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The dynamics of stock price and determination of investor's cash flows valuation.

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Abstract

We consider the dynamics of share price, that is, upward and downward movement of share price at time t. We made use of binomial derivative pricing process of stochastic processes. Our aim is to also determine investor's cash flows valuation generated from the investment. The investor invested her short position into N number of investment firms. The firms in turn invest the short position of the investor into the stock and bond markets in order to hedge out the risks associated with the investor's portfolio. We determine the value of the cash flows at time, t = 0 by finding the value of the discounted cash flows using a suitable discount rate. We assume that the discount rate is deterministic. We found out that with different values of the investment at the initial time from the investment firms over time yield almost the same percentage change (not the same value at time t) with the same interest rate. In this paper, we assume that there is no transaction costs.

Keywords
Investor; Cash flows valuation; Share price; Portfolio; Discount rate.

1.0 Introduction

1.1 Problem formulation

Consider the following problem. An investor invested $S_0^i, i = 1,2,\Lambda, N$, through investment firms at time $t = 0$. At time $t > 0$, investor is expected to have S_t^i from investment firm *i*. The investment firm invests the amount on behalf of the investor. Let S_0^i be the initial stock price, $u \; and \; d$ be the value of the movement of share price up and down respectively such that $0 < d < 1 < u$. In the evolution to next period of time, the share price will be either uS_0^i or dS_0^i. $uS_0^i \; and \; dS_0^i$ represent the values when the share price move upward and downward respectively at the first move [2] and [4].

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At time $t = 0$, the value of the investor investment is V_0. The investor will receive a payoff (uS_0^i) from firm i, if the share price goes up and (dS_0^i) if the share price goes down. As a result of the uncertainty of the share price, the investment firms have to hedge the investor's short position by purchasing σ_0^i share of stocks. If V_0 is greater than what should be used to purchase the σ_0^i shares of stock, the firms will invest the remaining amount in bond market at a continuously compounded interest rate r in a short interval of time. Hence, if $\sum_{i=1}^{N} \sigma_0^i S_0^i$ amount is invested into the stock market, then $V_0 - \sum_{i=1}^{N} \sigma_0^i S_0^i$ is the amount invested in bond market. If $V_0 - \sum_{i=1}^{N} \sigma_0^i S_0^i \leq 0$, it implies that some amount was borrowed to finance the investment which must be paid back at the end of the investment period, (see [1[[2], [3], [4] and [5].
Our aim is to determine the value of the investor's portfolio over time, which is given by

$$\sum_{i=1}^{N} \sigma_{t-1}^i S_t^{iu} + e^r \left(V_{t-1} - \sum_{i=1}^{N} \sigma_{t-1}^i S_{t-1}^i \right), t = 1, 2, ..., T \ .$$

This portfolio comprises the aggregate amount received from the investment firms at time *t*.

1.2 Valuation of the investor's portfolio

***Definition* 1.1**

The portfolio process is $\delta = (\delta_0, \delta_1, ..., \delta_{k+1})$, where

(i) δ_k is the number of shares in the stock market,

(ii) Each δ_k is $F_k - measurable$.

***Definition* 1.2**

Let V be the value of the portfolio process, then V is said to be self-financing if the following are all satisfies

(i) start with nonrandom initial wealth, $V_0 > 0$,

(ii) $V_{k+1} = \delta_k S_{k+1} + \exp[r](V_k - \delta_k S_k) = \exp[r]V_k + \delta_k (S_{k+1} - \exp[r]S_k)$

(iii) Then each V_k is $F_k - measurable.$

If the stock price rises, the value of the firm i portfolio is

$\sigma_0^i S_1^{iu} + e^r (V_0^i - \sigma_0^i S_0^i)$, where $V_0 = \sum_{i=1}^{N} V_0^i$ and the investor's short portion is not included.

Firm i need to have V_1^{iu} by choosing V_0^i *and* σ_0^i such that

$$V_1^{iu} = \sigma_0^i S_1^{iu} + e^r (V_0^i - \sigma_0^i S_0^i) \tag{1.1}$$

Hence, the investor's portfolio will be

$$\sum_{i=1}^{N} \sigma_0^i S_1^{iu} + e^r \left(V_0 - \sum_{i=1}^{N} \sigma_0^i S_0^i \right).$$

If the stock price falls, the value of the portfolio is $\sigma_0^i S_1^{id} + e^r (V_0^i - \sigma_0^i S_0^i)$. Similarly,

$$V_1^{id} = \sigma_0^i S_1^{id} + e^r\left(V_0^i - S_0^i\sigma_0^i\right) \qquad (1.2)$$

Hence, the investor's portfolio will be $\sum_{i=1}^{N}\sigma_0^i S_1^{id} + e^r\left(V_0 - \sum_{i=1}^{N}\sigma_0^i S_0^i\right)$. We now solve for the unknown, V_0^i *and* σ_0^i as follows: For σ_0^i, we subtract (1.2) from (1.1) to have

$$\sigma_0^i = \left(V_1^{iu} - V_1^{id}\right)/\left(S_1^{iu} - S_1^{id}\right) \qquad (1.3)$$

For V_0^i, we substitute (1.3) into either (1.1) or (1.2) as follows:

$$V_1^{iu} = \sigma_0^i S_1^{iu} + e^r\left(V_0^i - \sigma_0^i S_0^i\right) = \sigma_0^i S_1^{iu} + e^r V_0^i - e^r\sigma_0^i S_0^i$$

$$= \sigma_0^i\left(S_1^{iu} - S_0^i e^r\right) + e^r V_0^i$$

$$\therefore\ e^r V_0^i = V_1^{iu} - \sigma_0^i\left(S_1^{iu} - S_0^i e^r\right) = V_1^{iu} - \frac{V_1^{iu} - V_1^{id}}{S_1^{iu} - S_1^{id}}\left(S_1^{iu} - S_0^i e^r\right) = \frac{V_1^{iu}\left(S_1^{iu} - S_1^{id}\right) - \left(V_1^{iu} - V_1^{id}\right)\left(S_1^{iu} - S_0^i e^r\right)}{S_1^{iu} - S_1^{id}}$$

$$= \frac{\left(S_0^i e^r - S_1^{id}\right)V_1^{iu} + \left(S_1^{iu} - S_0^i e^r\right)V_1^{id}}{S_1^{iu} - S_1^{id}} = \frac{e^r - d}{u - d}V_1^{iu} + \frac{u - e^r}{u - d}V_1^{id}$$

Therefore
$$V_0^i = e^{-r}\left[\frac{e^r - d}{u - d}V_1^{iu} + \frac{u - e^r}{u - d}V_1^{id}\right]. \qquad (1.4)$$

This is the arbitrage price with payoff V_1^i at time period 1 for i firm.

Setting
$$p^* = \frac{e^r - d}{u - d},\ q^* = \frac{u - e^r}{u - d} = 1 - p^* \qquad (1.5)$$

Equation (1.4) now becomes
$$V_0^i = e^{-r}\left[p^* V_1^{iu} + q^* V_1^{id}\right], \qquad (1.6)$$

where, p^* *and* q^* are risk-neutral probabilities. Therefore,

$$V_0 = e^{-r}E_*\left[\sum_{i=1}^{N}V_1^i\right] = e^{-r}\sum_{i=1}^{N}E_*\left[V_1^i\right]. \qquad (1.7)$$

We now consider the arbitrage price with payoff V_2^i at time period 2 for firm i. At time period 1, the firm i has a portfolio that excludes the investor's short portion, valued at

$$V_1^i = \sigma_0^i S_1^i + e^r\left(V_0^i - \sigma_0^i S_0^i\right) \qquad (1.8)$$

From equation (1.8), we obtain
$$V_1^{iu} = \sigma_0^i S_1^{iu} + e^r\left(V_0^i - \sigma_0^i S_0^i\right) \qquad (1.9)$$
$$V_1^{id} = \sigma_0^i S_1^{id} + e^r\left(V_0^i - \sigma_0^i S_0^i\right)$$

At time period 1, firm i has V_1^i and can readjust her hedges. Suppose that firm i decides to hold σ_1^i shares of stock, they invest the remaining of her wealth, $\left(V_1^i - \sigma_1^i S_1^i\right)$ in the bond market. At time period 2, her wealth will be

$$V_2^i = \sigma_1^i S_2^i + e^r\left(V_1^i - \sigma_1^i S_1^i\right), \qquad (1.10)$$

from equation (1.10), we obtain

$$V_2^{iu^2} = \sigma_1^i(u)S_2^{iu^2} + e^r\left(V_1^{iu} - \sigma_1^i(u)S_1^{iu}\right) \qquad (i)$$
$$V_2^{iud} = \sigma_1^i(u)S_2^{iud} + e^r\left(V_1^{iu} - \sigma_1^i(u)S_1^{iu}\right) \qquad (ii) \qquad (1.11)$$

$$V_2^{idu} = \sigma_1^i(d)S_2^{idu} + e^r\left(V_1^{id} - \sigma_1^i(d)\,S_1^{id}\right) \qquad (iii)$$

$$V_2^{id^2} = \sigma_1^i(d)S_2^{id^2} + e^r\left(V_1^{id} - \sigma_1^i(d)\,S_1^{id}\right) \qquad (iv)$$

We now solve for the arbitrage price V_0 at time period $t = 0$ and hedging portfolio σ_0^i, $\sigma_1^i(u)$, $\sigma_1^i(d)$ as well as the values at time period 1, V_1^{iu} *and* V_1^{id}.

Subtracting 1.11(*ii*) from 1.11(*i*), we obtain

$$\sigma_1^i(u) = \left(V_2^{iu^2} - V_2^{iud}\right)/\left(S_1^{iu^2} - S_1^{iud}\right) \qquad (1.12)$$

To obtain the value of $V_2^{iu^2}$, we substitute (1.12) into 1.11(*i*) or 1.11(*ii*).

$$V_2^{iud} = \sigma_1^i(u)S_1^{iud} + e^r\left(V_1^{iu} - \sigma_1^i(u)S_1^{iu}\right) = \sigma_1^i(u)S_1^{iud} + e^rV_1^{iu} - e^r\sigma_1^i(u)S_1^{iu}$$

$$\therefore\ V_1^{iu}e^r = V_2^{iud} - \sigma_1^i(u)S_1^{iud} + e^r\sigma_1^i(u)S_1^{id} = V_2^{iud} - \sigma_1^i(u)\left(S_1^{iud} - e^rS_1^{id}\right)$$

$$= V_2^{iud} - \frac{V_2^{iu^2} - V_2^{iud}}{S_1^{iu^2} - S_1^{iud}}\left(S_1^{iud} - e^rS_1^{id}\right) = \frac{V_2^{iud}\left(S_1^{iu^2} - S_1^{iud}\right) - \left(V_2^{iu^2} - V_2^{iud}\right)\left(S_1^{iud} - e^rS_1^{id}\right)}{S_1^{iu^2} - S_1^{iud}}$$

$$= \frac{V_2^{iud}\left(S_1^{iu^2} - e^rS_1^{id}\right) + V_2^{iu^2}\left(e^rS_1^{id} - S_1^{iud}\right)}{S_1^{iu^2} - S_1^{iud}} = \frac{\left(u - e^r\right)V_2^{iud} + \left(e^r - d\right)V_2^{iu^2}}{u - d}$$

$$= \frac{\left(u - e^r\right)V_2^{iud}}{u - d} + \frac{\left(e^r - d\right)V_2^{iu^2}}{u - d} = p^*V_2^{iud} + q^*V_2^{iu^2}$$

$$\therefore\quad V_1^{iu} = e^{-r}\left[p^*V_2^{iud} + q^*V_2^{iu^2}\right] = e^{-r}E^*\left[V_2^i\right]. \qquad (1.13)$$

We deduce V_1^{id} from (1.13) as follows:

$$V_1^{id} = e^{-r}\left[p^*V_2^{idu} + q^*V_2^{id^2}\right] = e^{-r}E^*\left[V_2^i\right] \qquad (1.14)$$

We obtain V_0 by substituting (1.14) into (1.6),

$$V_1^i = e^{-r}\left[p^*V_1^{iu} + q^*V_1^{id}\right] = e^{-r}\left[p^*\left(e^{-r}\left\{p^*V_2^{iu^2} + q^*V_2^{iud}\right\}\right) + q^*\left(e^{-r}\left\{p^*V_2^{idu} + q^*V_2^{id^2}\right\}\right)\right]$$

$$= e^{-r}\left[e^{-r}\left(p^{*^2}V_2^{iu^2} + p^*q^*V_2^{iud}\right) + e^{-r}\left(p^*q^*V_2^{idu} + q^{*^2}V_2^{id^2}\right)\right]$$

$$= e^{-2r}\left[p^{*^2}V_2^{iu^2} + p^*q^*V_2^{iud} + p^*q^*V_2^{idu} + q^{*^2}V_2^{id^2}\right] = e^{-2r}\left[p^*V_2^{iu} + q^*V_2^{id}\right]^2 \qquad (1.15)$$

Therefore, at time t, we deduce that

$$V_0^i = e^{-rt}\left[p^*V_t^{iu} + q^*V_t^{id}\right]^t = e^{-rt}E^*\left[V_t^i\right] \qquad (1.16)$$

Hence,
$$V_0 = e^{-rt}E^*\left[\sum_{i=1}^{N}V_t^i\right] = e^{-rt}\sum_{i=1}^{N}E^*\left[V_t^i\right]. \qquad (1.17)$$

This is the binomial derivative pricing formula, (see [4], [8],and [9].

***Theorem* 1.1**

The discounted cash stock pricing process $\{\exp[-rt]S_t \mid F_t\}_{t=0}^n$ is a martingale under the risk-neutral probability measure $P*$.

Proof

$$E*\{\exp[-r(t+1)]S_{t+1} \mid F_t\} = \exp[-r(t+1)(p*u + q*d)S_t$$

$$= \left\{ \exp[-r(t+1)] \left[\frac{u(\exp[r]-d)}{u-d} + \frac{d(u-\exp[r]}{u-d} \right] S_t \right\}$$

$$= \left\{ \exp[-r(t+1)] \frac{u\exp[r]-ud+du-d\exp[r]}{u-d} S_t \right\} = \left\{ \exp[-r(t+1)] \frac{(u-d)\exp[r]}{u-d} S_t \right\}$$

$$= \{\exp[-r(t+1)]\exp[r]S_t\} = \{\exp[-rt]S_t\}.$$

The Theorem 1.2 below, shows that our portfolio is indeed self-financing.

***Theorem* 1.2**

The discounted self-financing portfolio value process $\{\exp[-rt]V_t \mid F_t\}_{t=0}^{n}$ is a martingale under the risk-neutral probability measure P^*.

Proof

$$\exp[-r(t+1)V_{t+1} = \exp[-rt]V_t + \delta_t[\exp[-r(t+1)S_{t+1} - \exp[-rt]S_t]$$

Hence, $\exp[-r(t+1)V_{t+1} = \exp[-rt]V_t + \delta_t[\exp[-r(t+1)S_{t+1} - \exp[-rt]S_t]$

$$E^*[\exp[-r(t+1)V_{t+1} \mid F_t] = E^*[\exp[-rt]V_t \mid F_t] + E^*[\exp[-r(t+1)]\delta_t S_{t+1} \mid F_t]$$
$$- E^*[\exp[-rt]\delta_t S_t \mid F_t]$$
$$= \exp[-rt]V_t + \delta_t E^*[\exp[-r(t+1)S_{t+1} \mid F_t] - \exp[-rt]\delta_t S_t] = \exp[-rt]V_t.$$

2.0 Valuation of cash flows generated from the investment

Let $(\Phi_t)_{t\geq 1}$ be the cash flows process generated from the investment. Then, at time $t = 0$

$$V_0 = E^*\left[\sum_{t=1}^{T}\sum_{i=1}^{N} e^{-rt} V_t^i\right] = E^*\left[\sum_{t=1}^{T} e^{-rt}\Phi_t\right], \qquad (2.1)$$

where $\Phi_t = \sum_{i=1}^{N} V_t^i$, t = 1,2,...,N.

To make this into a dynamic model, we introduce the value at time $t \geq 0$ as

$$V_t = E_t^*\left[\sum_{k=t+1}^{\infty} e^{-rk}\Phi_k\right] . \qquad (2.2)$$

where $E_t^*[\ \cdot\ ]$ is the expectation given information up to and including time t. By multiplying this expectation with e^{-rt} and splitting the expectation into two parts, we have

$$V_t\ e^{-rt} = E_t\left[\sum_{k=0}^{\infty}\Phi_k e^{-r(k-t)} e^{-rt}\right] - \sum_{k=0}^{t}\Phi_k e^{-rk} = E_t\left[\sum_{k=0}^{\infty}\Phi_k e^{-rk}\right] - \sum_{k=0}^{t}\Phi_k e^{-rk} . \qquad (2.3)$$

If $E\left|\sum_{k=1}^{\infty}\Phi_k e^{-rk}\right| < \infty$, then the first term on the left-hand side is a martingale and the value of the second one is known at time t. Equation (2.2) gives the following iterating equation

$$V_t = E_t\left[(\Phi_{t+1} + V_{t+1})e^{-r}\right] = e^{-r} E_t[\Phi_{t+1} + V_{t+1}] . \qquad (2.4)$$

Equation (2.4) implies that the value today is the expected discounted value of what we get tomorrow Φ_{t+1} plus the expected discounted value of having the right to the cash flow stream from the investment $\Phi_{t+2}, \Phi_{t+3}, \Lambda$ (which is the definition of V_{t+1}). When we continue iterations, we get for any $T > t$, addition of a martingale and an adapted process, i.e.

$$V_t = E_t\left[\sum_{k=t+1}^{T}\Phi_k e^{-r(k-t)}\right] + E_t\left[V_T e^{-rT}\right] \tag{2.5}$$

$$= E_t \lim_{T\to\infty}\left[\sum_{k=t+1}^{T}\Phi_k e^{-r(k-t)}\right] + E_t \lim_{T\to\infty}\left[V_T e^{-rT}\right] \cdot = E_t\left[\sum_{k=t+1}^{\infty}\Phi_k e^{-r(k-t)}\right]. \tag{2.6}$$

***Definition* 2.1**

Let $(\Omega, F, (F_t)_{t\in\aleph}, P)$ be a complete filtered probability space, and F_0 a trivial $\sigma-$algebra augmented with all null sets of F, then $F_\infty = F$ where $F_\infty = V_{t\geq 1} F_t$.

***Definition* 2.2**

A cash flows process generated from the investment $(\Phi_t)_{t\in\aleph}$ is a process that is adapted to the filtration $(F_t)_{t\in\aleph}$ and such that for each $t\in\aleph, |\Phi_t| < \infty$ almost surely.

***Definition* 2.3**

A cash flows process generated from the investment $(\Phi_t)_{t\in\aleph}$ that is, non-negative almost surely is referred to as a dividend (or payoff) process.

***Definition* 2.4**

A discount process is a process $g : \aleph\times\aleph\times\Omega \to \Re,$ which satisfies:

(i) $0 < g(s,t) < \infty$ almost surely for every $s,t\in\aleph.$

(ii) $g(s,t)$ *is* $F_{\max(s,t)}$ $-$measurable for every $s,t\in\aleph.$

(iii) $g(s,t) = g(s,u)g(u,t)$ almost surely for every $s,u,t\in\aleph.$

***Remark* 2.1**

In this paper, we consider $g(s,t)$ as a normal discount process, that is $r\geq 0.$ Hence, we express $g(s,t) = \exp[-r(t-s)], r\in\Re, t > s, t,s\in\aleph.$

The Lemma 2.1 below, gives us a sufficient condition for the value process to be finite almost surely.

***Lemma* 2.1**

Let $(\Phi_t)_{t\in\aleph}$ be a cash flows process generated from the investment and $E\left[|\sum_{k=1}^{\infty}\Phi_k \exp[-rk]|\right] < \infty$ almost surely. Then $|V_t| < \infty$ almost surely, for all $t\in\aleph.$

Proof

Given that $E\left[|\sum_{k=1}^{\infty}\Phi_k \exp[-rk]|\right] < \infty$ then

$$|V_t| = |E\left[\sum_{k=t+1}^{\infty}\Phi_t \exp[-(k-t)] | F_t\right]| \leq E\left[|\sum_{k=1}^{\infty}\Phi_k \exp[-rk]\exp[-rt] - \sum_{k=1}^{t}\Phi_k \exp[-rk]\exp[-rt]|| F_t\right]$$

$$\leq \exp[rt]E\left[|\sum_{k=1}^{\infty}\Phi_k \exp[-rk] - \sum_{k=1}^{t}\Phi_k \exp[-rk]|| F_t\right]$$

$$\leq \exp[rt](E\left[|\sum_{k=1}^{\infty}\Phi_k \exp[-rk]|| F_t\right] + |\sum_{k=1}^{t}\Phi_k \exp[-rk]|$$

$< \infty$ almost surely.

We then obtain the corollary 2.1 below for a dividend process.

***Corollary* 2.1**

Let $(\Phi_t)_{t\in\aleph}$ be a dividend process generated from the investment such that V_0 is finite, then V_t is finite almost surely, for every $t\in\aleph$.

Proof

Since $\Phi_t \geq 0$ almost surely for every $t\in\aleph$ and F_0 is the trivial $\sigma-$algebra augmented with the null sets.

$$E\left[\left|\sum_{k=1}^{\infty}\Phi_k\exp[-rk]\right|\right]=E\left[\sum_{k=1}^{\infty}\Phi_k\exp[-rk]\right]=V_0<\infty.$$

This implies that $$E\left[\left|\sum_{k=1}^{\infty}\Phi_k\exp[-r(k-t)]\right|\right]=E\left[\sum_{k=1}^{\infty}\Phi_k\exp[-r(k-t)]\right]=V_t<\infty$$

almost surely.

***Remarks* 2.2**

(i) Since $\exp[-rt]\ is\ F_t-measurable\ \ for\ all\ t\in\aleph$, we have

$$V_t=E\left[\sum_{k=t+1}^{\infty}\Phi_k\exp[-r(k-t)\mid F_t\right]=\exp[rt]E\left[\sum_{k=t+1}^{\infty}\Phi_k\exp[-rk]\mid F_t\right] \quad (2.7)$$

Multiplying bothsides by $\exp[-rt]$, we have

$$V_t\exp[-rt]=E\left[\sum_{k=0}^{\infty}\Phi_k\exp[-rk]\mid F_t\right]-\sum_{k=0}^{t}\Phi_k\exp[-rk].$$

(ii) $V_t\exp[-rt]$ is the value at time *t* discounted back to time 0.

(iii) If X is a random variable with $E\mid X\mid<\infty$, then $E[X\mid F_t], t=1,2,...$ is a uniformly integrable martingale (UIM), (see [6] and [8[). Thus, if $E\left[\left|\sum_{k=0}^{\infty}\Phi_k\exp[-rk]\right|\right]<\infty$, then $E\left|\sum_{k=0}^{\infty}\Phi_k\exp[-rk]\mid F_t\right|$ is a UIM. This leads to the Theorem below.

***Theorem* 2.1**

Let $(\Phi_t)_{t\in\aleph}$ be a cash flows process generated from the investment and exp[-*r*] a discount process. If $E\left[\left|\sum_{k=0}^{\infty}\Phi_k\exp[-rk]\right|\right]<\infty,\ then$ the discounted value process $V_t\exp[-rt]$ can be expressed as follows:

$$V_t\exp[-rt]=M_t-A_t,\ t\in\aleph,$$

where *M* is *UIM* and *A* is an adapted process (AP).

Proof

Given that $\left[\left|\sum_{k=0}^{\infty}\Phi_k\exp[-rk]\right|\right]<\infty$, almost surely and $E\left[\left|\sum_{k=0}^{\infty}\Phi_k\exp[-rk]\right|\right]<\infty.$

Let

$$M_t = E\left[\sum_{k=1}^{\infty}\Phi_k \exp[-rk] \mid F_t\right], t \in \aleph$$

$$A_t = \left[\sum_{k=1}^{t}\Phi_k \exp[-rk]\right], t \in \aleph\,.$$

Then, $V_t \exp[-rt] = M_t - A_t$, $t \in \aleph$.

Since $E\left[\,\left|\sum_{k=0}^{\infty}\Phi_k \exp[-rk]\right|\,\right] < \infty$, then M is a *UIM* and A is AP. Then M converges to

$E\left[\sum_{k=0}^{\infty}\Phi_k \exp[-rk] \mid F_\infty\right] = \sum_{k=0}^{\infty}\Phi_k \exp[-rk]$ almost surely as $t \to \infty$, (see [13] and [15]).

Now, $\lim_{t\to\infty} V_t \exp[-rt] = \lim_{t\to\infty} M_t - \lim_{t\to\infty} A_t = \lim_{t\to\infty} E\left[\sum_{k=1}^{\infty}\Phi_k \exp[-rk] \mid F_t\right] - \lim_{t\to\infty}\sum_{k=1}^{t}\Phi_k \exp[-rk]$

$= E\left[\sum_{k=1}^{\infty}\Phi_k \exp[-rk] \mid F_\infty\right] - \sum_{k=1}^{\infty}\Phi_k \exp[-rk] = M_\infty - A_\infty = 0.$ Since $M_\infty = A_\infty = \sum_{k=0}^{\infty}\Phi_k \exp[-rk]$ is finite almost surely.

Theorem 2.2 below, characterizes the relation between $\Phi, \exp[-r], \textit{and } V$ in terms of their values and differences. This gives three equivalent forms of defining the value process generated from the investment.

***Theorem* 2.2**

Let $(\Phi_t)_{t\in\aleph}$ be a cash flows process generated from the investment and exp[-*r*] a discount process. If $E\left[\,\left|\sum_{k=1}^{\infty}\Phi_k \exp[-rk]\right|\,\right] \le E\left[\sum_{k=1}^{\infty}|\Phi|\exp[-rk]\right] < \infty$, *then* the following three statements are equiavlent.

(i) For every $t \in N$,

$$V_t = E\left[\sum_{k=t+1}^{\infty}\Phi_k \exp[-r(k-t) \mid F_t\right] = \exp[rt]E\left[\sum_{k=t+1}^{\infty}\Phi_k \exp[-rk] \mid F_t\right]$$

(ii) For every $t \in N$,

$$M_t = V_t \exp[-rt] + \sum_{k=1}^{t}\Phi_k \exp[-rk]$$

is a *UIM*.

(iii) For every $t \in N$,

(a) $V_t = E\left[\exp[-r]\left(\Phi_{t+1} + V_{t+1}\right) \mid F_t\right].$

(b) $\lim_{T\to\infty} E\left[\exp[-r(T-t)V_T \mid F_t\right] = 0.$

Proof

Given $E\left[\,\left|\sum_{k=1}^{\infty}\Phi_k \exp[-rk]\right|\,\right] \le E\left[\sum_{k=1}^{\infty}|\Phi|\exp[-rk]\right] < \infty$. So $\left|\sum_{k=1}^{\infty}\Phi_k \exp[-rk]\right| < \infty$ almost surely. We are to show that (i) $\Leftrightarrow$ (ii) and (i) $\Leftrightarrow$ (iii).

(i) $\Leftrightarrow$ (ii):

The 'only if' part: This follows from the theorem above. For the 'if' part: From (ii) above, we have the following

$$-\exp[-r(k+1)]\Phi_{k+1} = \exp[-r(k+1)]V_{k+1} - \exp[-rk]V_k - M_{k+1} + M_k$$

and sum from t to T - 1:

$$-\sum_{k=t+1}^{T} \exp[-rk]\Phi_k = \exp[-rT]V_T - \exp[-rk]V_k - M_T + M_t .$$

Let $T \to \infty$ the term $V_T \exp[-rT] \to 0$ *almost surely* and $M_T \to M_\infty$ *almost surely* from the convergence result of UIMs (see [9]).

Hence, we have $V_t \exp[-rt] = \sum_{k=t+1}^{\infty} \exp[-rk]\Phi_k - M_\infty + M_t$ *almost surely*. By convergence of UIMs, we have that $E[M_\infty \mid F_t] = M_t$ *almost surely*. Taking conditional expectations with respect to F_t, we obtain

$$V_t = E\left[\sum_{k=t+1}^{\infty}\Phi_k \exp[-r(k-t) \mid F_t\right] = \exp[rt]E\left[\sum_{k=t+1}^{\infty}\Phi_k \exp[-rk] \mid F_t\right].$$

(i) $\Leftrightarrow$ (iii):

'only if' part: for $t \in N$, we obtain the following

$$V_t = E\left[\exp[-r]\Phi_{t+1} + \sum_{k=t+2}^{\infty}\Phi_k \exp[-r(k-t)] \mid F_t\right] = E\left[\exp[-r]\Phi_{t+1} + \exp[-r]\sum_{k=t+2}^{\infty}\Phi_k \exp[-r(k-t-1)] \mid F_t\right]$$

$$= E[\exp[-r](\Phi_{t+1} + V_{t+1}) \mid F_t].$$

Now let $T \geq t$. From

$$V_T = E\left[\sum_{k=T+1}^{\infty}\Phi_k \exp[-r(k-T) \mid F_T\right] = \exp[kT]E\left[\sum_{k=T+1}^{\infty}\Phi_k \exp[-rk] \mid F_T\right]$$

We obtain,

$$E[\exp[-r(T-t)\}V_T \mid F_t] = \exp[rt]E\left[\sum_{k=T+1}^{\infty}\Phi_k \exp[-rk] \mid F_t\right] = E\left[\sum_{k=T+1}^{\infty}\Phi_k \exp[-r(k-t) \mid F_t\right]$$

Since $\left|\sum_{k=T-1}^{\infty}\Phi_k \exp[-rk]\right| \leq \sum_{k=1}^{\infty}|\Phi_t|\exp[-rk]$ and $A \in F_t$, we obtain the following

$$\lim_{T\to\infty} E[\exp[-r(T-t)]V_T 1_A] = E[\lim_{T\to\infty} \exp[-r(T-t)]V_T 1_A] = 0.$$

Now, the 'if' part: we establish this by iterating (iii)(a) above. By doing that, we obtain the following

$$V_t = E\left[\exp[-r(T-t)]V_T + \sum_{k=t+1}^{T}\Phi_k \exp[-r(k-t)] \mid F_t\right]$$

$$= E[\exp[-r(T-t)]V_T \mid F_t] + \exp[rt]E\left[\sum_{k=t+1}^{T}\Phi_k \exp[-rk] \mid F_t\right]$$

As $T \to \infty$, the last term tends to 0 almost surely.

From (iiib), given that $\left[\left|\sum_{k=t+1}^{T}\Phi_k \exp[-rk]\right|\right] \le \left[\sum_{k=1}^{\infty}|\Phi_k|\exp[-rk]\right]$, and using the Dominated Convergence Theorem, we obtain

$$V_t = E\left[\sum_{k=t+1}^{\infty}\Phi_k \exp[-r(k-t) \mid F_t\right] = \exp[rt]E\left[\sum_{k=t+1}^{\infty}\Phi_k \exp[-rk] \mid F_t\right].$$

2.1 Numerical result

Table 2.1 shows the values of the parameters and payoffs as well as the percentage increase from the investor's investment in N = 10 numbers of investment firms. The amounts represent the values of the investment per stock in 5 years.

Table 2.1: The values of the parameters, stock and payoff in 5 years

i	S_0^i *(in Naira)*	V_0^i *(in Naira)*	*r(%)*	*u(%)*	*d(%)*	*p**	*q**	V_t^i *(in Naira)*	*Payoff amount (in Naira)*	*% increase*
1	50	080	15	18.0	07.0	0.93	0.07	169.36	089.36	111.70
2	47	075	15	20.0	05.0	0.85	0.15	158.78	083.78	111.71
3	62	095	15	20.0	11.0	0.88	0.12	201.12	106.12	111.71
4	69	102	15	24.0	10.0	0.77	0.23	215.93	113.93	111.70
5	45	063	15	21.0	09.0	0.84	0.16	133.37	070.37	111.70
6	48	068	15	20.5	10.2	0.86	0.14	143.96	075.95	111.69
7	70	105	15	18.9	06.5	0.89	0.11	222.29	117.29	111.70
8	41	059	15	27.0	05.0	0.66	0.34	124.90	065.90	111.69
9	38	055	15	25.0	04.0	0.70	0.30	116.44	061.44	111.70
10	43	062	15	29.0	06.0	0.63	0.37	131.25	069.25	111.69

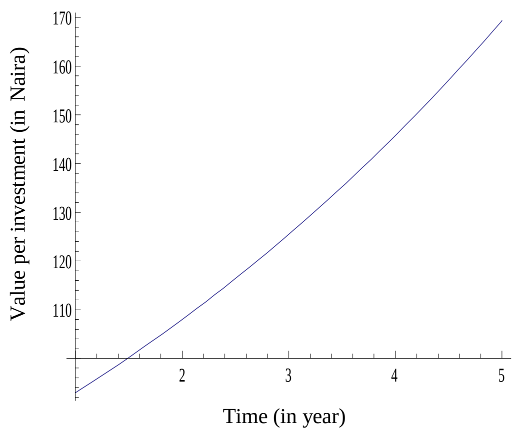

Figure 2.1: Value of investment for Firm 1

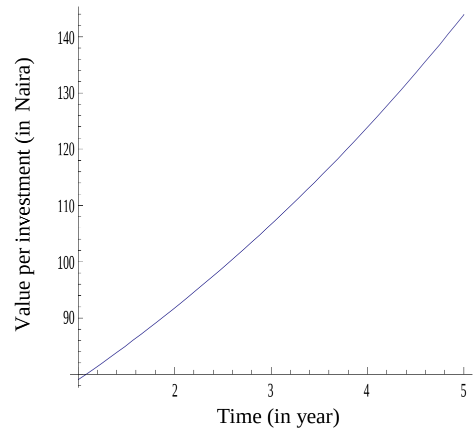

Figure 2.6: Value of investment for Firm 6

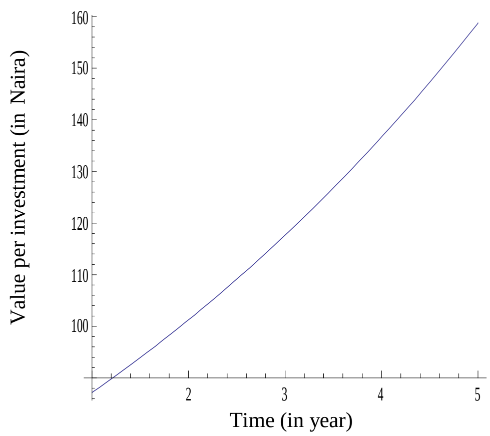

Figure 2.2: Value of investment for Firm 2

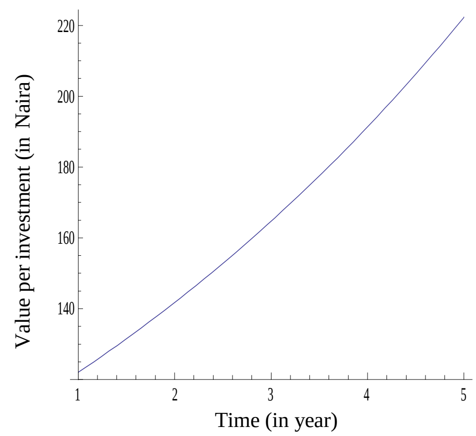

Figure 2.7: Value of investment for Firm 7

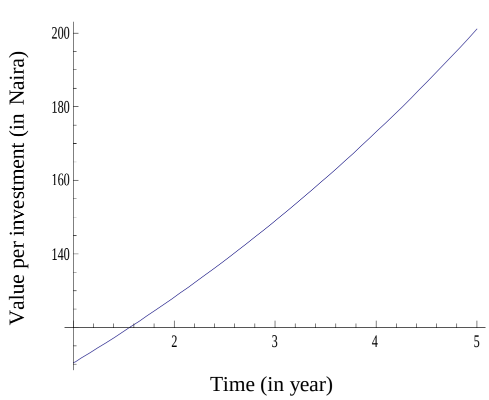

Figure 2.3: Value of investment for Firm 3

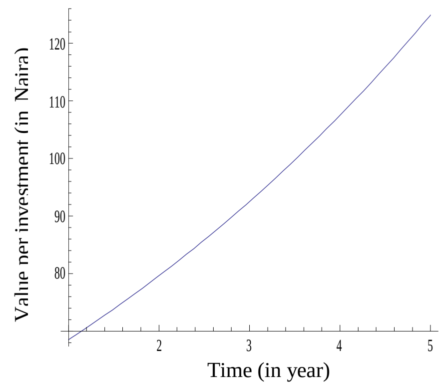

Figure 2.8: Value of investment for Firm 8

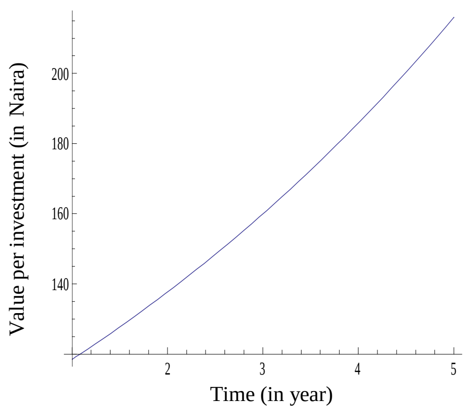

Figure 2.4: Value of investment for Firm 4

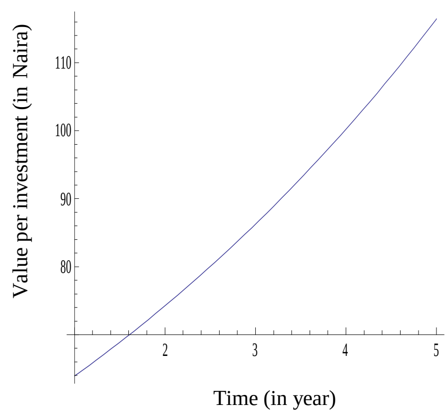

Figure 2.9: Value of investment for Firm 9

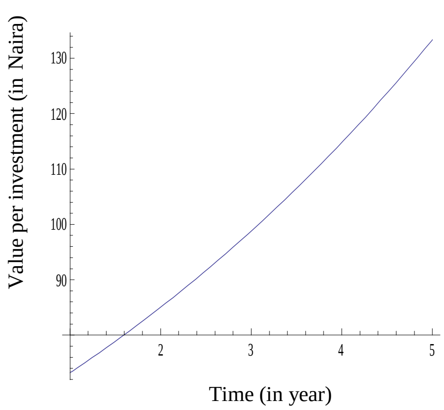

Figure 2.5: Value of investment for Firm 5

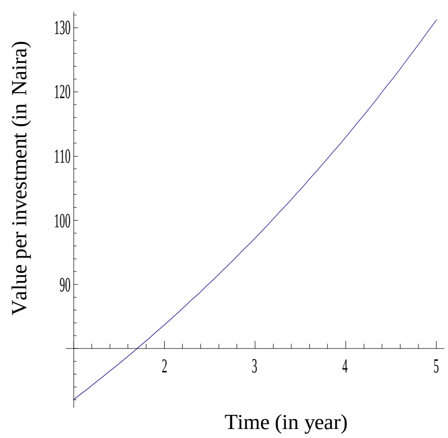

Figure 2.10: Value of investment for Firm 10

3.0 Discussion

The first column in Table 2.1, contained information arising from the past records of firm 1 and the value of the cash flows as well as the payoff from firm 1 in 5 years. The value S_0^1 is the value of a stock at the initial time of firm 1. V_0^1 is the value assigned to both stock and bond at the initial time. It implies that 50 naira goes to a stock and the remaining 30 naira goes to a bond in firm 1. The interest rate r is the proportion received for investing into the bond market.

The value u is the proportion of the investment received if the stock goes up with risk-neutral probability p^* . The value d is the proportion of the investment received if the stock

goes down with risk-neutral probability q^*. The values u, d, p^*, and q^*, in Table 1.1 are obtained from the past records of the firms. As time evolved, the value V_t per investment in 5 years is 169.36 naira. With this, we have a payoff of 89.36 naira and percentage increase of 111.70. Column 2 to 10 represent the information and payoff amounts for firm 2 to 10. The similar analysis also go to column 2 to 10.

Figure 2.1 represents the value of the investor's investment in firm 1 in 5 years, figure 2 represents the value of the investor's investment in firm 2 in 5 years and so on. In the Table 2.1, we observed that all the firms are having almost the same percentage increase in their investment. This shows that the firms operate within the same environment and with competent management. Firm 1 which has initial value of a portfolio of 80 naira, after 5 years the value of the portfolio rises to 169.36 naira, that is about 111.70% increment. Firm 2 with initial value of 75 naira has percentage increase after 5 years of 111.71. Firm 6, 8 and 10 have percentage increase of 111.69 each while firm 4, 5, 7 and 9 have almost the same percentage but different initial values portfolios. Also, firms 2 and 3 have almost the same percentage increase but different initial values portfolios. The aggregate of the investor's portfolio is obtained by adding up the cash flows generated from all the firms.

4.0 Conclusion

Investors are advised to maintain their investment with the firms since they are doing relatively well. Also, the cash flows from the firms in 5 years are given in table 2.1. The difference between the value at time $t = 5$ and the initial value of the portfolio gives the payoff from the investment. The sum of the payoffs from all the firms gives the gross total value of the investor's portfolio in 5 years of investment. We discovered that with different values per investment at the initial time from the investment firms over time yield almost the same percentage change after 5 years (not the same value) with the same interest rate, as we can see in Table 2.1 and Figures 2.1 – 2.10. This shows that the firms operate within the same environment.

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