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A flow in a trough: An integral equations formulation

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Abstract

The problem of fluid flow in an open trough driven by the base moving horizontally along its plane, was considered in* [2, 3]. *The approach used was based on finite difference technique which takes a lot of memory. This paper reformulates the problem in terms of integral equations and the resulting equations solved numerically. The integral reformulation is desirable as it uses less computer memory and is expected to give more accurate result as numerical integration is a more stable process than differentiation process.

1.0 Introduction

The steady state flow of a liquid in an open trough driven by the base is considered. The base is assumed to move along its plane at constant velocity. With the upper part of the liquid opened to the atmosphere and so in contact with the air above it, the upper boundary is a free surface whose shape has to be determined as part of the solution of the problem. This free surface is fixed by the stress equations at the interface and the volume constraints imposed on the problem.

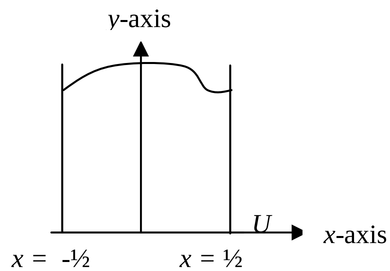

Figure 1.1

The problem has been considered in [2, 3, 4] and is a limiting case of the problem proved by Jean [1] theoretically to have a unique solution.

The numerical solution was sought in [3] for the problem using finite difference approach . Being a two dimensional problem, a lot of memory was needed to get a more accurate technique. In this paper this same problem is reformulated in form of integrals with

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hope to reduce the size of memory demand and the numerical process (i.e descritization process) undertaken to obtain solution of the problem. Unlike in the finite difference approach only the unknowns at the low boundary (and not those in the internal domain of the flow) need be known to determine the shape of the free surface. Moreover since integration unlike differentiation is a more stable process numerically, the hope is that a more accurate result might be obtained with moderate effort and storage size.

1.1 Formulation

Introducing the dimensionless variables *x,y, u, h, p* in place of the dimensional variables, *y´, u´, h´.*

$$y=\frac{y'}{a}, x=\frac{x'}{a}, u=\frac{u'}{U}, h=\frac{h'}{a}, p^*=\frac{p'+\rho' gy'}{\gamma/a}$$

the governing equation of the steady creeping flow considered here is governed by the biharmonic equation

$$\nabla^4\psi=0 \tag{1.1}$$

with the following boundary conditions. (assuming *s* and η are distances measured along the surface and normal to the surface of the flow boundary *dD).*

(a) No-slip and Non-penetration condition

$$\psi=0 \text{ on } dD \tag{1.2a}$$

$$-\frac{\partial\psi}{\partial\eta}=g(x)=\begin{matrix} 0 \text{ on } x=-1/2, 1/2 \\ f(x) \text{ on } y=0 \end{matrix} \tag{1.2b}$$

(b) shear stress

$$\nabla^2\psi=-2\kappa\frac{\partial\psi}{\partial\eta} \text{ on } dF \tag{1.3}$$

(c) Normal Stress

$$-ca(p+p_L)+Bh-2Ca\frac{\partial^2\psi}{\partial s\partial\eta}=\kappa \text{ on } dF \tag{1.4}$$

where *dF* is the interface (free surface) curve while

$$\kappa=\frac{d\theta(x,y)}{ds}=\frac{\frac{d^2h}{dx^2}}{(1+(\frac{dh}{dx})^2)^{3/2}}$$

is the curvature of the free surface and $Q(-1/2, h(-1/2))$ is the angle which the interface makes with the horizontal at its point of contact with the left vertical wall. The variable s, is the distance along the boundary measured in the anti-clockwise direction from the origin, while η is the distance along the outward normal to the boundary. The angle at the point (x, y) on the boundary dD which the tangent to the boundary curve makes with the horizontal is denoted by θ or sometimes by $\theta(x,y)$ to indicate the point at which the tangent is taken.

(d) Volume constraint

$$\int_{-1/2}^{1/2} h(x)dx = vol \tag{1.5}$$

(f) Contact angle condition:

$$\frac{dh}{dx}\Big|_{x=-1/2} = \tan\theta(-1/2, h(-1/2)),\ \frac{dh}{dx}\Big|_{x=1/2} = \tan\theta(1/2, h(1/2)) \tag{1.6}$$

(g) Pressure equation:

$$\frac{\partial p}{\partial s} = -\frac{\partial \nabla^2 \psi}{\partial \eta}, \frac{\partial p}{\partial \eta} = \frac{\partial \nabla^2 \psi}{\partial s} \tag{1.7}$$

where

$$u = -\frac{\partial \psi}{\partial y}, v = \frac{\partial \psi}{\partial x} \tag{1.8}$$

being the stream function

The three parameters *Ca*, *B*, volume are the capillary number, Bond number and the aspect ratio respectively and are defined as follows, $vol = \frac{vol'}{a}, B = \frac{d' g a^2}{\gamma}, Ca = \frac{\mu U}{\gamma}$. The variable p and p_L are defined as

$$p = p^*(x, h(x)) - p_L, p = p^*(-1/2, h(-1/2)) = p_L \tag{1.10a}$$

on *dF* so that

$$p(-1/2, h(-1/2)) = 0 \tag{1.10b}$$

2.0 Method of solution

The general solution of (1.1) can be expressed in the integral form as

$$\psi(\underline{x}_0) = \int_{dD} [\frac{\partial \omega(\underline{x})}{\partial \eta} G(\underline{x}, \underline{x}_0) - \frac{\partial G(\underline{x}, \underline{x}_0)}{\partial \eta}\omega + \frac{\partial \psi(\underline{x})}{\partial \eta}\nabla^2 G(\underline{x}, \underline{x}_0) - \psi(\underline{x})\frac{\partial \nabla^2 G(\underline{x}, \underline{x}_0)}{\partial \eta}]ds \tag{2.1}$$

where $\underline{x} = (x, y)$, $\omega = \nabla^2 \psi$ and $\underline{x}_0 = (x_0, y_0)$ is any point on or within the domain bounded by the boundary dD. The function $G(\underline{x}, \underline{x}_0)$ is the solution of $\nabla^4 G(\underline{x}, \underline{x}_0) = \delta(\underline{x}, \underline{x}_0)$, (δ($\underline{x}$, $\underline{x}_0$) being the usual dirac function) here taken as

$$G(\underline{x}, \underline{x}_0) = \frac{r(\underline{x}, \underline{x}_0)}{16\pi} \log \frac{r^+(\underline{x}, \underline{x}_0)}{r(\underline{x}, \underline{x}_0)} - \frac{y y_0}{4\pi} \tag{2.2}$$

where $r(\underline{x}, \underline{z}) = (x-u)^2 + (y-v)^2 = |\underline{x} - \underline{z}|^2$, $r^+(\underline{x}, \underline{z}) = (x-u)^2 + (y+v)^2$ (2..3)

$\underline{x} = (x, y), \underline{x}_0 = (x_0, y_0), \underline{z} = (u, v)$. The function $G(\underline{x}, \underline{x}_0)$ and its normal derivative vanish on line $y = 0$. Its tangential derivative on the line also vanishes. In view of the relation

$$\int_{dD} \frac{\partial \omega}{\partial \eta} G ds = -\int_{dD} \frac{\partial p}{\partial s} G ds \text{ (by equation (1.7))}$$

$$= \int_{dD} p \frac{\partial G}{\partial s} ds \text{ (after integration by parts} \quad (2.4)$$

and the boundary conditions (1.2a,b), (1.3), (1.7) inserted in the general solution equation (2.1) above becomes

$$\int_{dD-AC} p \frac{\partial G(\underline{x},\underline{x}_0)}{\partial s} ds - \int_{dD-dF} \frac{\partial G(\underline{x},\underline{x}_0)}{\partial \eta} \omega ds + \int_{dF} [2\kappa \frac{\partial G(\underline{x},\underline{x}_0)}{\partial \eta} + \nabla^2 G(\underline{x},\underline{x}_0)] \frac{\partial \psi(\underline{x})}{\partial \eta} ds + \int_{AC} f(x) \nabla^2 G(\underline{x},\underline{x}_0) ds = 0 \quad (2.5)$$

On the other hand applying the Green's third identity to the harmonic function $\omega = \nabla^2 \psi(\underline{x})$ one obtain the equation

$$(1/2)\omega(\underline{x}_0) + \int_{dD} [\omega(\underline{x}) \frac{\partial g(\underline{x},\underline{x}_0)}{\partial \eta} - \frac{\partial \omega(\underline{x})}{\partial \eta} g(\underline{x},\underline{x}_0)] ds = 0 \quad (2.6)$$

The function g is defined as $$g(\underline{x},\underline{x}_0) = \frac{1}{4\pi} \log \frac{r^+(\underline{x},\underline{x}_0)}{r(\underline{x},\underline{x}_0)} \quad (2.7)$$

being a solution of $\nabla^2 g(\underline{x},\underline{x}_0) = \delta(\underline{x},\underline{x}_0)$. The function g($\underline{x}$,$\underline{x}_0$) vanishes on line $y = 0$.

For each point $\underline{x}_0$ on the free surface *dF*, this equation (2.6) (by virtue of (1.3)) simplifies to the equation

$$-\kappa \frac{\partial \psi(\underline{x}_0)}{\partial \eta} + \int_{dD-dF} \omega(\underline{x}) \frac{\partial g(\underline{x},\underline{x}_0)}{\partial \eta} ds - 2\int_{dF} \kappa \frac{\partial g(\underline{x},\underline{x}_0)}{\partial \eta} \frac{\partial \psi(\underline{x})}{\partial \eta} ds - \int_{dD-AC} p \frac{\partial g}{\partial s} ds = 0 \quad (2.8)$$

whereas for every point $\underline{x}_0$ on the fixed vertical boundaries it takes the form:

$$(1/2)\omega(\underline{x}_0) + \int_{dD-dF} \omega(\underline{x}) \frac{\partial g(\underline{x},\underline{x}_0)}{\partial \eta} ds - 2\int_{dF} \kappa \frac{\partial g(\underline{x},\underline{x}_0)}{\partial \eta} \frac{\partial \psi(\underline{x})}{\partial \eta} ds - \int_{dD-AC} p \frac{\partial g}{\partial s} ds = 0 \quad (2.9)$$

and should be solved together with (2.1) for the unknown $\frac{\partial \psi}{\partial \eta}|_{\underline{x}_0}$ at each nodal point $\underline{x}_0$ on the interface, $\omega(\underline{x}_0)$ at each nodal point $\underline{x}_0$ on the fixed boundary and $p(\underline{x}_0)$ at each nodal point on the whole boundary. The values of these quantities are needed in (1.4), (1.5), (1.6) to obtain the constant p_L and the shape of the free boundary, *h*(*x*). The choice of *G* and *g* above makes it unnecessary to solve for the unknown quantities at the boundary.

In general $\underline{x}$ is a (variable) point on the boundary, while $\underline{x}_0$ is any point within the domain *D* or on the boundary *dD.*

3.0 Discretization

The boundary dD is approximated by a sequence of line segments dD_i ,

$$\underline{x} = t\underline{x}_{i-1} + (1-t)\underline{x}_i, \text{for } t\varepsilon[0,1]$$

joining points $\underline{x}_i, \underline{x}_{i+1}$ for all points $\underline{x}_i, (i = 1,2,\Lambda\ , N)$ on the boundary dD. The corner points of the boundary dD_i are taken to be some of those points. Defining the nodal point $\underline{x}_i^*$ as the mid-point of the boundary line segment dD_i, that is $\underline{x}_i^* = (1/2)(\underline{x}_i + \underline{x}_{i+1})$, for each i = 1,2,…, and suppose $p_i^*, \omega_i^*, h_i^*, \left(\frac{\partial \psi}{\partial \eta}\right)_i^*$ denote the values of $p, \omega, h, \frac{\partial \psi}{\partial \eta}$ at the nodal point $\underline{x}_i^*$ while $p_i, \omega_i, h_i \left(\frac{\partial \psi}{\partial \eta}\right)_i$ denote their values at the mesh point $\underline{x}_i$ then applying equations (2.1), (2.6), (1.6), (2.5), (2.8) at each nodal point $\underline{x}_i^*$ we have the equations below in methods 1 and 2. From (2.5) we have

$$\sum_{\underline{x}_j \varepsilon dD-Ac} c_{i,j}^* p_j^* - \sum_{\underline{x}_j \varepsilon dD-dF-AC} b_{i,j}\omega_j^* + \sum_{\underline{x}_j \varepsilon dF} c_{i,j}\left(\frac{\partial \psi}{\partial \eta}\right)_j^* = -\sum_{\underline{x}_j \varepsilon AC} a_{i,j}^* f(\underline{x}_j),\ i = 1,2,\Lambda\ , N \qquad (3.1)$$

From (2.8) we have $-\left(\kappa \frac{\partial \psi}{\partial \eta}\right)_i^* + \sum_{\underline{x}_j \varepsilon dD-dF} d_{i,j}\omega_j^* - 2\sum_{\underline{x}_j \varepsilon dF} f_{i,j}\left(\frac{\partial \psi}{\partial \eta}\right)_j^* - \sum_{\underline{x}_j \varepsilon dD-AC} b_{i,j}^* p_j^* = 0$ which may be re-written as

$$-\sum_{\underline{x} \varepsilon dD-AC} b_{i,j}^* p_j^* + \sum_{\underline{x}_j \varepsilon dD-dF} d_{i,j}\omega_j^* - \sum_{\underline{x}_j \varepsilon dF} (\kappa_j \delta_{i,j} + 2 f_{i,j})\left(\frac{\partial \psi}{\partial \eta}\right)_j^* = 0 \qquad (3.2)$$

for nodal points on the interface and from (2.9) we have

$$(1/2)\omega_i^* + \sum_{\underline{x}_j \varepsilon dD-dF} d_{i,j}\omega_j^* - 2\sum_{\underline{x}_j \varepsilon dF} f_{i,j}\left(\frac{\partial \psi}{\partial \eta}\right)_j^* - \sum_{\underline{x}_j \varepsilon dD-AC} b_{i,j}^* p_j^* = 0, i = 1,2,\Lambda\ , N$$

on the fixed vertical boundaries and this may be rewritten as

$$-\sum_{\underline{x} \varepsilon dD-AC} b_{i,j}^* p_j^* + \sum_{\underline{x}_j \varepsilon dD-dF} ((1/2)\delta_{i,j} + d_{i,j})\omega_j^* - 2\sum_{\underline{x}_j \varepsilon dF} f_{i,j}\left(\frac{\partial \psi}{\partial \eta}\right)_j^* = 0 \qquad (3.3)$$

where $a_{i,j} = \int_{dD_j} G(\underline{x}, \underline{x}_i^*)ds,\ b_{i,j} = \int_{dD_j} \frac{\partial G(\underline{x}, \underline{x}_i^*)}{\partial \eta} ds,\ c_{i,j} = \int_{dD_j} [2k\frac{\partial G(\underline{x}, \underline{x}_i^*)}{\partial \eta} + \nabla^2 G(\underline{x}, \underline{x}_i^*)]ds$

$$d_{i,j} = \int_{dD_j} \frac{\partial g(\underline{x}, \underline{x}_i^*)}{\partial \eta} ds,\ e_{i,j} = \int_{dD_j} g(\underline{x}, \underline{x}_i^*)ds,\ f_{i,j} = \int_{dD_j} k\frac{\partial g(\underline{x}, \underline{x}_i^*)}{\partial \eta} ds$$

$$a_{i,j}^* = \int_{dD_i} \nabla^2 G(\underline{x}, \underline{x}_i^*)ds,\ b_{i,j}^* = \int_{dD_i} \frac{\partial g(\underline{x}, \underline{x}_i^*)}{\partial s} ds,\ c_{i,j}^* = \int_{\underline{x} \varepsilon dD-AC} \frac{\partial G}{\partial s} ds \qquad (3.4)$$

and $\delta = \begin{cases} 1 & if \quad i=j \\ 0 & otherwise \end{cases}$. Equations (3.1), (3.2), (3.3) are systems of linear equations of the form

$$\begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} \\ A_{2,1} & A_{2,2} & A_{2,3} \\ A_{3,1} & A_{3,2} & A_{3,3} \end{pmatrix} u = C$$

where

$$A_{1,1} = [c^*_{i,j}], A_{1,2} = -[b_{i,j}], A_{1,3} = [c_{i,j}]$$

$$A_{2,1} = -[b^*_{i,j}], A_{2,2} = [d_{i,j}], A_{2,3} = -(k_j \delta_{i,j} + 2 f_{i,j})$$

$$A_{3,1} = -[b^*_{i,j}], A_{3,2} = (\frac{1}{2}\delta_{i,j} + d_{i,j}), A_{3,3} = -[2 f_{i,j}]$$

$$u = [p_1^*, p_2^*, \Lambda\ , p_n^*, \omega_1^*, \omega_2^*, \Lambda\ , \omega_m^*, \left(\frac{\partial\psi}{\partial\eta}\right)_1^*, \left(\frac{\partial\psi}{\partial\eta}\right)_2^*, \Lambda\ , \left(\frac{\partial\psi}{\partial\eta}\right)_k^*]^T$$

$$C = [a_1^* f_1(x_1), a_2^* f(x_2), \Lambda\ , a_n^* f(x_n), 0, 0.\Lambda\ , 0, 0, \Lambda\ , 0]$$

to solve for the unknown u (i.e the unknown p_j^* at the nodal point s $\underline{x}_j^*$ on the boundary

except those on the horizontal boundary ω_j^* at the nodal points $\underline{x}_j^*$ on the boundary except the free boundary. $\left(\frac{\partial\psi}{\partial\eta}\right)_j^*$ at the nodal points $\underline{x}_j^*$ on the interface). The differential equation (1.4) is discretized by first integrating it over the interval $[\underline{x}_i^*, \underline{x}_{i+1}^*]$ along the x direction and using the following approximations,

$$\int_{x_i^*}^{x_{i+1}^*} h dx = h_i (x_{i+1}^* - x_i^*), \left(\frac{dh}{dx}\right)_i^* = \left(\frac{dh}{dx}\right)_{x = x_i^*} = \frac{h_{i+1} - h_i}{x_{i+1} - x_i}$$

$$\int_{x_i^*}^{x_{i+1}^*} p dx = (1/2)(p_{i+1}^* + p_i^*)(x_{i+1}^* - x_i^*)$$

to have equation (1.4) becoming

$$\frac{h_{i+1} - h_i}{x_{i+1} - x_i}\left[1 + \left(\frac{h_{i+1} - h_i}{x_{i+1} - x_i}\right)^2\right]^{-1/2} - \frac{h_i - h_{i-1}}{x_i - x_{i-1}}\left[1 + \left(\frac{h_i - h_{i-1}}{x_i - x_{i-1}}\right)^2\right]^{-1/2}$$
$$- Bh_i(x_{i+1}^* - x_i^*) = -\frac{1}{2}Ca(p_{i+1}^* + p_i^* + p_L)(x_{i+1}^* - x_i^*) - 2CaZ_i \qquad (3.5)$$

where $Z_i = \int_{x_i^*}^{x_{i+1}^*} \frac{\partial^2\psi}{\partial\eta\partial s} dx = \left(\frac{\partial^2\psi}{\partial\eta\partial s}\right)_i \int_{x_i^*}^{x_{i+1}^*} dx = \frac{\left(\frac{\partial\psi}{\partial\eta}\right)_{i+1}^* - \left(\frac{\partial\psi}{\partial\eta}\right)_i^*}{|\underline{x}_{i+1}^* - \underline{x}_i^*|}(x_{i+1}^* - x_i^*) = \frac{\left(\frac{\partial\psi}{\partial\eta}\right)_{i+1}^* - \left(\frac{\partial\psi}{\partial\eta}\right)_i^*}{\sqrt{1 + \left(\frac{h_{i+1}^* - h_i^*}{x_{i+1}^* - x_i^*}\right)^2}}$

The values of h^*_{i+1}, h^*_i are replaced by $(1/2)(h_i + h_{i+1}), \quad (1/2)(h_{i-1} + h_i)$ respectively so that $h^*_{i+1} - h^*_i$ becomes $(1/2)(h_{i+1} - h_i)$. This equation is solved for h (using Newton's method for solving system of non-linear equations) together with conditions (1.6) which are approximated as

$$h_1 - h_0 = l\tan\theta(-1/2, h(-1/2)), h_n - h_{n-1} = -l\tan\theta(-1/2, h(-1/2)) \qquad (3.6)$$

where $h_n = h(1/2), \quad h_{n-1} = h(1/2 - l), \quad h_0 = h(-1/2) \quad h_1 = h(-1/2 + l)$ and the volume constraint condition from (1.5) (which on using trapezoidal rule to approximate the integral) becomes

$$Vol = \frac{1}{2}\sum_{i=0}^{i=n-1}(x_{i+1} - x_i)(h_{i+1} + h_i) = \frac{1}{2}(x_1 - x_0)h_0 + \frac{1}{2}\sum_{i=1}^{i=n-1}(x_{i+1} - x_{i-1})h_i + \frac{1}{2}(x_n - x_{n-1})h_n \quad (3.7)$$

4.0 Hydrostatic problem

The corresponding hydrostatic equation can be obtained from (1.4) by putting ca and to zero so that the pressure p, becomes zero leaving the equation,

$$\frac{\frac{d^2 h_0}{dx^2}}{[1 + (\frac{dh_0}{dx})^2]^{3/2}} = Bh_0$$

With replacing h_0 in (1.4) to denote the hydrostatic solution of the interface. This is to be be solved together with the conditions in 1.6.

5.0 Computational step

The equations above are solved iteratively and the procedure used is described as follows for a given vol and B.

1. Input the number of subdivision points needed on the vertical walls, horizontal wall and the interface and compute the total number N, of the subdivision points.
2. Divide the boundary into subintervals and store the subdivision points (x_i, y_i) for i=1,2,…N.. Compute the coordinates of the nodal points (x^*_i, y^*_i) of dD as well as its length $|dD_i|$ for i=1,2,…N.
3. Input the flow parameters (Ca,vol,B).
4. Solve numerically equation in 4.1 to obtain the hydrostatic solution. (This corresponds to the case when Ca=0).
5. Increase the current value of Ca for which solution is already obtained, by some amount, to say Ca* say.
6. Solve the problem for this value Ca*:
   - (a) Set the interface *h*(*x*), equal the one obtained for Ca*
   - (b) Solve the linear systems (3.1), (3.2) for the unknowns $p_i, \omega_i, (\frac{\partial\psi}{\partial\eta})_i$ and
   - (c) Calculate h_i from 3.6a, 3.6c and compute the new value of h_L from 3.7a
   - (d) Re-adjust the subdivision points, nodal points and the lengths $|dD_i|$ in the interface.

(e) Check the termination condition,

If iteration diverges then

(i) set Ca = (Ca + Ca*)/2

(ii) go to step 6

Else, if termination condition is met then

(i) Output result

(ii) If desired to compute for larger value of Ca then

(a) CA = Ca*

(b) Go to step 5

Otherwise terminate computation

Else go to step b

6.0 Results and observations

The above results presented in the graphs shows that the solution (the interface) approaches the hydrostatic solution as the capillary number becomes increasingly small.

The linear system resulting from the normal stress equation was solved at each round of equation by Gauss-elimination method. The choice of the normal stress equation for revising the free surface is madedue to the fact that we are interested in the flow with low capillarity number (that is, a flow with large surface tension).

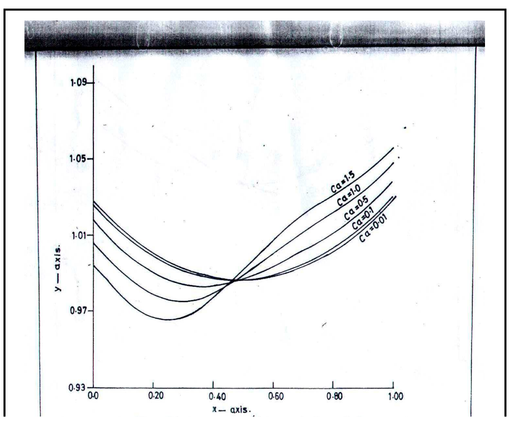


Figure 6.1: Interface curve. Nature of the limit Ca → 0, Vol = 1.0, Q = 10^0, B = 2.

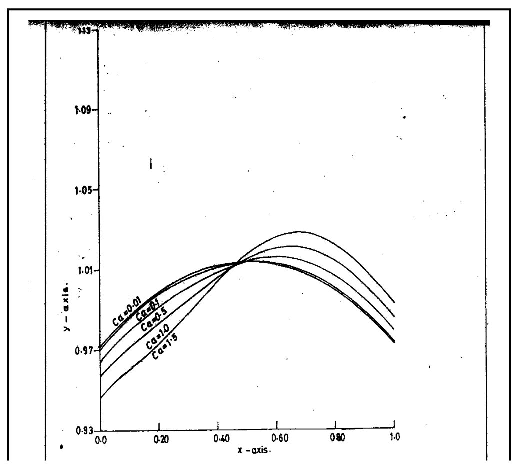
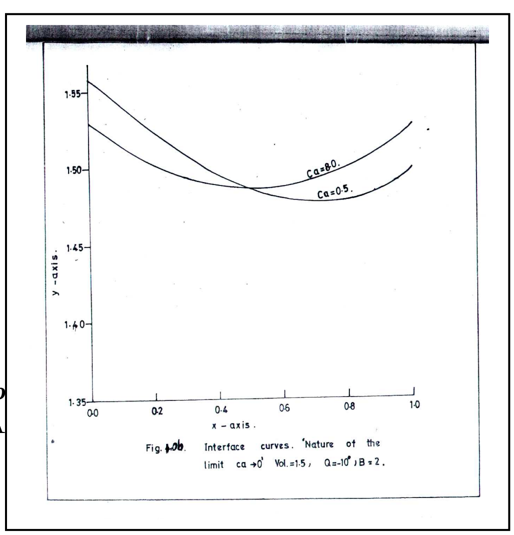


Figure 6.2: Interface curve. Effect of variation of Ca on the interface, Vol = 1.0, Q = 10^0, B = 2.



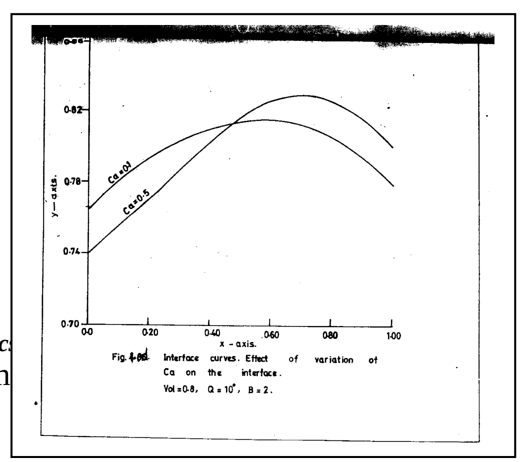


Figure 6.3: Interface curve. Nature of the limit Ca → 0, Vol = 1.5, Q = -10^0, B = 2.

Figure 6.4: Interface curve: Effect of variation of Ca on the interface. Vol = 0.8, Q = 10^0, B = 2.

7.0 Conclusion

The work above considers a very highly viscous flow with a free surface in a rectangular trough. The problem is turned into a dimensionless form making the flow to depend only on three parameters, namely the capillary number Bond number and the trough's aspect ratio. The problem in its dimensionless form is reformulated in term of integral-differential equations and the resulting equations solved numerically. The equations converge for small capillary number.

Appendix

Solving the non-linear system of equations (3.5), (3.6) and (3.7). The system of non linear equation $\underline{F}(\underline{h}^*) = 0$ where $\underline{h} = (h_0, h_1, \Lambda, h_n)$, $\underline{h}^* = (h_0, h_1, \Lambda, h_n, p_L)$,

$\underline{F}(\underline{h}^*) = (f_0(\underline{h}^*), f_1(\underline{h}^*), \Lambda, f_{n+1}(\underline{h}^*))$, $f_0(\underline{h}^*) = h_1 - h_0 - (x_1 - x_0)\tan\theta(-1/2, h(-1/2))$,

$$f_i(\underline{h}^*) = \frac{h_{i+1}-h_i}{x_{i+1}-x_i}\left[1+\left(\frac{h_{i+1}-h_i}{x_{i+1}-x_i}\right)^2\right]^{-1/2} - \frac{h_i-h_{i-1}}{x_i-x_{i-1}}\left[1+\left(\frac{h_i-h_{i-1}}{x_i-x_{i-1}}\right)^2\right]^{-1/2}$$

$$-Bh_i(x_{i+1}^* - x_i^*) + \frac{1}{2}Ca(p_{i+1}^* + p_i^* + p_L)(x_{i+1}^* - x_i^*) - 2CaZ_i = 0, \; for\; i = 1,2,\Lambda\; n-1$$

$f_n(\underline{h}^*) = h_{n-1} - h_{n-1} + (x_n - x_{n-1})\tan\theta(-1/2, h(-1/2))$,

$$f_{n+1}(\underline{h}^*) = Vol - \frac{1}{2}\sum_{i=0}^{i=n-1}(x_{i+1}-x_i)(h_{i+1}+h_i)$$

$$= Vol - [\frac{1}{2}(x_1 - x_0)h_0 + \frac{1}{2}\sum_{i=1}^{i=n-1}(x_{i+1}-x_{i-1})h_i \quad + \frac{1}{2}(x_n - x_{n-1})h_n]$$

is solved using Newton's method, which specifies that the system of linear equations $\underline{F}'(\underline{h}^{*(i)})\Delta\underline{h}^* = -\underline{F}(\underline{h}^{*(i)}), i = 0,1,2,\Lambda$ be solved for $\Delta\underline{h}^*$ where $\underline{h}^{*(i)} = (h_1^{(i)}, h_2^{(i)}, \Lambda,$ $h_n^{(i)}, p_L^{(i)})$, $\underline{h}^{*(i+1)} = \underline{h}^{*(i)} + \Delta\underline{h}^*$,

$$\underline{F}'(\underline{h}^*) = [a_{i,j}] = \left[\frac{\partial f_i}{\partial h_j}, \frac{\partial f_i}{\partial p_L}\right] = \begin{pmatrix} \frac{\partial f_0}{\partial h_0} & \frac{\partial f_0}{\partial h_1} & . & . & \frac{\partial f_0}{\partial h_n} & \frac{\partial f_0}{\partial p_L} \\ \frac{\partial f_1}{\partial h_0} & \frac{\partial f_1}{\partial h_1} & . & . & \frac{\partial f_1}{\partial h_n} & \frac{\partial f_1}{\partial p_L} \\ . & . & . & . & . & \\ . & . & . & . & . & \\ \frac{\partial f_{n+1}}{\partial h_0} & \frac{\partial f_{n+1}}{\partial h_1} & & & \frac{\partial f_{n+1}}{\partial h_n} & \frac{\partial f_{n+1}}{\partial p_L} \end{pmatrix}$$

This matrix $\underline{F}'(\underline{h}^*)$ is of form

$$
\begin{pmatrix}
* & * & & & & & & & & * \\
* & * & * & & & & & & & * \\
& * & * & * & & & & & & * \\
& & * & * & * & & & & & * \\
& & & * & * & * & & & & * \\
& & & & * & * & * & & & * \\
& & & & & * & * & * & & * \\
& & & & & & * & * & * & * \\
& & & & & & & * & * & * \\
* & * & * & * & * & * & * & * & * & *
\end{pmatrix}
$$

Note that

$$
\frac{\partial f_0}{\partial h_i} = \begin{cases} -1, \ if\ i = 0 \\ 1, \ if\ i = 1, \\ 0, otherwise \end{cases} \quad \frac{\partial f_0}{\partial p_L} = 0
$$

$$
\frac{\partial f_i}{\partial h_{i+1}} = \frac{1}{x_{i+1} - x_i} \cdot \frac{1}{\left[1 + \left(\dfrac{h_{i+1} - h_i}{x_{i+1} - x_i}\right)^2\right]^{3/2}}, \quad \frac{\partial f_i}{\partial h_{i-1}} = \frac{1}{x_i - x_{i-1}} \cdot \frac{1}{\left[1 + \left(\dfrac{h_i - h_{i-1}}{x_i - x_{i-1}}\right)^2\right]^{3/2}}
$$

$$
\frac{\partial f_i}{\partial h_i} = -\frac{\partial f_i}{\partial h_{i+1}} - \frac{\partial f_i}{\partial h_{i-1}} - B(x_{i+1}^* - x_i^*)
$$

$$
\frac{\partial f_i}{\partial h_j} = 0, \ for\ j \neq i+1, i, i-1, \quad \frac{\partial f_i}{\partial p_L} = \frac{1}{2} Ca(x_{i+1}^* - x_i^*)
$$

$$
\frac{\partial f_n}{\partial h_i} = \begin{cases} -1, \ if\ \ i = n-1 \\ 1, \ if \quad i = n, \\ 0, \ otherwise \end{cases} \quad \frac{\partial f_n}{\partial p_L} = 0
$$

$$
\frac{\partial f_{n+1}}{\partial h_i} = \begin{cases} \frac{1}{2}(x_1 - x_0), \ if\ i = 0 \\ (x_{i+1} - x_{i-1}), if\ i = n, \\ \frac{1}{2}(x_n - x_{n-1}) \ otherwise \end{cases} \quad \frac{\partial f_{n+1}}{\partial p_L} = 0
$$

We assume equally spaced interval in x with each subinterval of length l, (i.e $|x_{i+1} - x_i| = l$ for all i)

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