Journal of the Nigerian Association of Mathematical Physics
***Volume* 15 (November, 2009), pp 237 - 246**

The structure of white dwarf stars

Garba Babaji
Department of Physics
Bayero University
P.M.B.* 3011, *Kano- Nigeria

Abstract

A FORTRAN code to compute the structure of white dwarf Stars has been written. It is assumed that a good model for the matter in white dwarf stars is the free Fermi gas of electrons at zero temperature, treated with relativistic kinematics. The code written essentially solves numerically the two coupled first-order differential equations that determined the structure of the star for the given equation of state. The variation of mass density with distance from the center of the star is found to be directly proportional to the assigned density at the center of the dwarf and the value of the parameter η which characterizes the chemical composition of the dwarf, but inversely proportional to the distance from the center of the star. In general, the density decreases with increase in the distance. For a given central density, the radius of the hydrogen white dwarf is greater than that of the helium, carbon, or oxygen which are equal and greater than that of iron. Thus the radius increases with the parameter η. The so called Chandrasekhar mass limit has been found to be 1.144×10^{34} gm for hydrogen white dwarf, 2.861×10^{33} gm for helium, carbon, and oxygen white dwarf, and 2.464×10^{33} gm for iron white dwarf.

***Keywords*:**
White Dwarf, Equation of State for Degenerate Matter

1.0 Introduction

A white dwarf is a small star made up mainly of electron-degenerate matter. White dwarfs are very dense. It is estimated [10] that the average mass density of matter in a white dwarf is approximately 10^3 kg/cm^3. Today there are thousands of known white dwarfs, in fact they account for roughly 6% of all known stars in the solar neighborhood [9]. White dwarfs have a faint luminosity arising from the emission of stored heat.

At the expiration of the hydrogen-fusing lifetime of a main sequence star of low or medium mass, it will expand to a red giant which fuses helium to carbon and oxygen by the triple-alpha process. If a red giant has insufficient mass to generate the core temperatures required to fuse carbon, an inert mass of carbon and oxygen will build up at its center. After shedding its outer layers to form a planetary nebula, it will leave behind this core, which forms the remnant of the white dwarf [15]. Usually, therefore, white dwarfs are composed of carbon and oxygen. It is also possible that core temperatures suffice to fuse carbon but not neon, in which case an oxygen-neon-magnesium white dwarf may be formed [18]. Also, some helium

e-mail address: garbababaji@yahoo.com

Telephone: +234-7025120635

white dwarfs appear to have been formed by mass loss in binary systems.

The high mass densities in white dwarf stars are possible because white dwarf material is not composed of atoms bound by chemical bonds, but rather consists of a plasma of unbounded nuclei and electrons. There is therefore no obstacle to placing nuclei closer to each other than electron orbital. Compression of a white dwarf by gravitational force will increase the number of electrons in a given volume. Applying either the Pauli principle or the uncertainty principle, we can see that this will increase the kinetic energy of the electrons, causing pressure [1]. This electron degenerate pressure is what supports a white dwarf against gravitational collapse. It depends only on density and not temperature.

A white dwarf star has no internal source of energy since its material no longer under goes fusion reactions, thus it is not supported against gravitational collapse by the heat generated by fusion. However, it is supported against gravitational collapse only by electron degeneracy pressure, which enables it to be extremely dense. Degenerate pressure yields a maximum mass for non-rotating white dwarf, the Chandrasekhar mass limit – approximately 1.4 solar masses, beyond which it can not be supported by degeneracy pressure. A carbon-oxygen white dwarf that approaches this mass limit, typically by mass transfer from a companion star, may explode as a Type Ia supernova via carbon detonation [10].

Although most white dwarfs are thought to be composed of carbon and oxygen, spectroscopy typically shows that their emitted light comes from an atmosphere which is observed to be either hydrogen-dominated or helium dominated. The dominant element is usually at least 1,000 times more abundant than all other elements. The high surface gravity is thought to cause this purity by gravitationally separating the atmosphere so that heavier elements are at the bottom and lighter ones on top [12]. This atmosphere, the only part of the white dwarf visible to us, is thought to be the top of an envelope which is a residue of the star's envelope in the AGB phase and may also contain material accreted from the interstellar medium. The envelope is believed to consist of a helium-rich layer with mass no more than $1/100^{th}$ of the star's total mass, which, if the atmosphere is hydrogen-dominated, is overlain by a hydrogen-rich layer with mass approximately $1/10{,}00^{th}$ of the star's total mass [7] and [11].

It is simple to derive an approximate relationship between the mass and radii of white dwarfs using an energy minimization argument. It has been shown [2] and [8], assuming the electrons in a white dwarf to be non-relativistic, the mass, M and radius, R are related by

$$R = \frac{1}{M^{1/3}}, \tag{1.1}$$

while in the extreme relativistic approximation (i.e. kinetic energy of electrons = p c)

$$M_{\lim it} \approx N^2 \left(\frac{\hbar c}{G} \right), \tag{1.2}$$

where G is the gravitational constant, N the number of electrons per unit mass of the dwarf, c the speed of light, $\hbar$ the reduced Planck constant, and M_{limit} is the limiting mass of a white dwarf. In this work, a FORTRAN code which numerically solves the equation of state in a white dwarf is written. The code was then used to study the structure of white dwarf stars. The investigations carried out include determining the variation of mass density from the center to the surface of the dwarf, the mass and radius of the dwarf, the variation of mass with radius *etc*.

2.0 Theoretical background

2.1 Introduction

It is assumed that the matter in a white dwarf star is spherically symmetric, that the white dwarf star is not rotating, and that the effects of magnetic fields are not very important.

These assumptions are known to be accurate for many purposes [3]; [5], and [17] and so are justified. If the star is in mechanical (hydrostatic) equilibrium, the gravitational force on each bit of matter is balanced by the force due to spatial variation of electron degenerate pressure, *P*. The gravitational force acting on a unit volume of matter at a radius *r* is

$$F_{grav} = -\frac{Gm}{r^2}\rho, \tag{2.1}$$

where ρ(r) is the mass density, and m(r) is the mass of the star interior to the radius r:

$$m(r) = 4\pi\int_0^r \rho(r')r'^2 dr'. \tag{2.2}$$

The force per unit volume of matter due to changing pressure is *–dP/dr*. When the star is in equilibrium, the net force on each bit of matter is zero, therefore, from (2.1), we have

$$\frac{dP}{dr} = -\frac{Gm(r)}{r^2}\rho(r). \tag{2.3}$$

A differential relation between the mass and the density can be obtained by differentiating Eq. (2.2):

$$\frac{dm}{dr} = 4\pi r^2\rho(r). \tag{2.4}$$

Upon using the identity $\frac{dP}{dr} = \left(\frac{d\rho}{dr}\right)\left(\frac{dP}{d\rho}\right)$, (2.3) can be transformed to

$$\frac{d\rho}{dr} = -\left(\frac{dP}{d\rho}\right)^{-1}\frac{Gm}{r^2}\rho. \tag{2.5}$$

Equations (2.4) and (2.5) are two coupled first-order differential equations that determined the structure of the star for a given equation of state. The values of the dependent variables at r = 0 are $\rho = \rho_c$, the central density, and m = 0.

2.2 The equation of state

The assumed properties of matter in white dwarf star stated above means that a good model for the matter in white dwarf stars is the free Fermi gas of electrons at zero temperature, treated with relativistic kinematics. Such a model has been fully described [6]; [12]; [13] and [14], therefore, only a brief outline will given here.

The equation of state for matter in white dwarfs can be written as

$$P = \frac{1}{3}n_0 m_e x^4\beta', \tag{2.6}$$

where,

$$n_0 = \frac{m_e^3}{3\pi^2} = 5.89\times10^{29}\ cm^{-3}, \tag{2.7}$$

is the number density of electrons at which the Fermi momentum is equal to the electron mass, m_e,

$$x = \frac{p_f}{m_e} = \left(\frac{n}{n_0}\right)^3, \tag{2.8}$$

p_f being the Fermi momentum,

$$\beta' = \frac{d\beta}{dx}, \text{ and } \beta(x) = \frac{3}{8x^3}\left\{x(1+2x^2)(1+x^2)^{1/2} - \log\left[x+(1+x^2)^{1/2}\right]\right\}. \qquad (2.9)$$

But *x* can also be written as

$$x = \left(\frac{n}{n_0}\right)^{1/3} = \left(\frac{\rho}{\rho_0}\right)^{1/3}, \qquad (2.10)$$

where,

$$\rho_0 = \frac{m_p n_0}{\eta} = 9.79\times10^5 \eta^{-1}\ gm\ cm^{-3}, \qquad (2.11)$$

is the mass density of matter in which the electron number density is n_0, and η is the number of electrons per nucleon. η is equal to 0.5 for helium, carbon, and oxygen, 1.0 for hydrogen, and 0.464 for iron.

Differentiating (2.6) yields

$$\frac{dP}{d\rho} = \frac{dP}{dx}\frac{dx}{d\rho} = \eta\frac{m_e}{m_p}\gamma(x) \qquad (2.12)$$

where m_p is mass of proton (the difference between the masses of proton and neutron is neglected), and

$$\gamma(x) = \frac{1}{9x^2}\frac{d}{dx}\left(x^4\beta'\right) = \frac{x^2}{3(1+x^2)^{1/2}}. \qquad (2.13)$$

2.3 Scaling the equations

Very often, it is desirable to transform equations describing a physical system to dimensionless form, for a better physical insight and for numerical convenience. To do this transformation for the equations of the white dwarf star, we introduce a dimensionless radius, density, and mass variables as follows.

$$r = R_0\bar{r},\quad \rho = \rho_0\bar{r},\quad m = M_0\bar{m}, \qquad (2.14)$$

where the radius and mass scales, R_0 and M_0 are determined so as to get the needed convenience. Substituting (2.14) into (2.4 and 2.5) and using (2.12) yields after some rearrangement,

$$\frac{d\bar{m}}{d\bar{r}} = \left(\frac{4\pi R_0^3\rho_0}{M_0}\right)\bar{r}^2\bar{\rho}; \qquad (2.15)$$

$$\frac{d\bar{\rho}}{d\bar{r}} = -\left(\frac{GM_0}{R_0\eta(m_e/M_p)}\right)\frac{\bar{m}\bar{\rho}}{\gamma\bar{r}^2}. \qquad (2.16)$$

Now if M_0 and R_0 are chosen so that the coefficients in the parentheses in (2.15 and 2.16) are unity, then

$$R_0 = \left[\frac{\eta(m_e/M_p)}{4\pi G\rho_0}\right]^{1/2} = 7.72 \text{ x } 10^8\ \eta\ \text{ cm}, \qquad (2.17)$$

$$M_0 = 4\pi R_0^3\rho_0 = 5.67 \text{ x } 10^{33}\eta^2\ \text{ gm}, \qquad (2.18)$$

and the dimensionless coupled differential equations for the white dwarf are

$$\frac{d\bar{\rho}}{d\bar{r}} = -\frac{m\bar{\rho}}{\eta\bar{r}^2}; \qquad (2.19)$$

and

$$\frac{d\bar{m}}{d\bar{r}} = \bar{r}^2\bar{\rho}. \tag{2.20}$$

3.0 The programme

Essentially, the program written constructs a series of white dwarf models for a given electron fraction, η with central densities ranging in equal logarithmic steps between the values of DEN1 and DEN2 specified. For each model, the dimensionless equations, (2.19) and (2.20) for the mass and density are integrated by the fourth order Runge-Kutta algorithm. An empirically scaled radial step is used for each model, and initial conditions for the integration are determined by a Taylor expansion of the differential equations about the center of the star (i.e. $r = 0$). Integration proceeds until the density has fallen below 10^3 $gm\ cm^{-3}$.

For each run of the program, the inputs are the electron fraction, η the central density for the first model, DEN1, the central density for the last model, DEN2, and the number of models to construct. As each model is calculated, the radial step, central density, number of steps, total radius, and total mass are given as output. Also given for each model is the density as function of distance from the center of the star.

4.0 Results and discussion

In the graphs shown below, the parameter η, i.e. The number of electrons per nucleon is given as ye. It should be noted that η equals to 1.0 for hydrogen, 0.5 for helim, carbon, and oxygen, 0.464 for iron, and 0.4 for a hypothetical dwarf composed of more than one type of element.

4.1 Variation of mass density within the white dwarf star

The variation of mass density with distance from the center of the star are shown in Figures 4.1, 4.2, and 4.3. In each of these figures, (a) represents the hypothetical white dwarf while (b) represents a hydrogen white dwarf, also the distance from the center of the star is in the unit of radius of the earth, *i.e*. 6,400 km. It can be seen from the figures that the density is directly proportional to the assigned density at the center of the dwarf and the value of η, i.e. the composition of the dwarf but inversely proportional to the distance from the center of the star. In general, the density decreases with increase in the distance.

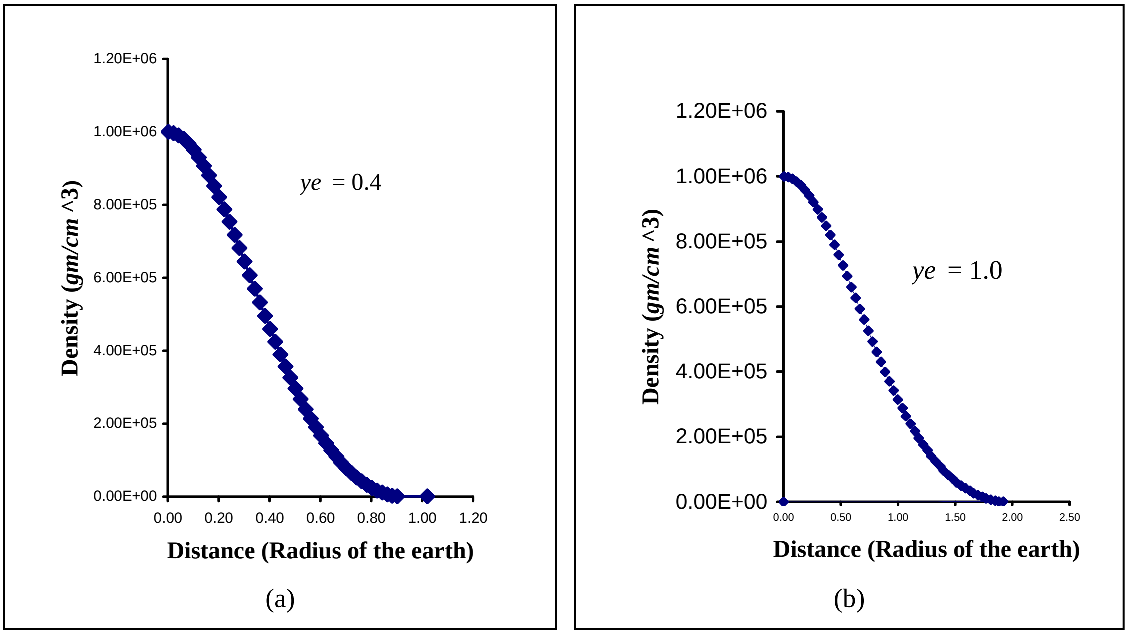

Figure 4.1: Variation of mass density with distance from the center of the star for a central density of 10^6 *gm cm*$^{-3}$

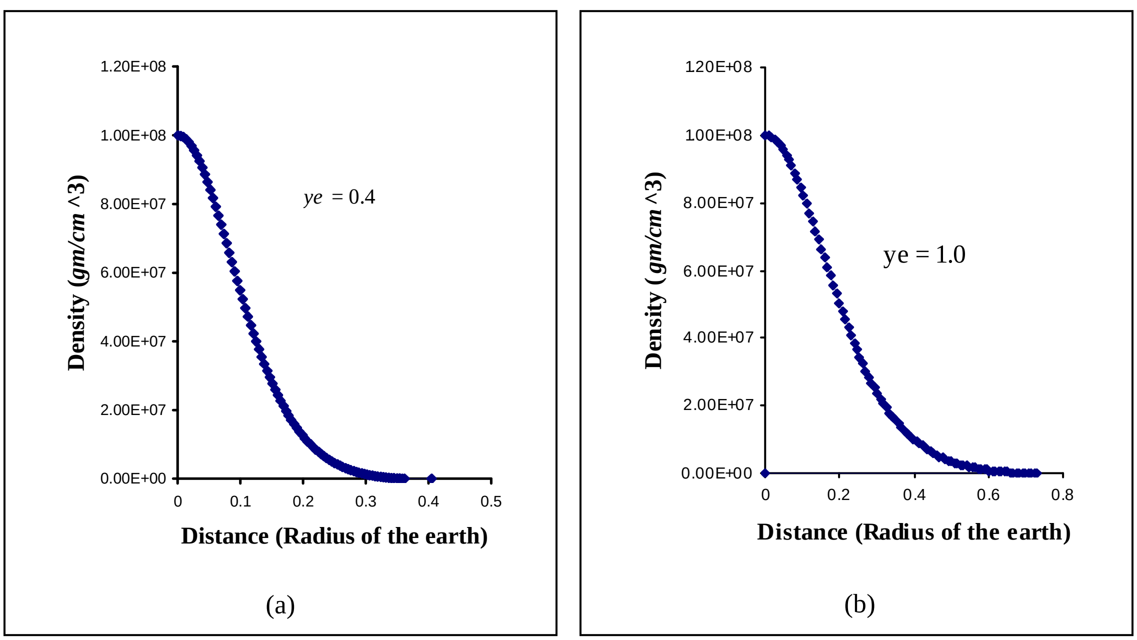

Figure 4.2: Variation of mass density with distance from the center of the star for a central density of 10^8 gm cm^{-3} .

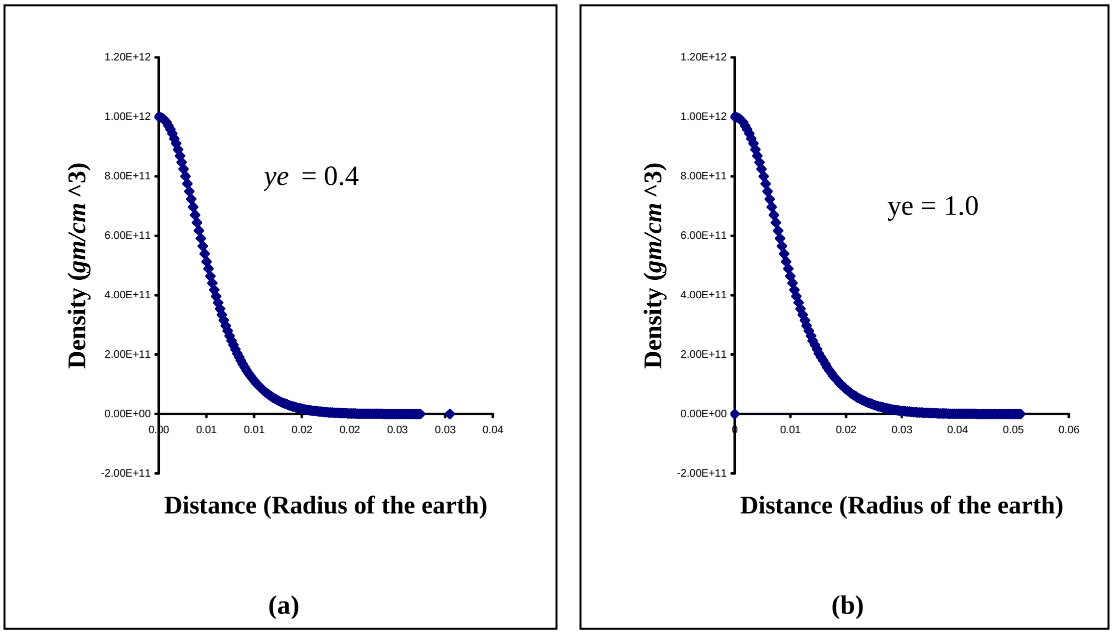

Figure 4.3: Variation of mass density with distance from the center of the star for a central density of 10^{12} gm cm^{-3}.

In order to quantitatively describe the variation of mass density with distance from the center of the star, the author finds it convenient to introduce the following four parameters, namely r_a, r_b, r_c, and r_0. These parameters are respectively the distance from the center of the star to the point where its density reduces to 95%, 50%, and 5%, of its value at the center, while r_0 is the distance from the center to the point where the density is equal to 1000 *gm cm*$^{-3}$. The computations for the integration of the two coupled equations of state are stopped at r_0. The values of these parameters are given in Table 4.1.

Table 4.1: The Values of the Parameters r_a, r_b, r_c, and r_0 for Hydrogen and a hypothetical white dwarf.

	η = 1.0			η = 0.4		
Central Density (*gm cm*$^{-3}$) →	10^6	10^8	10^{12}	10^6	10^8	10^{12}
r_a	0.188	0.0484	0.00262	0.102	0.0263	0.00142
r_b	0.778	0.199	0.00962	0.382	0.108	0.00522
r_c	1.55	0.469	0.0225	0.742	0.246	0.0122
r_0	1.92	0.731	0.0513	1.02	0.362	0.0274

Table 4.2: The ratios of the parameters r_a, r_b, r_c, and r_0 for hydrogen and a hypothetical white dwarf

	η = 1.0			η = 0.4		
Central Density (*gm cm*$^{-3}$) →	10^6	10^8	10^{12}	10^6	10^8	10^{12}
r_a/r_0	0.0979	0.0662	0.0510	0.1	0.07265	0.05182
r_b/r_0	0.405	0.2722	0.1875	0.3745	0.2983	0.1905
r_c/r_0	0.8072	0.6415	0.4385	0.7274	0.6795	0.4452

It can be seen from Figures 4.1, 4.2 and 4.3, that the density decreases with increasing distance from the center of the star. Near the center and far away from it, the rate of decrease in density is much less than elsewhere. The values of r_a/r_0, r_a/r_0, and r_a/r_0 given in Table 4.2 quantify the rate of decrease in density with distance from the origin. The values of these ratios are dependent mainly on the density at the center. The higher the central density, the lower these ratios are. These ratios are virtually independent of the composition of the star. For a given central density, the ratios for η = 1.0 and η = 0.4 are almost the same even though those for η = 1.0 are slightly less than those for η = 0.4.

It is interesting to note that r_a, r_b, r_c, and r_0 depend both on the composition and central density of the star. Table 4.1, and other data obtained by the author show that each of these parameters decrease with increase in central density, and that for a given central density they increase with increase in η.

4.2 The radius of a white dwarf star

The variation of radius of the white dwarf star with its central density is shown in Figure 4.4a,b. The radius depends on the composition and central density of the white dwarf star. For a given central density, the radius of the hydrogen white dwarf is greater than that of the helium, carbon, or oxygen which are equal and greater than that of iron. Thus the radius increases with the parameter η. For example, for a central density of 10^8 *gm* cm^{-3} the radius of hydrogen (η = 1.0) white dwarf is 7.312 × 10^8*cm*, that of helium, carbon, and oxygen (η = 0.5) is 4.306 × 10^8*cm*, while that of iron (η = 0.64) is 4.097 × 10^8*cm*. However, for a given white dwarf, (*i.e*. the parameter η constant) the radius decreases with increase in central density. For example, the radius of the hydrogen white dwarf decreases from 1.92 × 10^9*cm* for a central density of 10^6*gmcm*$^{-3}$ to 5.126 × 10^7*cm* for a central density of 10^{12}*gmcm*$^{-3}$.

4.3 The mass of a white dwarf star

The graphs for the variation of mass of a white dwarf star with its central density are shown in Figure 4.5a,b. The mass depends on both the composition and central density of the white dwarf star. For a given central density, the mass of a hydrogen white dwarf is greater than the mass of a helium, carbon, or oxygen which are equal and also greater than that of iron. Thus the mass increases with the parameter η. For instance, for a central density of 10^6 *gm* cm^{-3} the mass of hydrogen white dwarf is 6.1414 times greater than the mass of iron white dwarf, and 5.1387 greater than the mass of helium, carbon, and oxygen white dwarf.

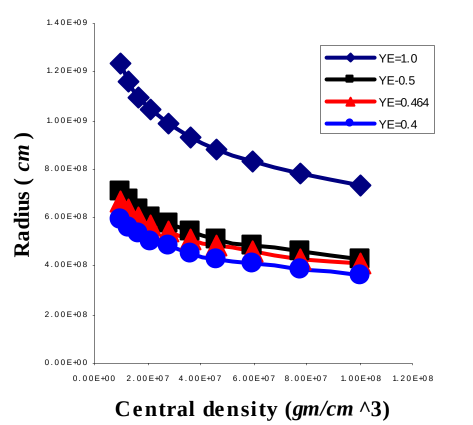

Figure 4.4a: Variation of radius with central density for central densities in the range of 10^7 to 10^8 *gm* cm^{-3}.

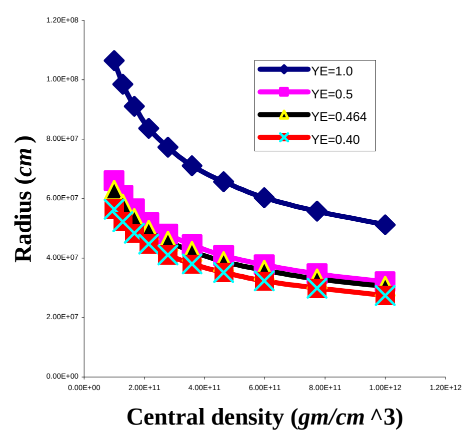

Figure 4.4b. Variation of radius with central density for central densities in the range of 10^{11} to 10^{12} *gm* cm^{-3}.

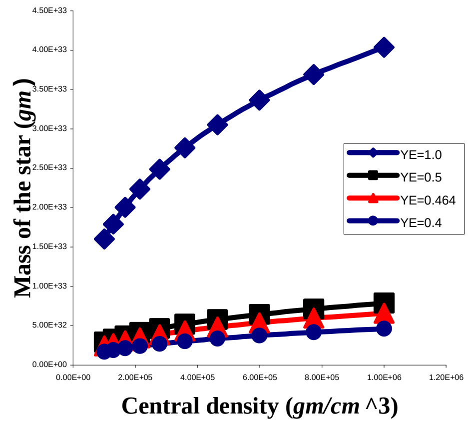

Figure 4.5a: Variation of mass with central density for central densities in the range of 10^5 to 10^6 gm cm^{-3}.

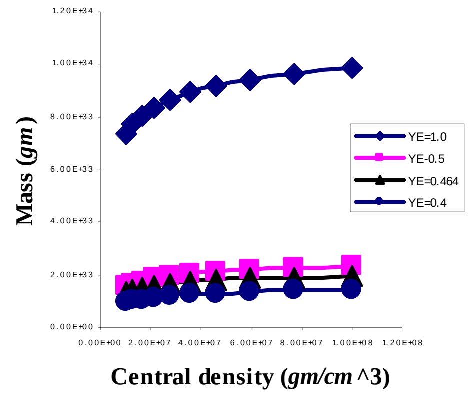

Figure 4.5b: Variation of mass with central density for central densities in the range of 10^7 to 10^8 *gm cm*$^{-3}$.

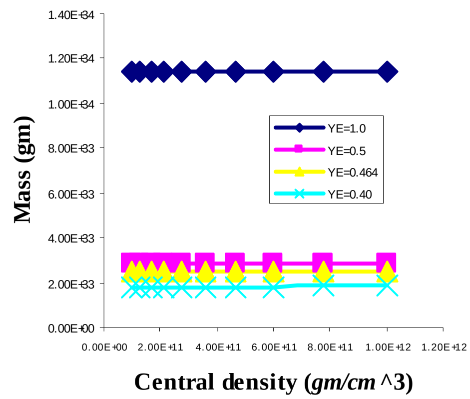

Figure 4.5c: Variation of mass with central density for central densities in the range of 10^{11} to 10^{12} *gm cm*$^{-3}$.

Starting from a central density of 10^5 *gm cm*$^{-3}$, the mass of a white dwarf star (*i.e.* for a given η) initially increases on increasing the central density but eventually becomes constant on attaining its limiting mass. In the case of the hydrogen white dwarf for example, its mass increases from 1.60×10^{33} *gm* to 4.04×10^{33} *gm* on increasing its central density from 10^5 *gm cm*$^{-3}$ to 10^6 *gm cm*$^{-3}$ and increases from 7.39×10^{33} *gm* to 9.86×10^{33} *gm* on increasing its central density from 10^7 *gm cm*$^{-3}$ to 10^8 *gm cm*$^{-3}$. Thus the rate of increase of mass decreases with increase in central density, so that as the density is increased the rate eventually becomes zero and the mass of the dwarf constant. The central density at which the mass becomes constant varies with the composition of the dwarf and is referred to by the author as the characteristic density. Table 4.3 below shows the maximum mass and the characteristic density of some white dwarfs. The results in Table 4.3 show that as the parameter η is increased, the limiting mass of the star decreases while the characteristic density increases.

Table 4.3. Maximum mass and characteristic central density.

Type of white dwarf	**Maximum Mass (*gm*)**	**Characteristic Density (*gm cm*$^{-3}$)**
Hydrogen ($\eta = 1.0$)	1.144×10^{34}	6.95×10^{11}
Helium, Carbon, and Oxygen ($\eta = 0.5$)	2.861×10^{33}	2.64×10^{13}
Iron ($\eta = 0.464$)	2.464×10^{33}	8.86×10^{13}
Hypothetical ($\eta = 0.4$)	1.831×10^{33}	2.64×10^{13}

5.0 Conclusion

The structure of hydrogen, helium, carbon, oxygen, iron, and a hypothetical white dwarf stars are investigated computationally. It has been found that the structure of white dwarf stars depend on their chemical composition and the value of the mass density at their center. The most outstanding findings in this work are given in Table 3 . It is generally known that there is a limiting mass for white dwarfs, the so-called Chandrasekhar mass limit, it is now found that each elemental type dwarf has its own limiting mass.

References

[1] Bean, R.(2007). *Degeneracy pressure*, Cornell University, Accessed on line Nov. 10, 2007.

[2] Chandrasekhar, S. (1931). The Maximum Mass of Ideal White Dwarfs, The Astrophysical Journal, pp81-82.

[3] Chandrasekhar, S. (1957). An Introduction to the Study of Stellar Structure, Dover (New York), Chapter 5.

[4] Chandrasekhar, S. (1984). Rev. Mod. Phys. **56**, 137.

[5] Chin, S. A. , Negele, J. W. and Koonin, S. E. (1984). Model White Dwarfs Structure, Ann. Phys. **157**, 140

[7] Chynoweth, K. (2006). *The Structure of White Dwarf Stars*, Statistical Mechanics, Spring , Accessed on line Nov.10, 2007.

[8] Fontaine, G. and Francois Wesemael, (2001). *White Dwarfs*, in Encyclopedia of Astronomy and Astrophysics, edited by Paul Murdin, Nature Publishing Group (New York) Chapter 11.

[9] Gilles, C. and Isabelle, B. (2000). Theory of Low-Mass Stars and Sub stellar Objects, Annual Review of Astronomy and Astrophysics 38, pp. 337-377.

[10] Henry,T. J. (2007), *The Hundred Nearest Star System*, RECONS. Accessed on line on the 12th August, 2007.

[11] Johnson, J.(2007), *Extreme Stars: White Dwarfs and Neutron Stars*, Ohio State University. Accessed on line on the 20th July, 2007.

[12] Kawaler, S. D. (1997). *White Dwarf Stars*, in Stellar remnants, edited by Georges Meynet and Daniel [13] Schaerer, Springer (Berlin) , chapter 17.

[13] Koester, D. and Chanmugam, G. (1990). *Physics of white dwarf stars*, Reports on Progress in Physics **53**, pp. 837-915.

[14] Koonin, S. E.. and Meredith, D. W.(1989). Computational Physics, Addison-Wesley Publication Company, (New York), pp.47-52

[15] Ostriker, J. P. and Bodenheimer, P. (1968). Equation of State in Stella Objects, ApJ, **151**, 1089.

[16] Richmond, M.(2007), *Late Stages of Evolution for Low-mass Stars*, Rochester Institute of Technology. Accessed on line February 20, 2007.

[17] Schatzman, E. (1958). *White Dwarfs*, North-Holland (Amsterdam) Chapter two.

[18] Shapiro, S. L. and Teukolsky, S. A. (1986). Black Holes, White Dwarfs, and Neutron Stars, J. Wiley and Sons (New York), Chapter 3.

[19] Werner, K., Hammer, N. J. , Nagel, T., Rauch, T., and Dreizler, S. *On Possible Oxygen/ Neon WhiteDwarfs: H1504+65 and the White Dwarf Donors in Ultra compact X-ray Binaries*, Proceedings of the 14th European Workshop on White Dwarfs held in July 19-23, 2004, edited by D. Koester and S. Moehler, and published by the Astronomical Society of the Pacific, 2005.pp. 165