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On congruence lattices

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Abstract

Investigations of the lattice of congruences on a semigroup have taken two different directions. One approach is to study special congruences on a semigroup, and describe their relative positions within the lattice of congruences. For some classes C_1 and C_2, it will happen that the intersection σ is, of course, the minimum C_1 congruence on S, and S/σ is a maximal homomorphic image of S of type C_1. For instance, it is easily seen that the intersection of all commutative congruences on any semigroup is a commutative congruence, and so every semigroup has a minimum commutative congruence. Similarly, every semigroup has a minimum band congruence (denoted β) and a minimum semilattice congruence (denoted η). We outline some results dealing with the lattice of congruences of a semigroup. It is clear that a modular lattices is a semimodular, but the converse, however, is not true.

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Complete lattice, modularity, homomorphic, isomomorhic

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1.0 Introduction

1.1 Preliminaries

Recall from [2], [5], [6] and [10] that an equivalence relation α on a semigroup *S is* called a *congruence* if $x\alpha y$ and $s\in S$ imply that *sxαsy* and *xsαys*. A congruence α, of partitions *S,* is the set S/α of α-classes which forms a semigroup, that is, a homomorphic image of *S*. Conversely, every homomorphic image of *S* is isomorphic to S/α for some congruence α. Thus, congruences play much the same role that normal subgroups do in group theory and ideals in ring theory.

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Suppose that ρ and σ are congruences on *S*, with $\rho \subseteq \sigma$. Then there is a unique homomorphism $\phi : S/\rho \to S/\alpha$ such that Figure 1.1 commutes.

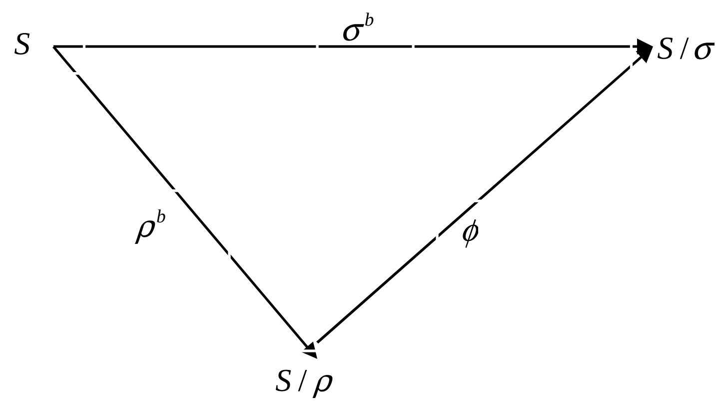

Figure 1.1: Homomorphism $\phi : S/\rho \to S/\alpha$

Define σ/ρ to be the relation on σ/ρ given by

$$\rho^b(x)\sigma/\rho\,\rho^b(y) \iff \sigma^b(x) = \sigma^b(y).$$

This relation is well-defined and follows from the fact that $\rho \subseteq \sigma$: if $x\rho x'$ and $y\rho y'$, then $x\sigma x'$ and $y\sigma y'$. It is easy to see that σ/ρ is congruence. And it then follows from the first isomorphism theorem that

$$(S/\rho)/(\sigma/\rho) \cong S/\sigma$$

giving an analog of the third isomorphism theorem.

***Definition* 1.1**

A lattice *L* is called *modular* if

$$a \le c \Rightarrow (a \vee b) \wedge c = a \vee (b \wedge c).$$

Now in any lattice, if it is true that $(a \vee b) \wedge c \ge a \vee (b \wedge c)$, *L* is modular if and only if

$$a \le c \Rightarrow (a \vee b) \wedge c \le a \vee (b \wedge c)$$

Equivalent to the condition of modularity is the conditions that

$$a \le c, a \wedge b = c \wedge b$$

and

$$a \vee b = c \vee b$$

which imply that *a* = *c*; that is, there is no sublattice isomorphic as shown in Figure 1.2.

***Definition* 1.2**

In a lattice, we say that *a covers b* if $a \phi b$ and there is no element *x* such that $a f x f b$. We write $a \phi b$ to indicate that *a* covers *b*.

***Definition* 1.3**

A lattice is *(upper) semimodular* if $a \phi a \wedge b$ and $b \phi a \wedge b$ together imply that $a \vee b \phi a$ and $a \vee b \phi b$.

It is natural to ask whether congruence lattices on various types of semigroups have either of these properties. It is fairly well known, for instance, that the lattice of normal subgroups of a group is modular, and thus the lattice of congruences on a group is modular. Although this fact has a group-theoretic proof, we will examine it from a semigroup-theoretic viewpoint.

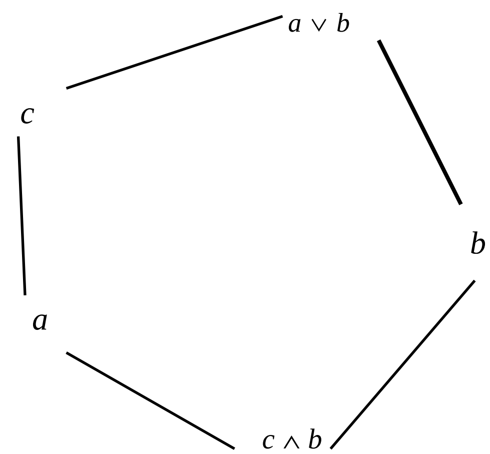

Figure 1.2: Non-isomorphic sublattice

***Lemma* 1.4**

If G is a group and ρ and σ are congruences, on G, then $\rho o \sigma = \sigma o \rho$.

Proof

Suppose $a \, \rho \, o \, \sigma \, b$. Then there is some $c \in G$ such that $a \, \rho \, c \, \sigma b$. Then

$$a = cc^{-1} a \sigma bc^{-1} a \rho bc^{-c} = b$$

and $a \, \sigma \, o \, \rho \, b$, showing that $\rho \, o \, \sigma \subseteq \sigma \, o \, \rho$. Similarly, $\sigma \, o \, \rho \subseteq \rho \, o \, \sigma$. ■

***Lemma* 1.5**

Suppose K is a sublattice of $\Lambda(S)$, where S is a semigroup and suppose that $\rho \, o \, \sigma = \sigma \, o \, \rho$ for all $\rho, \sigma \in K$. Then K is modular.

Proof

Suppose $\alpha, \beta, \gamma \in K$ with $\alpha \leq \gamma$. Since $\alpha \, o \, \beta = \beta \, o \, \alpha$, we have $\alpha \vee \beta = \alpha \, o \, \beta$. If $(x, y) \in (\alpha \vee \beta) \wedge \gamma$, then $(x, y) \in \gamma$ and there is some $z \in S$ such that $(x, z) \in \alpha$ and $(z, y) \in \beta$. But then $(z, x) \in \alpha \subseteq \gamma$, and so $(z, y) \in \gamma$ by transitivity. Thus $(z, y) \in \beta \wedge \gamma$. So $(x, z) \in \alpha$ and $(z, y) \in \beta \wedge \gamma$, and thus $(x, y) \in \alpha \, o \, (\beta \wedge \gamma) = \alpha \vee (\beta \wedge \gamma)$. ■

Combining these lemmas, we obtain

***Theorem* 1.6**

If S is a group, then $\wedge(S)$ is modular.

2.0 Some additional results

In this section we state, mostly without proof, some further results on congruences and congruence lattices.

***Theorem* 2.1** [1, Corollary 2]

If S is a semilattice, then $\wedge(S)$ is semimodular.

***Theorem* 2.2** [4]

If S is a completely simple semigroup, then $\wedge(S)$ is semimodular.

These results lead us to look at the congruence lattice of a semigroup that is constructed from groups and perhaps semilattices. For instance, is it true that the lattice of congruences on a a strong semilattice of groups is semimodular? The following example shows that this that this conjecture is not true.

***Example* 2.3**

Let S be the strong semilattice of groups $G_e \cup G_f$, where $G_e = \{e, a\}$ is a group with identity *e*, and $G_f = \{f, b\}$ is a group with identity *f*, and where the multiplication is defined by way of the isomorphism $\phi_{e,f} : G_e \to G_f$. The congruences on *S* are:

(*i*) $S \times S$

(*ii*) σ, the congruence whose classes are {*e*, *f*} and {*a*, *b*}

(*iii*) 0*s*

(*iv*) H, the congruence whose classes are {*e*, *a*} and {*f*, *b*}

(*v*) α, the congruence whose classes are {*e*}, {*a*} and {*f*, *b*}

It is easy to check that these congruences form a non-modular lattice. One the other hand, we have

***Theorem* 2.4**

If *S* is a regular semigroup, then *S* has a minimum group congruence.

Proof

Recall from [9] that a regular semigroup is a group if and only if it has a unique idempotent. Let $\Gamma(S)$ be the sublattice of $\wedge(S)$ consisting of all group congruences, and let $\sigma = \cap\, \Gamma(S)$. Then S/σ is a homomorphic image of *S*, and therefore is regular. And if S/σ has idempotents $\sigma^b(x)$ and $\sigma^b(y)$, then $x^2 \sigma x$ and $y^2 \sigma y$, so for every $\rho \in \Gamma(S)$, $x^2 \rho x$ and $y^2 \rho y$. But for each $\rho \in \Gamma(S), S/\rho$ has a unique idempotent, and so $x \rho y$ for all $\rho \in \Gamma(S)$. Thus $x \sigma y$, so there is a unique idempotent in S/σ, showing that S/σ is a group.

The following results of [3] and [4] typify the approach of investigating the position[redacted]of special congruences within $\wedge(S)$.

***Theorem* 2.5** [3, Theorems 1.3, 1.4]

If *S* is a regular semigroup, then

$$H^* \subset \beta \subseteq R^* \cap L^* \text{and } \eta = D^* = J^*$$

***Definition* 2.6**

Let *S* be a semigroup and $A \subseteq S$. We say that *A* is unitary if

$$a \in A,\ x \in S,\ ax \in A \Rightarrow x \in A$$

and

$$a \in A,\ x \in S,\ xa \in A \Rightarrow x \in A$$

***Theorem* 2.7** [3, Corollary 2.7]

If *S* is a regular semigroup, then $\sigma \cap \beta$ is the minimum *UBG* congruence on *S* (*UBG = band of groups in which E is unitary*)

Another approach is to attempt to use certain sublattices of $\wedge(S)$ to describe the global structure of $\wedge(S)$. In this direction, we note that, on a band of groups *S*, the following are sublattices of $\wedge(S)$.

B(*S*) = lattice of band congruences on *S*
M(*S*) = lattice of idempotent-separating congruences on *S*
D(*S*) = lattice of congruences on *S* contained in *D*.

***Theorem* 2.8** [11, Theorem 3.9]

Let *S* be a band of groups. Then the map

$$\phi : D(S) \rightarrow B(S) \times M(S)$$

given by

$$\psi(\rho) = (\rho \vee H, \rho \wedge H)$$

is an embedding.

***Definition* 2.9**

Let *S* be a regular semigroup. On $\wedge(S)$, define a relation θ by

$$(\rho_1, \rho_2) \in \theta \Leftrightarrow \rho_1 \cap (Es \times Es) = \rho_2 \cap (Es \times Es)$$

That is, two congruences are θ-related if they partition the idempotents in the same way. Reilly and Scheiblich proved in [7] that each θ-class of an inverse semigroup is a complete, modular sublattice of $\wedge(S)$. This was later extended by Scheiblich in [8]. Exploiting this θ relation, we define the following class of semigroups.

***Definition* 2.10**

A semigroup *S* is called θ-*modular* if each θ-class of $\wedge(S)$ is a modular sublattice of $\wedge(S)$.

***Theorem* 2.11** [11, Theorems 3.14, 3.15]

Let *S* be a band of groups. Then *S is* θ-modular if and only if the mapping $\psi : \wedge(S) \rightarrow B(S) \times M(S)$ given by

$$\psi(\rho) = (\rho \vee H, \rho \wedge H)$$

is an embedding.

It is immediate that the intersection of a nonempty collection of congruences on a semigroup *S* is a congruence on *S*. Moreover, if α is an equivalence relation on *S* and let α* denote $\bigcap \{\rho / \rho \text{ is a congruence on } S \text{ and } \alpha \subseteq \rho\}$, then α* is the smallest congruence containing α. Thus, the set ∧(*S*) is a *complete* lattice, in the sense that every nonempty subset has a meet and a join.

3.0 Conclusion

Not every semigroup has a minimum group congruence. For example, let *S* be the infinite cyclic semigroup on one generator. The *group* homomorphic images of *S* are precisely the groups $(Z_{n,+})$, and so there is no maximal group homomorphic image of *S*, and therefore no minimal group congruence.

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