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Multivariate semi-logistic distribution and processes

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Abstract

Multivariate semi-logistic distribution is introduced and studied. Some characterizations properties of multivariate semi-logistic distribution are presented. First order autoregressive minification processes and its generalization to k^{th} order autoregressive minification processes with multivariate semi-logistic distribution as marginal distribution are developed and studied.

Keywords
Minification process, Multivariate semi-logistic distribution, Autoregressive processes, Stationarity.

1.0 Introduction.

Logistic distribution has attracted the attention of many researchers due to the application of this distribution in various fields. Balakrishnan [3] discussed the application of logistic distribution in population growth, medical diagnosis and public health. A few more interesting uses of logistic distribution are in analysing of survival data. Balakrishnan [3] discussed the analysis of bioavailability data when successive samples are from logistic distribution. Univariate logistic distribution in its reduced form is given by

$$\overline{F}(x)=\frac{1}{1+\exp\{x\}},-\infty\le x\le\infty \tag{1.1}$$

and $f(x)=\dfrac{\exp\{x\}}{\left(1+\exp\{x\}\right)^2}, \quad -\infty\le x\le\infty.$

This distribution is symmetric about zero and it is closely resembles the normal distribution. Although multivariate data sets with logistic like marginals have always been around, it was not until 1961 that a bivariate logistic model was proposed. Gumbel [4] actually provided three bivariate logistic models, one of which has cumulative distribution function

$$F(x,y)=\frac{1}{1+\exp\{x\}+\exp\{y\}},-\infty\le x,y\le\infty. \tag{1.2}$$

Location and scale parameters can be introduced to generalize this expression. Gumbel [4] in his paper studied the regression properties and verified that the correlation coefficient is $\frac{1}{2}$. A

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multivariate extension of the Gumbel` s bivariate logistic is proposed by Malik and Abraham [7]. For more applications of logistic distribution, see [3] and [5]. The studies on minification processes began with the work of [12]. In his work, the observations are generated by the equation

$$X_n = K \min\left(X_{n-1}, \varepsilon_n\right), n \geq 1 \tag{1.3}$$

where $k > 1$ is a constant and $\{\varepsilon_n\}$ is an innovation process of independent and identically distributed random variables chosen to ensure that $\{X_n\}$ is a stationary Markov process with a given marginal distribution. Because of the structure of (1.3), the process $\{X_n\}$ is called minification process. Sim [11] developed a first order autoregressive Weibull process and studied its properties. Arnold [2[developed a logistic process involving Markovian minimization. Giving slight modifications to (1.3), several authors constructed other minification processes models. Yeh et al [14] considered a first order autoregressive minification process having Pareto marginal distribution. Arnold and Robertson [1] developed a minification process having logistic marginal distribution. Pillai and Jayakumar [9] discussed the autoregressive minification processes and the class of distribution of universal geometric minima. Lewis and McKenzie [6] provided us with minification processes and their transformations. Ristic [10] developed stationary bivariate minification processes. Miroslav, Biljana, Aleksandar and Miodrag [8] introduced and studied a stationary bivariate minification process. They defined the process as

$$X_n = K \min\left(X_{n-1}, Y_{n-1}, \varepsilon_{n1}\right)$$
$$Y_n = K \min\left(X_{n-1}, Y_{n-1}, \varepsilon_{n2}\right).$$

Thomas and Jayakumar [13] defined first order autoregressive bivariate semi-logistic process as

$$X_n = \min\left(X_{n-1}, \tfrac{1}{\alpha_1}\ln p, \varepsilon_n\right)$$
$$Y_n = \min\left(Y_{n-1}, \tfrac{1}{\alpha_2}\ln p, v_n\right),\ n \geq 1,\ 0 \leq p \leq 1,\ \ \alpha_1, \alpha_2 > 0,$$

where ε_n and v_n are real random variables with either both ∞ with probability *p* or both are finite with probability 1 - *p*. Hence, in that paper they also represented the process as

$$\left(\varepsilon_n, v_n\right) = \begin{cases} \left(-\infty, \infty\right) \text{with probability } p \\ \left(\omega_n, \tau_n\right) \text{with probability} 1 - p. \end{cases}$$

In this paper**,** we develop the multivariate logistic and semi logistic distributions as a follow up to bivariate logistic and semi- logistic distribution presented by [13]. The characterizations properties of semi-logistic distribution are investigated and studied. First order autoregressive minification process and its extension to k^{th} order autoregressive minification models are presented with multivariate semi-logistic distribution as a stationary marginal distribution.

This paper is organized as follows. Section 2, is devoted to introduction of multivariate logistic and semi-logistic distributions. The properties of the latter are also study in this section. In section 3, characterizations of semi-logistic distribution are obtained. First order autoregressive minification process model with semi-logistic distribution as its marginal distribution is constructed in section 4. We generalized the model to k^{th} order autoregressive minification model with multivariate semi-logistic distribution as a stationary marginal distribution in section 5.

2.0 Multivariate Semi-logistic Distribution

A random vector $\left(X_1, X_2, ..., X_n\right)$ defined on $\mathbb{R}^n$ is said to have multivariate semi-logistic distribution with parameters $\alpha_1, \alpha_2, ..., \alpha_n; p$ and we denote it by $\left(X_1, X_2, ..., X_n\right) \underline{\underline{d}}\ MSL\left(\alpha_1, \alpha_2, ..., \alpha_n; P\right)$ if its survival function is of the form

$$\overline{F}\left(x_1, x_2, ..., x_n\right) = P\left(X_1 > x_1, X_2 > x_2, ..., X_n > x_n\right) = \frac{1}{1+\eta\left(x_1, x_2, ..., x_n\right)} \tag{2.1}$$

where $\eta\left(x_1, x_2, ..., x_n\right)$ satisfies the functional equation

$$\eta\left(x_1, x_2, ..., x_n\right) = \frac{1}{p}\eta\left(x_1 + \tfrac{1}{\alpha_1}\ln p, x_2 + \tfrac{1}{\alpha_2}\ln p, ..., x_n + \tfrac{1}{\alpha_n}\ln p\right),$$

$$0 < p < 1,\ \alpha_1, \alpha_2, ..., \alpha_n > 0,\ -\infty < x_1, x_2, ..., x_n < \infty \tag{2.2}$$

***Lemma* 2.1**

The solution of the functional equation (2.2) is given by

$$\eta\left(x_1, x_2, ..., x_n\right) = \exp\left\{\alpha_1 x_1\right\} h_1\left(x_1\right) + \exp\left\{\alpha_2 x_2\right\} h_2\left(x_2\right) + ... + \exp\left\{\alpha_n x_n\right\} h_n\left(x_n\right) \tag{2.3}$$

where $h_i\left(x_i\right), i = 1, 2, ..., n,$ are periodic functions in $x_i,\ i = 1, 2, ..., n$ with period $\frac{1}{\alpha_i}\ln p, i = 1, 2, ..., n,$ respectively.

Proof

Considering equations (2.2) and (2.3), we have

$$\begin{aligned}
\eta\left(x_1, x_2, ..., x_n\right) &= \frac{1}{p}\eta\left(x_1 + \tfrac{1}{\alpha_1}\ln p, x_2 + \tfrac{1}{\alpha_2}\ln p, ..., x_n + \tfrac{1}{\alpha_n}\ln p\right) \\
&= \frac{1}{p}\left[\exp\left\{\alpha_1\left(x_1 + \tfrac{1}{\alpha_1}\ln p\right)\right\} h_1\left(x_1\right) + ... + \exp\left\{\alpha_n\left(x_n + \tfrac{1}{\alpha_n}\ln p\right)\right\} h_n\left(x_n\right)\right] \\
&= \frac{1}{p}\left[p.\exp\left\{\alpha_1 x_1\right\} h_1\left(x_1\right) + ... + p.\exp\left\{\alpha_n x_n\right\} h_n\left(x_n\right)\right] \\
&= \exp\left\{\alpha_1 x_1\right\} h_1\left(x_1\right) + \exp\left\{\alpha_2 x_2\right\} h_2\left(x_2\right) + ... + \exp\left\{\alpha_n x_n\right\} h_n\left(x_n\right) \\
&= \eta\left(x_1, x_2, ..., x_n\right).
\end{aligned}$$

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ence the proof is complete.

For example if $h_i\left(x\right) = \exp\left\{\beta\cos\left(\alpha_i x\right)\right\}, i = 1, 2, ..., n,$ it satisfies (2.2) with $p = \exp\left\{-2\pi\right\}$ and $\eta\left(x_1, x_2, ..., x_n\right)$ is monotone increasing function in $x_i,\ i = 1, 2, ..., n$ with $0 < \beta < 1$.

In particular, if we choose $h_i\left(x_i\right) = 1,\ i = 1, 2, ..., n,$ the $MSL\left(\alpha_1, \alpha_2, ..., \alpha_n; p\right)$ reduces to a multivariate logistic distribution with survival function

$$\overline{F}\left(x_1, x_2, ..., x_n\right) = \frac{1}{1+\exp\left\{\alpha_1 x_1\right\} + \exp\left\{\alpha_2 x_2\right\} + ... + \exp\left\{\alpha_n x_n\right\}}, \alpha_i > 0,\ \forall i \tag{2.4}$$

Gumbel [4] proved that the bivariate logistic distribution having distribution function (1.2) is asymmetric. Similarly, it can be shown that the survival function (2.4) is also asymmetric.

3.0 Characterizations Properties

In this section, we study some characterizations properties of multivariate semi-logistic distribution, $MSL(\alpha_1, \alpha_2, ..., \alpha_n; p)$, using the method of geometric minimization procedure.

Let $\{(X_i^{(1)}, X_i^{(2)}, ..., X_i^{(n)}), i \geq 1\}$ be a sequence of independent and identically distributed multivariate random vectors following $MSL(\alpha_1, \alpha_2, \ldots, \alpha_n; p)$ distribution and N be a geometric random variable with parameter p and

$$P(N = n) = pq^{n-1}, \; n = 1, 2, ..., 0 < p < 1, \; q = 1 - p \tag{3.1}$$

and N is independent of $\{(X_i^{(1)}, X_i^{(2)}, ..., X_i^{(n)}), i \geq 1\}$. Define

$$\begin{aligned} U_N^{(1)} &= \min\left(X_1^{(1)}, X_2^{(1)}, ..., X_N^{(1)}\right) \\ U_N^{(2)} &= \min\left(X_1^{(2)}, X_2^{(2)}, ..., X_N^{(2)}\right) \\ \ldots & \qquad \ldots \\ U_N^{(n)} &= \min\left(X_1^{(n)}, X_2^{(n)}, ..., X_N^{(n)}\right) \end{aligned} \tag{3.2}$$

***Theorem* 3.1**

Let

$$\left\{\left(X_i^{(1)}, X_i^{(2)}, ..., X_i^{(n)}\right), i \geq 1\right\}$$

be a sequence of independent and identically distributed multivariate random vectors with common survival function $\bar{F}(x_1, x_2, \ldots, x_n)$ and N is geometric random variable as is (3.1) which is independent of $\{(X_i^{(1)}, X_i^{(2)}, ..., X_i^{(n)}), i \geq 1\}$. Then the random vectors

$$\left(U_N^{(1)} - \tfrac{1}{\alpha_1}\ln p, U_N^{(2)} - \tfrac{1}{\alpha_2}\ln p, ..., U_N^{(n)} - \tfrac{1}{\alpha_n}\ln p\right) \text{ and } \left\{\left(X_i^{(1)}, X_i^{(2)}, ..., X_i^{(n)}\right), i \geq 1\right\}$$

are identically distributed if and only if $\{(X_i^{(1)}, X_i^{(2)}, ..., X_i^{(n)}), i \geq 1\}$ has a $MSL(\alpha_1, \alpha_2, \ldots, \alpha_n; p)$ distribution.

Proof

Let

$$\begin{aligned} \bar{H}(x_1, x_2, ..., x_n) &= P\left(U_N^{(1)} - \tfrac{1}{\alpha_1}\ln p > x_1, ..., U_N^{(n)} - \tfrac{1}{\alpha_n}\ln p > x_n\right) \\ &= P\left(U_N^{(1)} > x_1 + \tfrac{1}{\alpha_1}\ln p, ..., U_N^{(n)} > x_n + \tfrac{1}{\alpha_n}\ln p\right) \\ &= \sum_{n=1}^{\infty}\left[\bar{F}\left(x_1 + \tfrac{1}{\alpha_1}\ln p, ..., x_n + \tfrac{1}{\alpha_n}\ln p\right)\right]^n pq^{n-1}. \end{aligned}$$

That is,

$$\bar{H}(x_1, x_2, ..., x_n) = \frac{p\bar{F}\left(x_1 + \frac{1}{\alpha_1}\ln p, ..., x_n + \frac{1}{\alpha_n}\ln p\right)}{1 - q\bar{F}\left(x_1 + \frac{1}{\alpha_1}\ln p, ..., x_n + \frac{1}{\alpha_n}\ln p\right)} \tag{3.3}$$

Now if $\bar{F}(x_1, x_2 ..., x_n)$ is as in (2.1) and (2.2), the equation (3.3) becomes

$$\bar{H}(x_1, x_2, ..., x_n) = \frac{1}{1 + \eta(x_1, x_2, ..., x_n)} = \bar{F}(x_1, x_2, ..., x_n).$$

Conversely, let

$$\overline{H}(x_1, x_2, ..., x_n) = \overline{F}(x_1, x_2, ..., x_n).$$

Any arbitrary survival function can be represented as

$$\overline{F}(x_1, x_2, ..., x_n) = \frac{1}{1+\psi(x_1, x_2, ..., x_n)}. \quad (3.4)$$

where $(x_1, x_2, ..., x_n)$ is a monotonically increasing function in $x_i, \quad i = 1, 2, ..., n.$ Using the representation (3.4) in (3.3) with $\overline{H}(x_1, x_2, ..., x_n) = \overline{F}(x_1, x_2, ..., x_n),$ we get the equation $\psi(x_1, x_2, ..., x_n) = \frac{1}{p}\psi\left(x_1 + \frac{1}{\alpha_1}\ln p, x_2 + \frac{1}{\alpha_2}\ln p, ..., x_n + \frac{1}{\alpha_n}\ln p\right).$ his is the functional equation (2.2) satisfied by $MSL(\alpha_1, \alpha_2, \ldots, \alpha_n; p)$. Hence the proof is complete.

Let $\{N_k, k \geq\}$be a sequence of geometric random variables with parameters p_k, $0 < p_k < 1$. Define

$$\overline{F}_k(x_1, x_2, ..., x_n) = P\left(U_{k-1}^{(1)} > x_1, U_{k-1}^{(2)} > x_2, ..., U_{k-1}^{(n)} > x_n\right), \quad k = 2, 3, ...$$

$$= \frac{p_{k-1}\overline{F}_{k-1}(x_1, x_2, ..., x_n)}{1-(1-p_{k-1})\overline{F}_{k-1}(x_1, x_2, ..., x_n)} \quad (3.5)$$

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ere we refer $\overline{F}_k$ as the survival function of the geometric (p_{k-1}) minimum of independent and identically distributed random vectors with $\overline{F}_{k-1}$ common survival function.

***Theorem* 3.2**

Let $\left\{\left(X_i^{(1)}, X_i^{(2)}, \ldots, X_i^{(n)}\right), i \geq 1\right\}$ be a sequence of independent and identically distributed multivariate random vectors with common survival function $\overline{F}(x_1, x_2, ..., x_n)$. Define $\overline{F}_1 = \overline{F}$ and $\overline{F}_k$ as the survival function of the geometric (p_{k-1}) minimum of independent and identically distributed random vectors with common survival function $\overline{F}_{k-1}, \ k = 2, 3, ...$ Then

$$\overline{F}_k\left(x_1 + \sum_{j=1}^{k-1}\frac{1}{\alpha_1}\ln p_j, x_2 + \sum_{j=1}^{k-1}\frac{1}{\alpha_2}\ln p_j, ..., x_n + \sum_{j=1}^{k-1}\frac{1}{\alpha_n}\ln p_j\right) = \overline{F}(x_1, x_2, ..., x_n) \quad (3.6)$$

if and only if $(X_1, X_2, \ldots, X_n)$ have $MSL(\alpha_1, \alpha_2, ..., \alpha_n; p)$ distribution.

Proof

By the definition the survival function $\overline{F}_k$ satisfies equation (3.5). As in (3.4) we can write

$$\overline{F}_k\left(x_1,x_2,...,x_n\right)=\frac{1}{1+\psi_k\left(x_1,x_2,...,x_n\right)}, \qquad k=1,2,3,...$$

Substituting this in (3.5), we have

$$\psi_k\left(x_1,x_2,...,x_n\right)=\frac{1}{p_{k-1}}\psi_{k-1}\left(x_1,x_2,...,x_n\right), \qquad k=2,3,...$$

Using this relation recursively, we get

$$\psi_k\left(x_1,x_2,...,x_n\right)=\frac{1}{\Pi_{j=1}^{k-1}p_j}\psi_1\left(x_1,x_2,...,x_n\right),$$

since

$$F_1=F \text{ implies } \psi_1=\psi.$$

This implies

$$\psi_k\left(x_1+\sum_{j=1}^{k-1}\tfrac{1}{\alpha_1}\ln p_j, x_2+\sum_{j=1}^{k-1}\tfrac{1}{\alpha_2}\ln p_j,...,x_n+\sum_{j=1}^{k-1}\tfrac{1}{\alpha_n}\ln p_j\right)=\frac{1}{\Pi_{j=1}^{k-1}p_j}\psi_1\left(x_1+\sum_{j=1}^{k-1}\tfrac{1}{\alpha_1}\ln p_j, x_2+\sum_{j=1}^{k-1}\tfrac{1}{\alpha_2}\ln p_j,...,x_n+\sum_{j=1}^{k-1}\tfrac{1}{\alpha_n}\ln p_j\right) \quad (3.7)$$

gives us (3.6) if we replace ψ_k *by* η_k and if we assume that η_1 satisfies (2.2). On the other hand, we assume that (3.6) holds. By the hypothesis of the theorem we have (3.7). Therefore (3.6) and (3.7) together lead to the equation,

$$\frac{1}{1+\frac{1}{\Pi_{j=1}^{k-1}p_j}\psi_1\left(x_1+\sum_{j=1}^{k-1}\tfrac{1}{\alpha_1}\ln p_j, x_2+\sum_{j=1}^{k-1}\tfrac{1}{\alpha_2}\ln p_j,...,x_n+\sum_{j=1}^{k-1}\tfrac{1}{\alpha_n}\ln p_j\right)}-\overline{F}\left(x_1,x_2,...,x_n\right)=\frac{1}{1+\eta\left(x_1,x_2,...,x_n\right)}$$

This implies that

$$\psi\left(x_1,x_2,...,x_n\right)=\frac{1}{\Pi_{j=1}^{k-1}p_j}\psi\left(x_1+\sum_{j=1}^{k-1}\tfrac{1}{\alpha_1}\ln p_j, x_2+\sum_{j=1}^{k-1}\tfrac{1}{\alpha_2}\ln p_j,...,x_n+\sum_{j=1}^{k-1}\tfrac{1}{\alpha_n}\ln p_j\right)$$

which is same as (2.2). Hence the proof of this theorem is complete.

***Lemma* 3.1**

We say that a random vector $\left(X_1,X_2,...,X_n\right)$ R^n has multivariate semi extreme value distribution if its survival function is

$$\overline{F}\left(x_1,x_2,...,x_n\right)=\exp\left\{-\eta\left(x_1,x_2,...,x_n\right)\right\}$$

where $\eta\left(x_1,x_2,...,x_n\right)$ satisfies the functional equation (2.2).

The following theorem gives the relationship between multivariate semi-logistic distribution and multivariate semi-extreme value distribution.

***Theorem* 3.3**

If $\left\{\left(X_i^{(1)}, X_i^{(2)}, \ldots, X_i^{(n)}\right), i \geq 1\right\}$ are independent and identically distributed $MSL(\alpha_1, \alpha_2, \ldots, \alpha_n; p)$ random vectors then

$$\left(Z_n^{(1)}, Z_n^{(2)}, \ldots, Z_n^{(n)}\right) = \left\{\min\left(X_1^{(1)} - \tfrac{1}{\alpha_1}\ln\tfrac{1}{n}, X_1^{(2)} - \tfrac{1}{\alpha_1}\ln\tfrac{1}{n}, \ldots, X_1^{(n)} - \tfrac{1}{\alpha_1}\ln\tfrac{1}{n}\right), \ldots, \min\left(X_n^{(1)} - \tfrac{1}{\alpha_n}\ln\tfrac{1}{n}, X_n^{(2)} - \tfrac{1}{\alpha_n}\ln\tfrac{1}{n}, \ldots, X_n^{(n)} - \tfrac{1}{\alpha_n}\ln\tfrac{1}{n}\right)\right\},$$

$$\alpha_1, \alpha_2, \ldots, \alpha_n > 0, \; n > 1, \; n > \alpha_i \left(i = 1, 2, \ldots, n\right)$$

is asymptotically distributed as multivariate semi-extreme value.

Proof

If the random vector $\left(X_1, X_2, \ldots, X_n\right)$ is distributed as $MSL\left(\alpha_1, \alpha_2, \ldots, \alpha_n; p\right)$ then

$\overline{F}\left(x_1, x_2, \ldots, x_n\right) = \dfrac{1}{1 + \eta\left(x_1, x_2, \ldots, x_n\right)}$ where $\eta\left(x_1, x_2, \ldots, x_n\right)$ satisfies (2.2).

$$\overline{G}\left(x_1, x_2, \ldots, x_n\right) = P\left\{\min\left(X_1^{(1)} - \tfrac{1}{\alpha_1}\ln\tfrac{1}{n}, \ldots, X_1^{(n)} - \tfrac{1}{\alpha_1}\ln\tfrac{1}{n}\right) > x_1, \ldots, \min\left(X_n^{(1)} - \tfrac{1}{\alpha_n}\ln\tfrac{1}{n}, \ldots, X_n^{(n)} - \tfrac{1}{\alpha_n}\ln\tfrac{1}{n} > x_n\right)\right\}$$

$$= \left[\overline{F}\left(x_1 + \tfrac{1}{\alpha_1}\ln\tfrac{1}{n}, x_2 + \tfrac{1}{\alpha_2}\ln\tfrac{1}{n}, \ldots, x_n + \tfrac{1}{\alpha_n}\ln\tfrac{1}{n}\right)\right]^n$$

where the minimum is taken in component wise

$$\overline{G}\left(x_1, x_2, \ldots, x_n\right) = \left[\frac{1}{1 + \frac{\eta\left(x_1, x_2, \ldots, x_n\right)}{n}}\right]^n.$$

Taking limit when n approaches infinity, we have

$$G\left(x_1, x_2, \ldots, x_n\right) = \exp\left\{-\eta\left(x_1, x_2, \ldots, x_n\right)\right\}.$$

This gives the Proof of this theorem.

4.0 First order autoregressive multivariate semi-logistic process (MSL-AR(1))

Let

$$\left\{\left(\varepsilon_{n1}, \varepsilon_{n2}, \ldots, \varepsilon_{nn}\right), n \geq 1\right\}$$

be a sequence of independent and identically distributed multivariate real random vectors. Then

the (MSL-AR(1)) minification process $\left\{\left(X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(n)}\right), n \geq 0\right\}$ is defined as

$$X_n^{(1)} = \min\left(X_{n-1}^{(1)} - \tfrac{1}{\alpha_1}\ln p, \varepsilon_{n1}\right)$$

$$X_n^{(2)} = \min\left(X_{n-1}^{(2)} - \tfrac{1}{\alpha_2}\ln p, \varepsilon_{n2}\right)$$

$$\ldots \qquad \ldots$$

$$X_n^{(n)} = \min\left(X_{n-1}^{(n)} - \tfrac{1}{\alpha_n}\ln p, \varepsilon_{nn}\right), \; n \geq 1, 0 \leq p \leq 1, \; \alpha_1, \alpha_2, \ldots, \alpha_n > 0 \qquad (4.1)$$

Assume that $\left(X_0^{(1)}, X_0^{(2)}, ..., X_0^{(n)}\right)$ is independent of $\left\{\left(\varepsilon_{n1}, \varepsilon_{n2}, ..., \varepsilon_{nn}\right)\right\}$. Then it follows that

$\left\{\left(X_n^{(1)}, X_n^{(2)}, ..., X_n^{(n)}\right), n \geq 0\right\}$ is a multivariate Markov sequence. The $\varepsilon_{n1}, \varepsilon_{n2}, ..., \varepsilon_{nn}$ are real random variables assume that either all are infinite with probability p or all are finite with probability 1 - *p*. Hence they represented the process as

$$\left(\varepsilon_{n1}, \varepsilon_{n2}, ..., \varepsilon_{nn}\right) = \begin{cases} \text{infinite with probability } p \\ \left(\omega_{n1}, \omega_{n2}, ..., \omega_{nn}\right) \text{with probability} 1 - p \end{cases} .$$

***Theorem* 4.1**

Assume that

$$\left(X_0^{(1)}, X_0^{(2)}, ..., X_0^{(n)}\right) \underline{\underline{d}} \left(\omega_{11}, \omega_{12}, ..., \omega_{1n}\right).$$

The process defined by (4.1) is stationary if and only if it has multivariate semi-logistic distribution.

Proof

Consider equation (4.1),

$$\overline{G}_n\left(x_1, x_2, ..., x_n\right) = P\left(X_n^{(1)} > x_1, X_n^{(2)} > x_2, ..., X_n^{(n)} > x_n\right)$$

$$= \overline{G}_{n-1}\left(x_1 + \tfrac{1}{\alpha_1}\ln p, x_2 + \tfrac{1}{\alpha_2}\ln p, ..., x_n + \tfrac{1}{\alpha_n}\ln p\right)\left[p + (1-p)\overline{F}\left(x_1, x_2, ..., x_n\right)\right] \quad (4.2)$$

where $\overline{F}\left(x_1, x_2, ..., x_n\right)$ is the survival function of $\left(\omega_{11}, \omega_{12}, ..., \omega_{1n}\right)$

Assume that $\left\{\left(X_n^{(1)}, X_n^{(2)}, ..., X_n^{(n)}\right), n \geq 0\right\}$

is stationary and $\left(X_0^{(1)}, X_0^{(2)}, ..., X_0^{(n)}\right) \underline{\underline{d}} \left(\omega_{11}, \omega_{12}, ..., \omega_{1n}\right)$. Then for $n = 1$, (4.2) gives

$$\overline{G}_0\left(x_1 + \tfrac{1}{\alpha_1}\ln p, x_2 + \tfrac{1}{\alpha_2}\ln p, ..., x_n + \tfrac{1}{\alpha_n}\ln p\right) = \frac{\overline{F}\left(x_1, x_2, ..., x_n\right)}{p + (1-p)\overline{F}\left(x_1, x_2, ..., x_n\right)} \quad (4.3)$$

If we write

$$\overline{F}\left(x_1, x_2, ..., x_n\right) = \frac{1}{1 + \eta\left(x_1, x_2, ..., x_n\right)},$$

the equation (4.3) gives the relation

$\eta\left(x_1, x_2, ..., x_n\right) = \tfrac{1}{p}\eta\left(x_1 + \tfrac{1}{\alpha_1}\ln p, x_2 + \tfrac{1}{\alpha_2}\ln p, ..., x_n + \tfrac{1}{\alpha_n}\ln p\right)$. That is $\overline{F}\left(x_1, x_2, ..., x_n\right)$ is the form of (2.2) and hence by (4.2), the random vector $\left(X_1^{(1)}, X_1^{(2)}, ..., X_1^{(n)}\right)$ is distributed as *MSL*(α_1, α_2, …, α_n; *p*). By induction, we can show that

$$\left\{\left(X_n^{(1)}, X_n^{(2)}, ..., X_n^{(n)}\right), n \geq 0\right\}$$

is a $MSL(\alpha_1, \alpha_2, \ldots, \alpha_n; p)$ Markov sequence.

Conversely, assume that $\left\{\left(\varepsilon_{n1}, \varepsilon_{n2}, \ldots, \varepsilon_{nn}\right), n \geq 0\right\}$ is a sequence of independent and identically distributed $MSL(\alpha_1, \alpha_2, \ldots, \alpha_n; p)$ random vectors and

$$\left(X_0^{(1)}, X_0^{(2)}, \ldots, X_0^{(n)}\right) \underline{\underline{d}} \left(\omega_{11}, \omega_{12}, \ldots, \omega_{1n}\right).$$

For $n = 1$, (4.2) leads to

$$\overline{G}_1\left(x_1, x_2, \ldots, x_n\right) = \overline{F}\left(x_1 + \tfrac{1}{\alpha_1}\ln p, x_2 + \tfrac{1}{\alpha_2}\ln p, \ldots, x_n + \tfrac{1}{\alpha_n}\ln p\right)\left[p + (1-p)\overline{F}\left(x_1, x_2, \ldots, x_n\right)\right].$$

If we write

$$\overline{F}\left(x_1, x_2, \ldots, x_n\right) = \frac{1}{1 + \eta\left(x_1, x_2, \ldots, x_n\right)}$$

and applying (2.2), we have

$$\overline{G}_1\left(x_1, x_2, \ldots, x_n\right) = \frac{1}{1 + \eta\left(x_1, x_2, \ldots, x_n\right)}.$$

That is, $\left(X_1^{(1)}, X_1^{(2)}, \ldots, X_1^{(n)}\right)$ has $MSL\left(\alpha_1, \alpha_2, \ldots, \alpha_n; p\right)$ distribution. Again using (4.2), by induction we can prove that $\left\{\left(X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(n)}\right), n \geq 0\right\}$ is a sequence of $MSL\left(\alpha_1, \alpha_2, \ldots, \alpha_n; p\right)$ random vectors. That is $\left\{\left(X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(n)}\right), n \geq 0\right\}$ is a stationary $MSL\left(\alpha_1, \alpha_2, \ldots, \alpha_n; p\right)$ sequence. This gives the complete proof.

***Corollary* 4.1**

Let

$$\left\{\left(U_n^{(1)}, U_n^{(2)}, \ldots, U_n^{(n)}\right), n \geq 0\right\}$$

be an arbitrary random vector with survival function $\overline{G}_0\left(u_1, u_{2,}, \ldots, u_n\right)$ *and* $\left\{\left(\omega_n^{(1)}, \omega_n^{(2)}, \ldots, \omega_n^{(n)}\right), n \geq 0\right\}$ is a sequence of independent and identically distributed $MSL\left(\alpha_1, \alpha_2, \ldots, \alpha_n; p\right)$ random vectors. Then the multivariate sequence $\left\{\left(X_n^{(1)}, X_n^{(2)}, \ldots, X_n^{(n)}\right), n \geq 0\right\}$ defined by (4.1), converges in distribution to $MSL(\alpha_1, \alpha_2, \ldots, \alpha_n; p)$ as $n \to \infty$

Proof

The model defined in (4.1) and the relations (4.2), (2.1) and (2.2) together lead to

$$G_n(x_1,x_2,...,x_n)=G_0\left(x_1+\tfrac{1}{\alpha_1}\ln p, x_2+\tfrac{1}{\alpha_2}\ln p,...,x_n+\tfrac{1}{\alpha_n}\ln p\right)\left[\frac{1+p^n\eta(x_1,x_2,...,x_n)}{1+\eta(x_1,x_2,...,x_n)}\right]\to\frac{1}{1+\eta(x_1,x_2,...,x_n)},$$ as $n\to\infty$.

5.0 Autoregressive multivariate semi-logistic process of order k (MSL-AR(k))

In this section, we rather attempt to give the generalization of first order autoregressive minification process presented in the previous section. Let $\{(\varepsilon_{n1},\varepsilon_{n2},...,\varepsilon_{nk}), n\geq 1\}$ be a sequence of independent and identically distributed multivariate real random vectors. Then the (MSL-AR(*k*)) minification process $\left\{\left(X_n^{(1)},X_n^{(2)},...,X_n^{(k)}\right), n\geq 0\right\}$ is given as:

$$\begin{aligned}
X_n^{(1)} &= \min\left(X_{n-1}^{(1)},X_{n-1}^{(2)},...,X_{n-1}^{(k)},\varepsilon_{n1}\right)\\
X_n^{(2)} &= \min\left(X_{n-1}^{(1)},X_{n-1}^{(2)},...,X_{n-1}^{(k)},\varepsilon_{n2}\right)\\
&\cdots \qquad\qquad \cdots\\
X_n^{(k)} &= \min\left(X_{n-1}^{(1)},X_{n-1}^{(2)},...,X_{n-1}^{(k)},\varepsilon_{nk}\right)
\end{aligned}$$

with $\{(\varepsilon_{n1},\varepsilon_{n2},...,\varepsilon_{nk}), n\geq 1\}$ defined as in Section 4.

Note that Theorem 4.1 as well as Corollary 4.1 also hold for the multivariate minification processes of order *k* that have multivariate semi-logistic distribution as stationary marginal distribution.

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