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Finitely generated commutative Noetherian semigroups

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Abstract

We provide a short and more direct proof that a commutative semigroup is finitely generated if its lattice of congruences is Noetherian.

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1.0 Introduction

Let R be a unitary commutative ring and S a commutative monoid. Gilmer proves in [3] that the monoid ring $R[S]$ is Noetherian if and only if R is Noetherian and S is finitely generated. The proof consists of three parts:

(*i*) If $R[S]$ is Noetherian, then R is Noetherian and Cong S, the lattice of congruenes of S, is Noetherian.

(*ii*) If Cong S is Noetherian, then S is finitely generated (as a semigroup or as a monoid).

(*iii*) If R is Noetherian and S is finitely generated then $R[S]$ is Noetherian.

By far, the hardest part of this proof is the pure monoid theory represented by (*ii*) in this list. We will say that a monoid (or semigroup) is Noetherian if its lattice of congruences is Noetherian. Then (*ii*) says that any Noetherian monoid is finitely generated. The proof of this is due to Budach [1] and fills Chapter 5 of [3]. It depends on a primary decomposition theorem for congruences on Noetherian semigroups proved by Drbohlav in [2].

The purpose of this paper is to provide a shorter and more direct proof of this result. In fact, it is just as easy to show that any Noetherian semigroup is finitely generated, a result which Gilmer obtained in [4] by reducing to the monoid case.

2.0 Main results

We obtain with some definitions, notation and basic properties of partially ordered sets and semigroups.

***Definition* 2.1**

Let L be a partially ordered set. Then L is **Artinian** if every nonempty subset of L has

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a minimal element (equivalently, L satisfies the descending chain condition, d.c.c.), and L is **Noetherian** if every nonempty subset of L has a maximal element (equivalently, L satisfies the ascending chain condition, a.c.c.).

***Definition* 2.2**

If $\sigma K \to L$ is a strictly increasing (decreasing) map between partially ordered sets and L is Noetherian, then K is Noetherian (Artinian).

***Definition* 2.3**

A lower set of L is a subset $D \subseteq L$ such that for all $x, y \in L$, if $x \le y$ and $y \in D$, then $x \in D$. We write $\Downarrow L$ for the set of lower sets of L ordered by inclusion. L embeds in $\Downarrow$L via the map $x \alpha \; \{y \in L | \; y \le x\}$, hence if $\Downarrow L$ is Artinian, then so is L. The lower set generated by a subset $A \subseteq L$ is $\{x \in L$ | there exists $a \in A$, such that $x \le a$ }.

The proof of the main theorem of this paper proceeds by reducing the question about finite generation of semigroups to the following purely order theoretic result:

***Lemma* 2.1**

Let L be a partially ordered set. If L is Noetherian and $\Downarrow L$ is Artinian, then L is finite.

Proof

Suppose to the contrary that L is infinite. Since L is Noetherian we can construct an infinite sequence $\{a_n \mid n \in \mathrm{N}\}$ of distinct elements of L such that a_1 is maximal in L, and for all $n \ge 2$, a_n is maximal in L given that $\{a_1, a_2, a_3, \mathrm{K} \; , a_{n-1}\}$. For $n \in N$, let D_n for all $n \ge 1$, there must be some n such that $D_n = D_{n+1}$. In particular, $a_n \in D_{n+1}$. This means that $a_n \le a_m$ for some $m > n$. But $a_n = a_m$ is not possible because the elements in the sequence are distinct, and $a_n \le a_m$ is not possible since a_n is maximal in L given that $\{a_1, a_2, a_3, \mathrm{K} \; , a_{n-1}\}$, a set which also contains a_m. Thus we have a contradiction. ■

***Remark* 2.1**

This lemma follows also from the standard result [6], [7], [8, 1.4] that, if $\Downarrow$L is Artinian, then any infinite sequence in L is Noetherian, then no such infinite strictly increasing sequence, and so L cannot be infinite.

For the definitions and basic properties of commutative semigroups we refer the reader to [3]. If S is a commutative semigroup, we will write Cong S for the set of congruences of S ordered in the usual way: $\sim \le \sim'$ if for all $x, y \in S, x \sim y$ implies $x \sim' y$. If Cong S is Noetherian, we say that S is a **Noetherian semigroup**. The smallest congruence in Cong S is equality, also e, the **identity congruence**. The largest congruence is the **universal congruence** defined by $x \sim y$ for all $x, y \in S$. For a fixed congruence ~, Cong (S/~) is isomorphic to the subset $\{\sim' | \sim' \ge \sim\}$ of Cong S. In particular, if S is Noetherian then so is S/~. The subsemigroup generated by an element a or subset A of S will be written $\langle a \rangle$ or $\langle A \rangle$. In this paper "(finitely) generated" means "(finitely) generated as a semigroup".

Define a relation ≤ on S by $x \le y$ if $x = y \; or \; x + s = y$ for some $s \in S$. It is easy to see that ≤ is reflexive and transitive. Since it is possible to have $x \le y \le x$ but $x \ne y$, the relation ≤ is not, in general, a partial order on S.

One important case in which ≤ is a partial order on *S* is when every element is an idempotent, that is, $b = 2b$ for all $b \in S$. In this circumstance (*S*, ≤) is a (join-) semilattice in which + and ∨ coincide. See, for example, [5, 1.3.2].

In proving that Noetherian semigroups are finitely generated, certain congruences which behave well with respect to generating sets are the key: A congruence ~ on a semigroup *S* satisfies * or is a ***-congruence** if it has the following property: If *Y* is a subset of *S* whose image in *S*/~ is itself, then *S* is generated by *Y* and is finite.

Note that the identity congruence satisfies *, and that *S* is finitely generated if and only if the universal congruence satisfies *.

***Lemma* 2.2**

Let *S* be a Noetherian semigroup. If the identity congruence is the only *-congruence on *S*,< then *S* is trivial.

Proof

(*i*) (*S*, ≤) *is a partially ordered set.* Since ≤ is reflexive and transitive, it remains only to show that $a \le b \le a$ implies $a = b$ for $a, b \in S$. If $a \le b \le a$, then either $a = b$ or $a = b + t_1$ and $b = a + t_2$ for some $t_1, t_2 \in S$.

In the second case, set $T = \langle t_1, t_2 \rangle$ and define the congruence ~ by $x \sim y$ if $x = y$ or there exist $t, t' \in T$ such that $x = y + t$ and $y = x + t'$. By construction, we have $a \sim b$, so to prove $a = b$ it suffices to show that ~ is a *-congruence.

Suppose that the image of $Y \subseteq S$ generates *S*/~, then for any element $x \in S$ we have $x \sim y$ for some $y \in \langle Y \rangle$ or $x = y + t \in \langle t_1, t_2, Y \rangle$ for some $t \in T$. Thus $S = \langle t_1, t_2, Y \rangle$ and ~ is a *-congruence.

(*ii*) $\Downarrow (S, \le)$ *is Artinian. In particular,* (*S*, ≤) *is Artinian.* For $D \in \Downarrow (S, \le)$, definite the congruence $\tilde{D}$ by $x\tilde{D}y$ if either $x = y$ or $x \notin D$ and $y \notin D$. It is easy to show that the map $D \,\alpha\, \tilde{D}$ from $\Downarrow (S, \le)$ to Cong *S* is decreasing.

This map is, infact strictly decreasing when restricted to proper lower sets of $(S, \le)$. If $D, E \in \Downarrow (S, \le)$ with $D \subset E \subset S$, then for any $x \in E \setminus D$ and $y \in S \setminus E$ we have $x \underset{D}{\sim} y$ but not $x\tilde{E}y$. Therefore $\tilde{D} > \tilde{E}$.

Since Cong *S* is Noetherian, this implies that the set of proper lower sets of *S* is Artinian. It follows immediately that $\Downarrow (S, \le)$ is Artinian.

(*iii*) (*S*, ≤) *is a semilattice, that is,* $b = 2b$ for all $b \in S$. For $b \in S$ define the congruence ~ by $x \sim y$ if either $x = y$ or $b \le x, y$ and $x + mb = y + nb$ for some $m, n \in N$. By construction we have $b \sim 2b$, so to prove $b = 2b$ it suffices to show that ~ is a *-congruence.

Suppose the image of $Y \subseteq S$ generates *S*\~. We will show that $S = \langle b, Y \rangle$. If, to the contrary $S \ne \langle b, Y \rangle$, we have $x \sim y$ for some $y \in \langle Y \rangle$. Since $x \ne y \in \langle Y \rangle$, we must have $b \le x, y$ and $x + mb = y + nb$ for some $m, n \in N$. Since $x \ne b$, there is some x' such that $x = b + x'$. The element x' cannot be in $\langle b, Y \rangle$ since that would imply the same for *x*. By the minimality of *x* we have $x' = x$, that $x = X + b$. From this we get $x + mb = y + nb \in \langle b, Y \rangle$, a contradiction.

(iv) (S, ≤) *is Noetherian.* For an element $s \in S$ define the congruence $\tilde{s}$ by $x\tilde{s}y$ if $s + x = s + y$. It is easy to check that, since S is a semiattice, the map $s\, \alpha\, \tilde{s}$ from (S, ≤) to Cong S is strictly increasing. Since Cong S is Noetherian, so is (S, ≤).

(v) *S is trivial.* We now have that **(**S, ≤) is Noetherian and $\Downarrow (S, \leq)$ is Aritinian, so from Lemma 2.1, we know that S is finite. But in this case, the universal congruence on S satisfies *. Thus the universal congruence is also the identity congruence, meaning that S is trivial. ■

***Theorem* 2.1**

Any Noetherian semigroup is finitely generated.

Proof

Let S be a Noetherian semigroup. Let ≈ be a maximal *-congruence on S and $S' = S / \approx$. We show that the only *-congruence on S' is represented by a congruence ~ on S such that $\approx \leq \sim$. If $Y \subseteq S$ generates S/~ and ~ satisfies * with respect to S', then Y is a finite set and generates $S / \approx$. But then, since ≈ satisfies *, Y is a finite set and also generates S. Thus ~ is a *-congruence with respect to S. By the maximality of ≈, we have ≈ = ~, that is ~ represents the identify congruence on S'. ■

3.0 Conclusion

Since (S, ≤) is partially ordered, the complement of a proper lower set is an ideal of S and vice versa. Moreover, for a proper lower set D, the congruence $\tilde{D}$ is the Rees congruence associated to the ideal $S\backslash D$. Hence we have also proved that the set of ideals of S ordered by inclusion is Noetherian, a fact which is true in any Noetherian semigroup. See [3, 5.1].

Since S' is a Noetherian semigroup whose only *congruence is the identity congruence, Lemma 2.2 implies that S' is trivial. It follows immediately that ≈ is the universal congruence and hence S is finitely generated.

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