Journal of the Nigerian Association of Mathematical Physics
Volume **15 (November, 2009), pp 247 - 252**

Gravitational time dilation and length contraction in fields exterior to static oblate spheroidal mass distributions

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Abstract

Here, we use our new metric tensor exterior to a massiv3e oblate spheroid to study the gravitational phenomena of time dilation and length contraction. It turns out most profoundly that, the above phenomena hold good in the gravitational field exterior to an oblate spheroid. We then use the oblate spheroidal Earth to exemplify our findings in approximate gravitational fields.

Keywords
Gravitation, world line element, time dilation, length contraction

1.0 Introduction

The Sun and planets in the solar system are known to be oblate spheroidal in geometry [1-4]. The oblate spheroidal geometries of these bodies have effects on their gravitational fields and hence the motions of test particles and photons in these fields. Here, we study the gravitational phenomena of time dilation and length contraction exterior to static homogeneous oblate spheroids.

The covariant metric tensor in the gravitational field of a static homogeneous oblate spheroid in oblate spheroidal coordinates (η, ξ, ϕ) has been recently obtained [2-4] as;

$$g_{00} = \left(1 + \frac{2}{c^2} f(\eta, \xi)\right) \tag{1.1}$$

$$g_{11} = -\frac{a^2}{1+\xi^2-\eta^2}\left[\eta^2\left(1+\frac{2}{c^2} f(\eta,\xi)\right)^{-1} + \frac{\xi^2(1+\xi^2)}{(1-\eta^2)}\right] \tag{1.2}$$

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$$g_{12} \equiv g_{21} = -\frac{a^2\eta\xi}{1+\xi^2-\eta^2}\left[1-\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\right] \qquad \text{s}(1.3)$$

$$g_{22} = -\frac{a^2}{1+\xi^2-\eta^2}\left[\xi^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}+\frac{\eta^2(1-\eta^2)}{(1+\xi^2)}\right] \qquad (1.4)$$

$$g_{33} = -a^2(1+\xi^2)(1-\eta^2) \qquad (1.5)$$

$$g_{\mu\nu} = 0; otherwise \qquad (1.6)$$

where $f(\eta,\xi)$ is an unknown function determined by the mass distribution and a is a constant parameter. It is also well established that the unknown function $f(\eta,\xi)$ is equal to Newton's gravitational scalar potential exterior to an oblate spheroid in approximate gravitational fields. We will use this metric tensor and our computed values [4] for Newton's gravitational scalar potential exterior to the Earth to study gravitational time dilation and length contraction in this gravitational field.

2.0 Gravitational time dilation

The world line element for empty space in oblate spheroidal coordinates [5] is defined as

$$c^2d\tau^2 = c^2dt^2 - \frac{a^2\left(\eta^2+\xi^2\right)}{\left(1-\eta^2\right)}d\eta^2 - \frac{a^2\left(\eta^2+\xi^2\right)}{\left(1+\xi^2\right)}d\xi^2 - a^2\left(1-\eta^2\right)\left(1+\xi^2\right)d\phi^2 \qquad (2.1)$$

where τ is the proper time. Now, consider a clock at rest at a fixed point in empty space; then $d\eta = d\xi = d\phi \equiv 0$ and hence the line element, equation (2.1) reduces to

$$d\tau = dt \qquad (2.2)$$

From equation (2.2) we conveniently conclude that the clock in empty space keeps the same time as proper time.

The world line element in the gravitational field exterior to a massive oblate spheroid [6] is given as

$$c^2d\tau^2 = c^2g_{00}dt^2 - g_{11}d\eta^2 - 2g_{12}d\eta d\xi - g_{22}d\xi^2 - g_{33}d\phi^2 \qquad (2.3)$$

Now, let us consider a clock at rest at a fixed point (η,ξ,ϕ) in the gravitational field exterior to an oblate spheroidal mass. Equivalently, $d\eta = d\xi = d\phi \equiv 0$ and the world line element for this gravitational field; equation (2.3) reduces to

$$d\tau^2 = g_{00}dt^2 \qquad (2.4)$$

or more explicitly
$$d\tau = \left[1+\frac{2}{c^2}f\left(\eta,\xi\right)\right]^{\frac{1}{2}}dt \qquad (2.5)$$

or equivalently
$$dt = \left[1+\frac{2}{c^2}f\left(\eta,\xi\right)\right]^{-\frac{1}{2}}d\tau \qquad (2.6)$$

Expanding the right hand side of equation (2.6) gives

$$dt = \left[1-\frac{1}{c^2}f\left(\eta,\xi\right)+\ldots\right]d\tau \qquad (2.7)$$

From equation (2.7), we can conveniently deduce that $dt > d\tau$ (time dilation). Thus coordinate time of a clock in this gravitational field is dilated relative to proper time. In other words, a clock in this gravitational field keeps dilated time relative to the proper time unlike the corresponding expression (2.2) for empty space.

As an illustration, consider two events at fixed points exterior to the static homogenous oblate spheroidal Earth along the equator; separated in this gravitational field by coordinate time dt and proper time $d\tau$.

The gravitational scalar potential along the equator exterior to the Earth has been obtained [4] and results are shown in table 2.1.

Table 2.1: Gravitational scalar potential at points along the equator exterior to the earth

ξ	**Radial distance along the equator from center (*km*)**	**$f(\eta,\xi)$ along the equator ($\times 10^7 Nmkg^{-1}$)**
ξ_0(12.91)	6,378	-6.2079113
$2\xi_0$(24.02)	12,723	-3.0985880
$2\xi_0$(36.03)	19,075	-2.0650627
$4\xi_0$(48.04)	25,430	-1.5486230
$5\xi_0$(60.05)	31,784	-1.2388340
$6\xi_0$(72.06)	38,140	-1.0323325
$7\xi_0$(84.07)	44,495	-0.8848413
$8\xi_0$(96.08)	50,851	-0.7742277
$9\xi_0$(108.09)	57,207	-0.6881971
$10\xi_0$(120.1)	63,562	-0.6193740

Substituting the values for the gravitational scalar potential in table 1 into the equation for gravitational time dilation (2.7); (approximate fields) yields the results presented in table 2.2 and figure 2.1.

Table 2.2: Coordinate Time along the equator in the field exterior to the earth as a factor of proper time

Fixed point along the equator	**Radial distance along the equator (*km*)**	***dt* as factor of *dτ***
ξ_0	6,378	1.306170
$2\xi_0$	12,723	1.122655
$3\xi_0$	19,075	1.076871
$4\xi_0$	25,430	1.055996
$5\xi_0$	31,784	1.044042
$6\xi_0$	38,140	1.036296
$7\xi_0$	44,495	1.030867
$8\xi_0$	50,851	1.026852
$9\xi_0$	57,207	1.023761
$10\xi_0$	63,562	1.021308

Thus from equation (2.7), Table 2.2 and Figure 2.1, we conclude that the clock runs more slowly at a smaller distance from the massive oblate spheroidal body. In other words, clocks will run slower at lower gravitational potentials (deeper within a gravity well). This was first confirmed experimentally in the laboratory by the Hafele-Keating experiment. Today, there are numerous direct measurements of gravitational time dilation using atomic clocks [7], while ongoing validation is provided as a side-effect of the operation of Global Positioning System (GPS) [8].

3.0 Gravitational length contraction

The world line element for empty space in oblate spheroidal coordinates is given as equation (2.1). The space part of this metric is given as

$$ds^2 = \frac{a^2\left(\eta^2+\xi^2\right)}{\left(1-\eta^2\right)}d\eta^2 + \frac{a^2\left(\eta^2+\xi^2\right)}{\left(1+\xi^2\right)}d\xi^2 + a^2\left(1-\eta^2\right)\left(1+\xi^2\right)d\phi^2 \qquad (3.1)$$

Now, consider two points for which the angular coordinates; η and ϕ are constant (along the equatorial line), then, the space part or distance element ds is given as

$$ds = \frac{a\xi}{\left(1+\xi^2\right)^{\frac{1}{2}}}d\xi \qquad (3.2)$$

We now recall that the world line element in the gravitational field exterior to an oblate spheroidal mass is given as equation (2.3). The space part of this line element is given as

$$ds^2 = g_{11}d\eta^2 + 2g_{12}d\eta d\xi + g_{22}d\xi^2 + g_{33}d\phi^2 \qquad (3.3)$$

Also, recall [4] that $-1 \le \eta \le 1$, $0 \le \xi \le \infty$ and $0 \le \phi \le 2\pi$ and the surface of an oblate spheroid is defined by the equation $\xi = \xi_0$ for some constant ξ_0. Suppose we have two points for which the angular coordinates η and ϕ are same such that $d\eta = d\phi \equiv 0$. Then, the space part of the metric, equation (3.3) takes the form:

$$ds^2 = g_{22}d\xi^2 \qquad (3.4)$$

or

$$ds = \left(g_{22}\right)^{\frac{1}{2}}d\xi \qquad (3.5)$$

Equation (3.5) can be written more explicitly as

$$ds = \left(\frac{a^2}{1+\xi^2-\eta^2}\left[\xi^2\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1} + \frac{\eta^2(1-\eta^2)}{(1+\xi^2)}\right]\right)^{\frac{1}{2}}d\xi \qquad (3.6)$$

Along the equatorial line, $\eta = 0$ and equation (3.6) becomes

$$ds = a\xi\left(\frac{1}{1+\xi^2}\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-1}\right)^{\frac{1}{2}}d\xi \qquad (3.7)$$

or
$$ds = a\xi\left(1+\xi^2\right)^{-\frac{1}{2}}\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-\frac{1}{2}}d\xi \qquad (3.8)$$

Thus, in the neighborhood of a massive oblate spheroidal body along the equator, two points for which η and ϕ are same now has a separation which is different from the corresponding separation in empty space.

By binomially expanding the second bracket on the right hand side of equation (3.8), it can be shown trivially that $ds > d\xi$. In other words, the coordinate distance separating these two points is contracted in this gravitational field. Thus, we can write

$$d\xi = \left(a\xi\right)^{-1}\left(1+\xi^2\right)^{\frac{1}{2}}\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{\frac{1}{2}}ds \qquad (3.9)$$

as our expression for gravitational length contraction along the equator in this gravitational field.

Now, as an illustration of this gravitational phenomenon, let us consider a long stick lying radially along the equator in the approximate gravitational field of a static homogenous oblate spheroidal mass such as the Earth. Let the ξ-coordinates of the ends be ξ_1 and ξ_2, where $\xi_2 > \xi_1$; then we can find the expression for its proper length as follows;

From equation (3.9) we deduce that the proper length s is given as

$$s = a\int_{\xi_1}^{\xi_2}\xi\left(1+\xi^2\right)^{-\frac{1}{2}}\left(1+\frac{2}{c^2}f(\eta,\xi)\right)^{-\frac{1}{2}}d\xi \qquad (3.10)$$

or
$$s = a\int_{\xi_1}^{\xi_2}\xi\left(1+\xi^2\right)^{-\frac{1}{2}}\left(1-\frac{1}{c^2}f(\eta,\xi)+...\right)d\xi \qquad (3.11)$$

Substituting the approximate expression of $f(\eta,\xi)$ along the equator [4] ;

$$f(\eta,\xi) \approx \frac{B_0}{3\xi^3}\left(1+3\xi^2\right)i + \frac{B_2}{30\xi^3}\left(7+15\xi^2\right)i \qquad (3.12)$$

into (3.11) and simplifying yields

$$s = a\int_{\xi_1}^{\xi_2}\xi\left(1+\xi^2\right)^{-\frac{1}{2}}\left(1-\frac{i}{c^2}\left[\left(\frac{2B_0+B_2}{2}\right)\frac{1}{\xi}+\left(\frac{10B_0+7B_2}{30}\right)\frac{1}{\xi^3}+...\right]\right)d\xi \qquad (3.13)$$

Considering terms to the order of c^{-2} and integrating equation (3.13) gives the expression for the proper length of the stick as

$$s = a\left\{\left(1+\xi^2\right)^{\frac{1}{2}} - i\left(\frac{2B_0+B_2}{2c^2}\right)\ln\left|\xi+\sqrt{1+\xi^2}\right| + i\left(\frac{10B_0+7B_2}{30c^2}\right)\frac{\left(1+\xi^2\right)^{\frac{1}{2}}}{\xi}\right\}_{\xi_1}^{\xi_2} \qquad (3.14)$$

or equivalently

$$s=a\left\{\left(1+\xi_2^2\right)^{\frac{1}{2}}-\left(1+\xi_1^2\right)^{\frac{1}{2}}-i\left(\frac{2B_0+B_2}{2c^2}\right)\ln\left|\frac{\xi_2+\sqrt{1+\xi_2^2}}{\xi_1+\sqrt{1+\xi_1^2}}\right|+i\left(\frac{10B_0+7B_2}{30c^2}\right)\left[\frac{\left(1+\xi_2^2\right)^{\frac{1}{2}}}{\xi_2}-\frac{\left(1+\xi_1^2\right)^{\frac{1}{2}}}{\xi_1}\right]\right\} \quad (3.15)$$

Now, substituting the values of the constants B_0 and B_2 for the Earth from [4] into equation

(3.15) gives the following expression for the proper length of a long stick lying radially along the equator in the gravitational field of the homogenous oblate spheroidal Earth as;

$$s=a\left\{\left(1+\xi_2^2\right)^{\frac{1}{2}}-\left(1+\xi_1^2\right)^{\frac{1}{2}}+8.265013\times10^{-9}\ln\left|\frac{\xi_2+\sqrt{1+\xi_2^2}}{\xi_1+\sqrt{1+\xi_1^2}}\right|-2.755130\times10^{-9}\left[\frac{\left(1+\xi_2^2\right)^{\frac{1}{2}}}{\xi_2}-\frac{\left(1+\xi_1^2\right)^{\frac{1}{2}}}{\xi_1}\right]\right\} \quad (3.16)$$

Setting $\xi_2 = 2\xi_0$ and $\xi_1 = \xi_0$ and substituting the value of ξ_0 and a for the Earth [4] gives $s = 6414181.705m$. To compute the corresponding length between $2\xi_0$ and ξ_0 along the equator in the gravitational field, we use equation (3.16) as follows:

The distance of the equatorial point $2\xi_0$ from the centre of the Earth [4] is given by

$$2x_0 = a\left(1+4\xi_o^{\ 2}\right)^{\frac{1}{2}} \quad (3.17)$$

Substituting the value of ξ_0 and a for the Earth [4] gives $2x_0 = 12723020.11205m$.

Also, the distance of the equatorial point ξ_0 from the centre of the Earth (which is the equatorial radius) is given by

$$x_0 = a\left(1+\xi_o^{\ 2}\right)^{\frac{1}{2}} \quad (3.18)$$

Substituting the value of ξ_0 and a for the Earth [4] gives $x_0 = 6377998.927m$.Thus, the separation of this points is $2x_0 - x_0 = 6345021.185m$. From this analysis, we conclude that the coordinate length of the stick ($6345021.185m$) is smaller than its proper length ($6414181.705m$) and thus it is contracted in length.

4.0 Conclusion

The expressions for gravitational time dilation and length contraction were obtained respectively as equations (2.6) and (3.9). The oblate spheroidal Earth gives satisfactory approximate values on the equatorial plane in its exterior field.

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