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A rational second order difference equation with convergence solutions

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Abstract

A review of second order difference equation is presented with instructive analysis of second-order rational difference equations. After classifying the various types of these equations and introducing some preliminary results, we systematically investigate each equation for convergence of solutions to the equilibrium. The convergence rate of solutions of a second order rational difference equation has been treated. We also investigate the rate of convergence of solutions of some special cases of the equation xn+1= (α+βxn+γxn−1)/(A+Bxn+Cxn−1), n = 0,1,…, with positive parameters and nonnegative initial conditions which gives precise results about the rate of convergence of the solutions that converge to the equilibrium.

Keywords

Rational, Second Order, Difference Equation, Convergence of Solution.

1.0 Introduction.

A second order difference equation is similar to a second order differential equation involving the unknown function *y*, its derivatives *y*' and *y*'', and the variable *x*. The solving of a second order difference equation is very similar to the method of solving a second order differential equation [6]. Kalabu and Kulenovi [4] systematically investigated each equation for semicycles, invariant intervals, boundedness, periodicity and global stability. They also presented prototype results towards the development of the basic theory of the global behaviour of solutions of nonlinear difference equations of order greater than one. The techniques and results are also extremely useful in analyzing the equations in the mathematical models of various biological systems and other applications [4, 7].

2.0 Difference equation

A difference equation is an equation involving the differences between successive values of a function of an integer variable. It can be regarded as the discrete version of a differential equation. For example the difference equation $f(n+1) - f(n) = g(n)$ is the discrete version of the differential equation $f'(x) = g(x)$. We can see difference equation from at least three points of views: as sequence of number, discrete dynamical system and iterated function.

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It is the same thing but we look at different angle. Difference equation is a sequence of numbers that are generated recursively using a rule to relate each number in the sequence to previous numbers in the sequence as presented in Figure 2.1, {1, 1, 2, 3, 5, 8, 13, 21,…}. The Sequence {1, 1, 2, 3, 5, 8, 13, 21,….} is called Fibonacci sequence, generated with rule $y(k+2) = y(k+1) + y(k)$ for $k = 0, 1, 2, 3,\ldots$, and initial value $y(0) = y_0 = 1$.

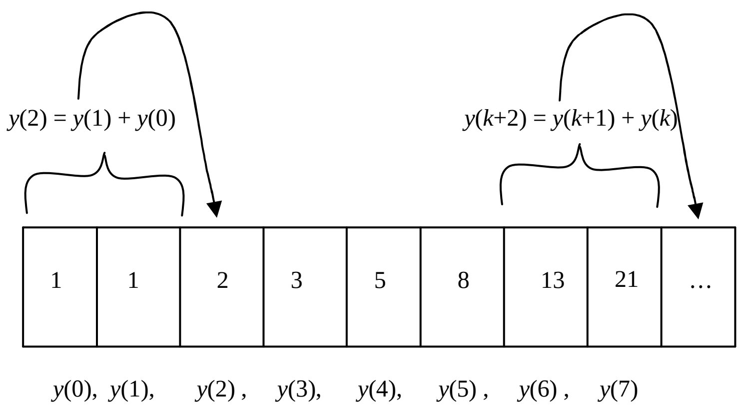

Figure 2.1: Sequence of difference equation.

Sequence {3, 5, 7, 9,...} has rule $y(k+1) = 2y(k) + 3$ for $k = 0, 1, 2, 3,\ldots$,. Both sequences have initial value $y(0) = y_0 = 0$. A general second-order difference equation is of the form

$$x_{t+2} = f(t, x_t, x_{t+1}). \tag{2.1}$$

This second-order difference equation has a unique solution and by successive calculation we can see that given x_0 and x_1 there exists a uniquely determined value of x_t for all $t \geq 2$.The solving of a second order difference equation is very similar to the method of solving a second order differential equation [6]. A second order difference equation of the form

$$y(x + 2) + ay(x + 1) + by(x) = 0 \tag{2.2}$$

has the following characteristic equation which is written out as

$$\lambda^2 + a\lambda + b = 0 \tag{2.3}$$

Consider the following cases.

2.1 Real and distinct roots

Case I:

The roots λ_1 and λ_2 of the quadratic equation (2.3) are real and distinct. Then the general solution of the original finite –difference equation is expressed as

$$y(x) = \theta_1(x)\lambda_1^x + \theta_2(x)\lambda_2^x, \tag{2.4}$$

where $\theta_1(x)$ and $\theta_2(x)$ are arbitrary periodic functions with unit period, $\theta_k(x) = \theta_k(x+1)$, k = 1, 2. If $\theta_k \equiv$ constant, it follows from (2.4) that there are particular solutions

$$y(x) = C_1\lambda_1^x + C_2\lambda_2^x \text{b} \tag{2.5}$$

where C_1 and C_2 are arbitrary constants.

2.2 Equal roots

Case II

The quadratic equation (2.3) has equal roots, $\lambda = \lambda_1 = \lambda_2$. In this case, the general solution of the functional equation is given by

$$y = [\theta_1(x) + x\theta_2(x)]\lambda^x \tag{2.6}$$

2.3 Complex roots

Case III

In the case of complex conjugate roots, $\lambda = \rho(\cos\beta \pm i\sin\beta)$, the general solution of the functional equation is expressed as:

$$y = \theta_1(x)\rho^x c\text{os}(\beta x) + \theta_2\rho^x \sin(\beta x), \tag{2.7}$$

where $\theta_1(x)$ and $\theta_2(x)$ are arbitrary periodic functions with unit period.

3.0 Rational difference equation

The convergence rate of solutions of a second order rational difference equation is now investigated:

$$y_{n+1} = \frac{\alpha + \beta y_n + \lambda y_{n-1}}{A + By_n + Cy_{n-1}}, \; n = 0,1,\ldots, \tag{3.1}$$

where the parameters α, β, λ, A, B, and C are positive real numbers and the initial conditions y_1, y_0 are arbitrary nonnegative real numbers [2].

. In our study, related nonlinear second-order rational difference equations which were investigated (2,5,6,7,8,9,10) in order to determine their contributions and it was discovered that the study of these equations was quite challenging. Three special cases of equations (2.1, 3.1) are considered at n = 0, 1,…,

$$y_{n+1} = \frac{B}{y_n} + \frac{C}{y_{n-1}}, \tag{3.2}$$

$$y_{n+1} = \frac{px_n + x_{n-1}}{qx_n + x_{n-1}}, \tag{3.3}$$

$$y_{n+1} = \frac{px_n + x_{n-1}}{q + x_{n-1}}, \tag{3.4}$$

where all the parameters are assumed to be positive and the initial conditions x_{-1}, x_0 are arbitrary positive real numbers. It can now be shown that the asymptotics of solutions that converge to the equilibrium depends on the character of the roots of the characteristic equation of the linearized equation evaluated at the equilibrium [4]. The results on asymptotics of (1.2, 9), (1.3, 10), and (1.4, 11) will show all the complexity of the asymptotics of the general equation (3.1).
Here we give some necessary definitions and results that we will use later.

Let I be an interval of real numbers and let $f \in C^1$[I * I, I]. Let $\bar{y} \in I$ be an equilibrium point of the difference equation

$$y_{n+1} = f(x_n, x_{n-1}), \; n = 0, 1, \ldots, \tag{3.5}$$

that is $\bar{y} = f(\bar{y}, \bar{y})$. Let

$$s = \frac{\partial f}{\partial u}(\bar{y}, \bar{y}), \; t = \frac{\partial f}{\partial v}(\bar{y}, \bar{y}), \tag{3.6}$$

denote the partial derivatives of $f(u,v)$ evaluated at an equilibrium $\bar{y}$ of (3.5). Then the equation

$$Z_{n+1} = sZ_n + tZ_{n-1}, \; n = 0, 1,\ldots \tag{3.7}$$

is called the linearized equation associated with (3.5) about the equilibrium point $\bar{y}$.

3.1 Linearized stability

***Theorem* 3.1**

(a) If both roots of the quadratic equation:

$$\lambda^2 - s\lambda - t = 0 \tag{3.8}$$

lie in the open unit disk $|\lambda| < 1$, then the equilibrium $\bar{y}$ of (3.5) is locally asymptotically stable.

(b) If at least one of the roots of (3.8) has an absolute value greater than one, then the equilibrium $\bar{y}$ of (3.5) is unstable.

(c) A necessary and sufficient condition for both roots of (3.8) to lie in the open unit disk $|\lambda| < 1$ is

$$|s| < 1-t < 2. \tag{3.9}$$

In this case, the locally asymptotically stable equilibrium $\bar{y}$ is also called a sink.

(d) A necessary and sufficient condition for both roots of (3.8) to have absolute values greater than one is

$$|t| > 1,\ |s| < |1-t|. \tag{3.10}$$

(e) A necessary and sufficient condition for one root of (3.8) to have an absolute value greater than one and for the other to have an absolute value less than one is

$$s^2 + 4t > 0,\ |s| > |1-t| \tag{3.11}$$

In this case, the unstable equilibrium $\bar{y}$ is called a saddle point.

The set of points whose orbits converge to an attracting equilibrium point or, periodic point is called the basin of attraction, [1].

***Definition* 3.1**

Let **T** be a map on R^2 and let **p** be an equilibrium point or a periodic point for **T**. The basin of attraction of **p**, denoted by Φ_p, is the set of points $y \in R^2$ such that $|T^k(x) - T^k(p)| \to 0$, as $k \to \infty$, that is,

$$\Phi_p = \{\, y \in R^2 : |T^k(x) - T^k(p)| \to 0, \text{ as } k \to \infty \}, \tag{3.12}$$

The positive and negative semicycles of a solution of (3.5) relative to an equilibrium point $\bar{y}$ is now defined.

A positive semicycle of a solution $\{y_n\}$ of (3.5) consists of a set of terms $\{y_r, y_{r+1}, \ldots, y_m)$, all greater than or equal to the equilibrium $\bar{y}$, with $r \geq -1$ and $m \leq \infty$ and such that either $r = -1$ or $r > -1$, $y_{r-1} < \bar{y}$, and either $m = \infty$ or $m < \infty$, $y_{m+1} < \bar{y}$.

A negative semicycle of a solution $\{y_n\}$ of (3.5) consists of a set of terms $\{y_r, y_{r+1}, \ldots, y_m)$, all less than the equilibrium $\bar{y}$, with $r \geq -1$ and $m \leq \infty$ and such that either $r = -1$ or $r > -1$, $y_{r-1} \geq$ y, and either $m = \infty$ or $m < \infty$, $y_{m+1} \geq \bar{y}$.

***Theorem* 3.2**

This theorem is a slight modification of the result obtained in [5]. It is assumed that *f* :

$$[0,\infty]\times[0,\infty] \to [0,\infty] \tag{3.13}$$

is a continuous function satisfying the following properties:

(a) there exist *L* and U, $0 < L < U$, such that

$$f(L,L) \geq L,\ f(U,U) \leq U, \tag{3.14}$$

and $f(y, Z)$ is non-decreasing in y and Z in $[L,U]$;

(b) the equation

$$f(y,y) = y \tag{3.15}$$

has a unique positive solution in [L,U]. Then (3.5) has a unique equilibrium $\bar{y} \in [L,U]$ and every solution of (3.5) with initial values y_{-1}, $y_0 \in [L,U]$ converges to $\bar{y}$.

Proof

Set $h_0 = L, H_0 = U,$ (3.16)

and for $i = 1, 2, \ldots,$

Set $H_i = f(H_{i-1}, H_{i-1}),\ h_i = f(h_{i-1}, h_{i-1}),$ (3.17)

Now observe that for each $i \geq 0$,

$$h_0 \leq h_1 \leq \ldots \leq h_i \leq \ldots \leq H_i \leq \ldots \leq H_1 \leq H_0,$$

The next two theorems give precise information about the asymptotics of linear non-autonomous difference equations. Consider the scalar kth-order linear difference equation

$$y(n+k) + p_1(n)y(n+k-1) + \ldots + p_k(n)y(n) = 0, \tag{3.18}$$

where k is a positive integer and p_i: ¢ $^+ \rightarrow$ £ for $i = 1, \ldots, k$. Assume that

$$q_i = \lim_{k\to\infty} p_i(n),\ i = 1, \ldots, k, \text{ exist in £} \tag{3.19}$$

Consider the limiting equation of (3.18)),

$$y(n+k)+q_1y(n+k-1) + \ldots + q_ky(n) = 0. \tag{3.20}$$

Then the following results describe the asymptotics of solutions of (3.18) [3].

***Theorem* 3.3 (Poincare's theorem**.)

Consider (3.18) subject to condition (3.19). Let $\lambda_1, \ldots, \lambda_k$ be the roots of the characteristic equation

$$\lambda^k + q_1\lambda^{k-1} + \ldots + q_k = 0 \tag{3.21}$$

of the limiting equation (3.20), and suppose that

$$|\lambda_i| \neq |\lambda_j| \text{ for } i \neq j \tag{3.22}$$

If $y(n)$ is a solution of (3.18), then either $y(n) = 0$ for all large n or there exists an index $j \in \{1, \ldots, k\}$ such that

$$\lim_{n\to\infty} \frac{y(n+1)}{y(n)} = \lambda_j. \tag{3.23}$$

The related results were obtained by Perron, which was improved upon by Pituk [4].

***Theorem* 3.4**

Suppose that (3.18) holds. If $y(n)$ is a solution of (3.19), then either $y(n) = 0$ eventually, or

$$\lim_{n\to\infty}(|y_j(n)|)^{1/n} = |\lambda_j|, \tag{3.24}$$

where $\lambda_1, \ldots, \lambda_k$ are the (not necessarily distinct) roots of the characteristic equation (3.21).

4.0 Rate of convergence of equation

The rate of convergence of equation

$$y_{n+1} = \frac{B}{y_n} + \frac{C}{y_{n-1}} \tag{4.1}$$

Equation (3.2) has a unique equilibrium point $y = \sqrt{B+C}$. The linearized equation associated with (3.2) about x is

$$Z_{n+1} + \frac{B}{B+C} z_n + \frac{C}{B+C} z_{n-1} = 0,\ n = 0,1,\ldots, \tag{4.2}$$

This equation was considered in [5], where the method of full limiting sequences was used to prove that the equilibrium is globally asymptotically stable for all values of parameters *B* and *C*. Here, we want to establish the rate of this convergence. The characteristic equation

$$\lambda^2 + \frac{B}{B+C}\lambda + \frac{C}{B+C} = 0,\ n = 0,1,\ldots, \tag{4.3}$$

that corresponds to (4.2) has roots

$$\lambda_{\pm} = \frac{-B \pm \sqrt{B^2 - 4C(B+C)}}{2(B+C)} \tag{4.4}$$

***Theorem* 4.1**

All solutions of (3.2) which are eventually different from the equilibrium satisfy the following.

(1) If the condition

$$C < \frac{B}{2(1+\sqrt{2})} \tag{4.5}$$

holds, then

$$\lim_{n\to\infty} \frac{y_{n+1} - \bar{y}}{y_n - \bar{y}} = \lambda_+ \quad \text{or} \quad \lim_{n\to\infty} \frac{y_{n+1} - \bar{y}}{y_n - \bar{y}} = \lambda_-, \tag{4.6}$$

where λ_+ and λ_- are the real roots given by (4.4). In particular, all solutions of (3.2) oscillate.

(ii) If the condition

$$C = \frac{B}{2(1+\sqrt{2})} \tag{4.7}$$

holds, then

$$\lim_{n\to\infty} \left(|y_n - \bar{y}|\right)^{1/n} = \frac{B}{2(B+C)} \tag{4.8}$$

(iii) If the condition $C > \dfrac{B}{2(1+\sqrt{2})}$ holds, then

$$\lim_{n\to\infty} \left(|y_n - \bar{y}|\right)^{1/n} = |\lambda_{\pm}| \tag{4.9}$$

where λ_+ and λ_- are the complex roots.

Proof

We have

$$y_{n+1} - \bar{y} = \frac{B}{y_n} + \frac{C}{y_{n-1}} - \bar{y} = -\frac{B}{y_n\bar{y}}(y_n - \bar{y}) - \frac{C}{y_{n-1}\bar{y}}(y_{n-1} - \bar{y}) \tag{4.10}$$

Set $t_n = y_n - \bar{y}$. Then we obtain:

$$t_{n+1} + p_n t_n + q_n t_{n-1} = 0, \tag{4.11}$$

where $$p_n = \frac{B}{y_n \bar{y}},\ q_n = \frac{C}{y_{n-1}\bar{y}} \qquad (4.12)$$

Since the equilibrium is a global attractor, we obtain

$$\lim_{n\to\infty} p_n = \frac{B}{B+C},\ \lim_{n\to\infty} q_n = \frac{C}{B+C} \qquad (4.13)$$

Thus, the limiting equation of (3,2) is the linearized equation (4.2) whose characteristic equation is (4.5). The discriminant of this equation is given by

$$D = B^2 - 4C(B+C) = (B - 2\sqrt{C(B+C)})(B + 2\sqrt{C(B+C)}) \qquad (4.14)$$

Conditions (4.5), (4.7) and (4.9) are the conditions for $D > 0$, $D = 0$, and $D < 0$, respectively. Now, statement (i) follows as an immediate consequence of Poincare's theorem and statements (ii) and (iii) follow as the consequences of Theorem 3.4. Finally, the statement on oscillatory solutions follows from the asymptotic estimate (3.6) and the fact that in the case D > 0 both roots λ_+ and $\lambda_- < 0$.

Figure 4.1 visualizes the regions for the different asymptotic behaviour of solutions of (1.2).

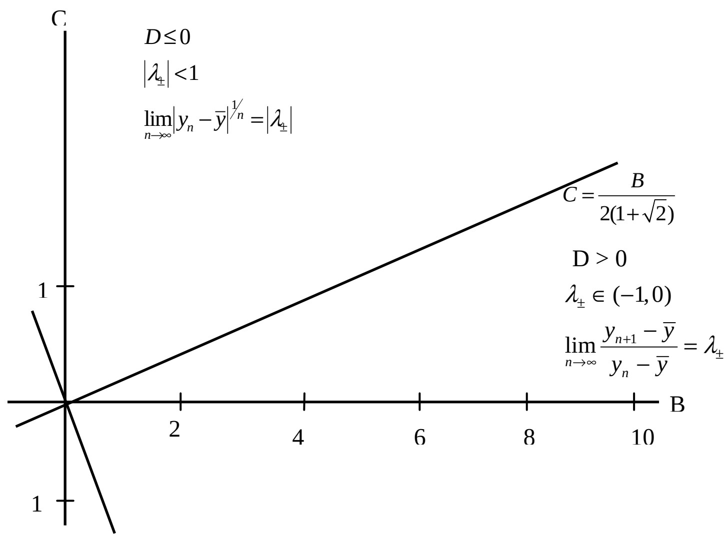

Figure 4.1: Regions for the different asymptotic behavior of solutions of (9).

5.0 Conclusion

We investigated the rate of convergence of solutions of some special cases of the equation $x_n + 1 = (\alpha+\beta x_n + \gamma x_{n-1})/(A+Bx_n + Cx_{n-1})$. The rate of convergence of the equation to equilibrium with positive parameters and nonnegative initial conditions was determined based on certain theorems such as Poincare's theorem.

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