t,w) \tag{4.4}$$

We observe that P(.) is a non-negative continuous function on the half interval $[0,\infty)$. Also Z satisfies

$$Z^{|}(t,w) = -\sum_{j=1}^{n} P_j(t,w) Z(t-\sigma_j, w), \; t>0 \tag{4.5}$$

Let the subset $\Omega^* \subseteq \Omega$ exists such that for all $w \in \Omega$,

$$\Omega^* = \left\{ w \in \Omega : \int_{h(t)}^{t} \sum_{j=1}^{n} P_j(s,w) ds = \frac{1-c}{e} \right\}, \; w.p.1 \tag{4.6}$$

Then as P(.) and h(t) = t-r satisfy the hypothesis of proposition 2.1 on a path-wise basis. It follows that the path Z(.,w) is oscillatory. If not, we assume on the contrary that there exists a non-oscillatory solution Z(t) which is eventually positive, that is, there exists $t_1 > 0$ such that $Z(t)>0 \; for \; all \; t \geq t_1$. We have that

$$\frac{d}{dt}[Z(t) - cZ(t-r)] \leq 0, \; for \; all \; t > t_1 + \sigma = T \tag{4.7}$$

From (4.7), we have two possible options:

(i) $\quad Z(t) - cZ(t-r) \leq 0, \; for \; t > T$

(ii) $\quad Z(t) - cZ(t-r) > 0, \; for \; t > T$

Now consider (i): Suppose (i) holds, then there exists a constant $\delta > 0$ such that $Z(t) - cZ(t-r)$ $\leq -\delta, \; t > T$ resulting in

$Z(t) \leq -\delta + cZ(t-r) \leq -\delta + c[-\delta + cZ(t-2r)] \leq \delta[c + c^2 + \Lambda + c^n] + c^{n+1} Z(t-(n+1)c)$

We let $\|\psi\| = \underset{t\in[T-r,T]}{Sup} |\psi|$. Then for $t \geq T$ and a large enough n, we have

$$Z(t) \leq -\delta(c + c^2 + ... + c^n) + c^{n+1}\|\psi\|, \; 0 < c < 1 \tag{4.8}$$

which implies that Z(t) < 0. This is a contradiction showing that (i) is impossible.

Consider (ii):

Suppose (2) holds, define $w(t) = \dfrac{Z(t-r) - cZ(t-2r)}{Z(t) - cZ(t-r)} \geq 1$. Now rewriting (2.2) as

$$Z^{|}(t) - cZ(t-r) + \sum_{j=1}^{n} P_j(t) Z_j(t-\sigma_j) = 0 \tag{4.9}$$

Dividing through by [$Z(t) - c\,Z(t$-$r)$] then integrating gives

$$\begin{aligned} Log\,[w(t)] &= \int_{t-r}^{t} \frac{\sum_{j=1}^{n} P_j(s) Z_j(s-\sigma_j)}{Z(s) - cZ(s-r)} ds \\ &= \int_{t-r}^{t} \frac{\sum_{j=1}^{n} P_j(s) Z_j(s-\sigma_j) - cZ_j(s-\sigma_j - r) cZ_j s - \sigma_j - r}{Z(s) - cZ(s-r)} ds \\ &\geq \int_{t-r}^{t} \sum_{j=1}^{n} P_j(s) w(s) ds + \int_{t-r}^{t} \frac{\sum_{j=1}^{n} P_j(s) cZ_j(s-\sigma_j - r)}{Z(s) - cZ(s-r)} ds \end{aligned} \tag{4.10}$$

Taking advantage of the periodicity of $P_i(s)$ in (4.10), we obtain

$$Log[w(t)] \geq \int_{t-r}^{t} \sum_{j=1}^{n} P_j(s)w(s)ds - c\int_{t-r}^{t} \frac{Z(s-r)-cZ^{|}(s-2r)}{Z(s)-cZ(s-r)}ds$$

$$\int_{t-r}^{t} \sum_{j=1}^{n} P_j(s)w(s)ds - c\int_{t-r}^{t} w(s)\frac{d}{ds}\{\log[Z(s-r)-cZ(s-2r)]\}ds \tag{4.11}$$

Let t^* be a number such that $t\text{-}r < t^* < t$ and $\int_{t-r}^{t^*} \sum_{j=1}^{n} P_j(s)ds = \frac{P_0}{2}$, $\int_{t^*}^{t} \sum_{j=1}^{n} P_j(s)ds = \frac{P_0}{2}$ We prove that $w(t)$ is bounded above. Integrating (4.9) over (t^*, t), we have

$$Z(t)-cZ(t-r)-\left[Z(t^*)-cZ(t^*-r)\right]+\int_{t^*}^{t} \sum_{j=1}^{n} P_j(s)Z_j(s-\sigma_j)ds = 0$$

which follows that

$$x(t^*)-cZ(t^*-r) \geq \int_{t^*}^{t} \sum_{j=1}^{n} P_J(s)Z_j(s-\sigma_j)ds > \int_{t^*}^{t}[\sum_{j=1}^{n} P_J(s)Z_j(s-\sigma_j)-cZ_j(s-\sigma_j-r)]ds$$

$$\geq \left[Z_j(t-\sigma_j)-cZ_j(t-\sigma_j-r)\right]\int_{t^*}^{t} \sum_{j=1}^{n} P_J(s)ds = \left(x_j(t-\sigma_j)-cZ_j(t-\sigma_j-r)\right)\frac{P_0}{2} \tag{4.12}$$

Again integrating (4.9) over $[t\text{-}r, t^*]$, we get

$$Z(t^*)-cZ(t^*-r)-[Z(t-r)-cZ(t-2r)]+\int_{t-r}^{t^*} \sum_{j=1}^{n} P_j(s)Z_j(s-\sigma_j)ds = 0$$ which implies

$$Z(t-r)-cZ(t-2r) \geq \int_{t-r}^{t^*} \sum_{j=1}^{n} P_j(s)\left[Z_J(s-\sigma_j)-cZ_j(s-\sigma_j-r)\right]ds$$

$$\geq \left[Z_j(t^*-\sigma_j)-cZ(t^*-r-\sigma_j)\right] \tag{4.13}$$

Combining (4.12) and (4.13) and consequent of the fact that Z(t) –cZ(t-r) is non-increasing, we have

$$Z(t^*)-cZ(t^*-r) \geq [Z(t-r)-cZ(t-2r)]\left(\frac{P_0}{2}\right) \geq \left[Z(t^*-r)-cZ_j(t^*-\sigma_j-r)\right]\left(\frac{P_0}{2}\right)^2$$

$$\geq \left[Z(t^*-r)-cZ(t^*-2r)\right]\frac{P_0^2}{4}$$

Therefore
$$w(t^*) = \frac{Z(t^*-r)-cZ(t^*-2r)}{Z(t^*)-cZ(t^*-r)} \leq \frac{4}{P_0^2} \quad for\ all\ t^* \geq T \tag{4.14}$$

We define
$$\underset{t\to\infty}{Lim}\inf w(t) = \alpha \tag{4.15}$$

Since $\alpha < \infty$, it follows from (4.11) that

$$Log(\alpha) \geq P_0\alpha + \underset{t\to\infty}{Lim}\inf\left(-c\int_{t-r}^{t} w(s)\frac{d}{ds}[\log Z(s-r)-cZ(s-2r)]ds\right) \tag{4.16}$$

From (4.13), we have

$$Log(\alpha) \geq P_0\alpha - w(\theta(t))c\int_{t-r}^{t} \frac{d}{ds}[\log(Z(s-r)-cZ(s-2r))]ds,\ \theta(t) \in [t-r,t]$$

$$\geq P_0\alpha + w(\theta(t))cLog[w(t-r)]$$

T

$$\geq P_0\alpha + \alpha cLog(\alpha) \quad (4.17)$$

$$\geq P_0\alpha + cLog(\alpha) \quad (4.18)$$

he implication of (4.18) is that $(1-c)\frac{\log(\alpha)}{\alpha} \geq P_0$ *leading to* $\frac{1-c}{e} \geq P_0 = \int_{t-r}^{t}\sum_{j=1}^{n} P_j(s)ds$

which contradicts condition G_3 of proposition 2.1. Therefore *Z* is oscillatory *w. p.*1. So the trajectory *X*(.,*w*) is oscillatory. By (3.1). It follows that the subset $\Omega^* \subseteq \Omega$ is an almost sure event. Hence the solution *X* of the NSDDE (1.3) is oscillatory w. p.1 on $[0,\infty)$.

Integrating the deviation (2.3) for $t > \bar{t}$, over [t-r,t], (See [1]), we have

$$\int_{t-r}^{t} P(s)ds = \int_{t-r}^{t} -b\exp\left(-\left(I - \frac{\mu^2}{2}\right)\Gamma\right)\exp(-\mu(B(s)-B(s-r)))ds$$

T

$$\geq -b\max\left(1, \exp\left(-\left(I - \frac{\mu^2}{2}\right)\right)\Gamma\right)\int_{t-r}^{t}\exp(-\mu(B(s)-B(s-r)))ds \quad (4.19)$$

he crucial factor which generates oscillation in the NSDDE (1.3) is the large enough deviation in the increments of the Brownian motion. This holds if

$$\underset{t\to\infty}{Lim\,Sup}\int_{t-r}^{t}\exp(-\mu(B(s)-B(s-r)))ds = \infty \quad (4.20)$$

In the classical equation (2.1) (where noise is absent), if the time lags are small enough, the integral in (4.20) is made so small that the condition of proposition 2.2 holds and at that ssssmoment the classical neutral differential equation (2.1) has a non-oscillatory solution. However, the presence of multiplicative noise in the SNDDE ensures that the integral in (4.20) holds irrespective of the magnitude of the time lags. Although the noise has not completely replaced the time lags as the cause of oscillation, we note that the time lags are no longer the sole cause of oscillation in the SNDDE.

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