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Null controllability of nonlinear neutral volterra integrodifferential systems with infinite delay

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Abstract

Sufficient conditions are developed for the null controllability of nonlinear neutral Volterra integrodifferential systems with infinite delay. It is shown that if the uncontrolled system is uniformly asymptotically stable, and if the linear system is controllable, then the nonlinear infinite neutral system is null controllable

Keywords
Controllability, neutral Volterra integrodifferential system, infinite delay, uniform asymptotic stability.

1.0 Introduction

It is well known that the future state of realistic models in the natural sciences, economics and engineering depend not only on the present but on the past state and the derivative of the past state [3]. Such models which contain past information are called hereditary systems.

Neutral functional differential equations are characterized by a delay in the derivative. Equations of this type have applications in many areas of applied mathematics [6]. A good guide concerning the literature for neutral functional differential equations is the Hale and Verduyn [5] book and the references therein.

Controllability is the property of being able to steer between two arbitrary points in the state space. On the other hand, null controllability is the property of being able to steer all points exactly to the origin. This has important connections with the concept of stabilizability [2].

Investigation into the controllability of functional differential systems to the origin has attracted great attention in recent years with the growing interest in disease control models in which the number of infected individuals is desired to be controlled to zero [7]. Balachandran and Dauer [1] studied the null controllability of nonlinear infinite delay systems with time varying multiple delays in control whereas Balachandran and Leelamani [2] investigated the null controllability of neutral evolution integrodifferential systems with infinite delay. Iyai [8] discussed the Euclidean null controllability of infinite neutral differential systems. Sinha [11]

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derived a set of sufficient conditions for the null controllability of nonlinear infinite delay systems with restrained controls. Onwuatu [10] studied the null controllability of nonlinear infinite neutral system. Eke [4] established a set of conditions for the null controllability of linear control systems. Umana and Nse [13] studied the null controllability of nonlinear integrodifferential systems with delays whereas Umana [12] discussed the relative null controllability of linear systems with multiple delays in state and control.

The aim of this paper is to study the null controllability of nonlinear neutral Volterra integrodifferential systems with infinite delay on a finite interval $J=[t_0,t_1]$, $t_1>t_0$, when its linear system is assumed controllable and its uncontrolled system is uniformly asymptotically stable with the assumption that the perturbation function satisfies some smoothness and growth conditions.

2.0 Preliminaries

Let n and m be positive integers, R the real line $(-\infty,\infty)$. We denote by R^n, the space of real $n-$tuples with the Euclidean norm defined by $|\cdot|$. If J is any interval of R, the usual Lebesgue space of square integrable functions from J to R^n will be denoted by $L_2(J,R^n)$.

Let $\gamma \ge h > 0$ be a given real number and let $B = B\left([-\gamma,0],R^n\right)$ be the Banach space of functions which are continuous and bounded on $[-\gamma,0]$ with $\|\phi\| = \sup_{-\gamma\le s<0}|\phi(s)|$, $\phi \in B\left([-\gamma,0],R^n\right)$.

Consider the nonlinear neutral Volterra integrodifferential systems with infinite delay of the following form

$$\frac{d}{dt}\left[x(t)-\int_{-\infty}^{t}C(t-s)x(s)ds-g(t)\right]=Ax(t)+\int_{-\infty}^{t}G(t-s)x(s)ds$$
$$+B(t)u(t)+f\left(t,x(t),u(t)\right) \qquad (2.1)$$

$x(t)=\phi(t),\quad t\in(-\infty,0]$ where the initial function ϕ is continuous and bounded on R^n and where $x\in R^n$, $u\in R^m$, $C(t)$ is an $n\times n$ continuously differentiable matrix valued function, $G(t)$ is an $n\times n$ continuous matrix, and $B(t)$ is a continuous $n\times m$ matrix function, A is a constant $n\times n$ matrix and f and g are respectively continuous and absolutely continuous $n-$vector functions.

We shall show that if the free system

$$\frac{d}{dt}\left[x(t)-\int_{-\infty}^{t}C(t-s)x(s)ds-g(t)\right]=Ax(t)+\int_{-\infty}^{t}G(t-s)x(s)ds \qquad (2.2)$$

is uniformly asymptotically stable, and if the linear control system

$$\frac{d}{dt}\left[x(t)-\int_{-\infty}^{t}C(t-s)x(s)ds-g(t)\right]=Ax(t)+\int_{-\infty}^{t}G(t-s)x(s)ds+B(t)u(t) \qquad (2.3)$$

is completely controllable, then system (2.1) is null controllable provided the continuous function *f* satisfies some smoothness and growth conditions.

Equivalently, system (2.1) takes the form

$$\frac{d}{dt}\left[x(t) - \int_{-\infty}^{t} C(t-s)x(s)ds - g(t) - \int_{-\infty}^{0} C(t-s)f(s)ds\right] = Ax(t) + \int_{-\infty}^{t} G(t-s)x(s)ds$$
$$+ \int_{-\infty}^{0} G(t-s)f(s)ds + B(t)u(t) + f\big(t, x(t), u(t)\big) \qquad (2.4)$$

The solution of (2.4) can be written as in Wu [14]:

$$x(t) = Z(t)\left[x(0) - g(0) - \int_{-\infty}^{0} C(-s)f(s)ds\right] + g(t) + \int_{-\infty}^{0} C(t-s)f(s)ds$$
$$+ \int_{0}^{t} \dot{Z}(t-s)\left[g(s) + \int_{-\infty}^{0} C(s-\tau)d\tau\right]ds + \int_{0}^{t} Z(t-s)\left[\int_{-\infty}^{0} G(s-\tau)f(\tau)d\tau\right]ds$$
$$+ \int_{0}^{t} Z(t-s)\left[B(s)u(s) + f\big(s, x(s), u(s)\big)\right]ds \qquad (2.5)$$

where $\dot{Z}(t-s) = \partial/\partial t\, Z(t-s)$ and $Z(t)$ is an $n \times n$ continuously differentiable matrix satisfying the equation

$$\frac{d}{dt}\left[Z(t) - \int_{0}^{t} C(t-s)Z(s)ds - \int_{-\infty}^{0} C(t-s)Z(s)ds\right] = AZ(t) + \int_{-\infty}^{t} G(t-s)Z(s)ds$$
$$+ \int_{-\infty}^{0} G(t-s)Z(s)ds$$

with $Z(0) = I$, the identity matrix.

In (2.5) set the matrix function $Z(t-s)B(s) = Y(t,s)$ and define the controllability matrix W by

$$W(t_0, t) = \int_{t_0}^{t} Y(t,s)Y^T(t,s)ds$$

where T denotes matrix transpose.

***Definition* 2.1**

The system (2.1) is said to be null controllable on J if for each $\phi \in B\big([-\gamma, 0], R^n\big)$, there exists a $t_1 \geq t_0$, $u \in L_2\big([t_0, t_1], IU\big)$, IU a compact convex subset of R^m, such that the solution $x(t) = x(t, t_0, \phi, u, f)$ of (2.1) satisfies $x_{t_0}(\cdot, t_0, \phi, u, f) = \phi$ and $x(t_1, t_0, \phi, u, f) = 0$.

3.0 Main results

***Theorem* 3.1**

For system (2.1) assume that the constraint set IU is an arbitrary compact subset of R^n, and that

(i) the solution (2.2) is uniformly asymptotically stable so that the solution of (2.2) satisfies $\|x(t, t_0, \phi, 0, 0)\| \leq Me^{-\alpha(t-t_0)}\|\phi\|$ for some $\alpha > 0$, $M > 0$,

(ii) the linear control system (2.3) is controllable,

(iii) the continuous function f satisfies the following smoothness and growth conditions $\big|f\big(t, x(\cdot), u(\cdot)\big)\big| \leq \exp(-\beta t)\pi\big(x(\cdot), u(\cdot)\big)$, for all $\big(t, x(\cdot), u(\cdot)\big) \in [t_0, \infty) \times B \times L_2$,

where $\int_{t_0}^{\infty} \pi\big(x(\cdot), u(\cdot)\big)ds \leq K < \infty$ and $\beta - \alpha \geq 0$, then system (2.1) is Euclidean null controllable.

Proof

By (ii), $W^{-1}(t_0,t_1)$ exists for each $t_1 > t_0$. Suppose the pair of functions x, u form a solution pair to the set of integral equations

$$u(t) = -Y^T(t_1,t)W^{-1}(t_0,t)\Big\{ Z(t)\Big[x(0) - g(0) - \int_{-\infty}^{0} C(-s)f(s)ds\Big] + g(t)$$

$$+\int_{-\infty}^{0} C(t-s)f(s)ds + \int_{t_0}^{t} \dot{Z}(t-s)\Big[g(s) + \int_{-\infty}^{0} C(s-\tau)f(\tau)d\tau\Big]ds$$

$$+\int_{t_0}^{t} Z(t-s)\Big[\int_{-\infty}^{0} G(s-\tau)f(\tau)d\tau\Big]ds + \int_{t_0}^{t} Z(t-s)f\big(s,x(s),u(s)\big)ds\Big\} \qquad (3.1)$$

for some suitably chosen $t_1 \geq t \geq t_0$, and

$$x(t) = Z(t)\Big[x(0) - g(0) - \int_{-\infty}^{0} C(-s)f(s)ds\Big] + g(t) + \int_{-\infty}^{0} C(t-s)f(s)ds$$

$$+\int_{t_0}^{t} \dot{Z}(t-s)\Big[g(s) + \int_{-\infty}^{0} C(s-\tau)f(\tau)d\tau\Big]ds + \int_{t_0}^{t} Z(t-s)\Big[\int_{-\infty}^{0} G(s-\tau)f(\tau)d\tau\Big]ds$$

$$+\int_{t_0}^{t} Y(t,s)u(s)ds + \int_{t_0}^{t} Z(t-s)f\big(s,x(s),u(s)\big)ds \qquad (3.2)$$

$$x(t) = \phi(t), \quad t \in [t_0-\gamma, t_0].$$

Then u is square integrable on $[t_0-\gamma, t_1]$ and x is a solution of (2.1) corresponding to u with initial state $x(t_0) = \phi$. Also, $x(t_1) = 0$. Now it is shown that $u:[t_0,t_1] \to IU$ is in a compact constraint subset of R^m, that is $|u| \leq a$ for some constant $a > 0$. By (i), $\left|Y^T(t_1,t_0)W^{-1}(t_0,t_1)\right| \leq k_1$ for some $k_1 > 0$, and $\left|Z(t)\Big[x(0) - g(0) - \int_{-\infty}^{0} C(-s)f(s)ds\Big] + g(t)\right|$ $\leq k_2 \exp\big[-\alpha(t_1,t_0)\big]$ for some $k_2 > 0$. Hence,

$$|u(t)| \leq k_1\Big[k_2 \exp\big[-\alpha(t_1,t_0)\big] + \int_{t_0}^{t_1} M \exp\big[-\alpha(t_1-s)\big]\exp(-\beta s)p\big(x(\cdot),u(\cdot)\big)ds\Big].$$

Thus

$$|u(t)| \leq k_1\Big[k_2 \exp\big[-\alpha(t_1-t_0)\big] + KM\exp(-\alpha t_1)\Big], \qquad (3.3)$$

since $\beta - \alpha \geq 0$ and $s \geq t_0 \geq 0$. From (3.3), t_1 can be chosen so large that $|u(t)| \leq a$, $t \in [t_0,t_1]$ which proves that u is an admissible control for this choice of t_1. We now prove the existence of a solution pair of the integral equations (3.1) and (3.2). Let B be the Banach space of all functions

$$(x,u):[t_0-h,t_1]\times[t_0-h,t_1] \to R^n \times R^m$$

where $x \in B\left([t_0-h,t_1],R^n\right)$ and $u \in L_2\left([t_0-h,t_1],R^m\right)$ with the norm defined by $\|(x,u)\| \le \|x\|_2 + \|u\|_2$, where $\|x\|_2 = \left\{\int_{t_0-h}^{t_1} |x(s)|^2\, ds\right\}^{1/2}$ and $\|u\|_2 = \left\{\int_{t_0-h}^{t_1} |u(s)|^2\, ds\right\}^{1/2}$. Define the operator $T: B \to B$ by $T(x,u) = (y,v)$, where

$$v(t) = -Y^T(t_1,t)W^{-1}(t_0,t_1)\Big\{ Z(t)\Big[x(0) - g(0) - \int_{-\infty}^{0} C(-s)f(s)ds\Big] + g(t)$$
$$+ \int_{-\infty}^{0} C(t-s)f(s)ds + \int_{t_0}^{t} \dot{Z}(t-s)\Big[g(s) + \int_{-\infty}^{0} C(s-\tau)f(\tau)d\tau\Big]ds$$
$$+ \int_{t_0}^{t} Z(t-s)\Big[\int_{-\infty}^{0} G(s-\tau)f(\tau)d\tau\Big]ds + \int_{t_0}^{t} Z(t-s)\, f\big(s,x(s),u(s)\big)ds\Big\} \qquad (3.4)$$

for $t \in J = [t_0,t_1]$; $y(t) = Z(t)\Big[x(0) - g(0) - \int_{-\infty}^{0} C(-s)f(s)ds\Big] + g(t) + \int_{-\infty}^{0} C(t-s)f(s)ds$

$$+ \int_{t_0}^{t} \dot{Z}(t-s)\Big[g(s) + \int_{-\infty}^{0} C(s-\tau)f(\tau)d\tau\Big]ds + \int_{t_0}^{t} Z(t-s)\Big[\int_{-\infty}^{0} G(s-\tau)f(\tau)d\tau\Big]ds$$
$$+ \int_{t_0}^{t} Y(t,s)v(s)ds + \int_{t_0}^{t} Z(t-s)\, f\big(s,x(s),u(s)\big)ds \qquad (3.5)$$

for $t \in J$ and $y(t) = \phi(t)$, $t \in [t_0-\gamma, t_0]$.

From equation (3.3) it is clear that $|v(t)| \le a$, $t \in J$ and also $v:[t_0-h,t_0] \to IU$, so $|v(t)| \le a$. Hence $\|v\|_2 \le a(t_1+h-t_0)^{1/2} = \beta_0$.

Next, $|y(t)| \le k_2 \exp[-\alpha(t_1-t_0)] + k_4 \int_{t_0}^{t} |v(s)|ds + KM\exp(-\alpha t_1)$ where $k_4 = \sup|Y(t,s)|$.

Since $\alpha > 0$, $t \ge t_0 \ge 0$, we deduce that $|y(t)| \le k_2 + k_4 a(t_1-t_0) + KM \equiv \beta$, $t \in J$ and $|y(t)| \le \sup|\phi(t)| \equiv \delta$, $t \in [t_0-\gamma, t_0]$. Hence, if $\lambda = \max\{\beta,\delta\}$, then $\|y\|_2 \le \lambda(t_1 + h + t_2)^{1/2} \equiv \beta_1$.

Let $r = \max\{\beta_0,\beta_1\}$. Then if we let $Q(r) = \left\{(x,u) \in B: \|x\|_2 \le r,\ \|u\|_2 \le r\right\}$ we have proved that $T: Q(r) \to Q(r)$. Since $Q(r)$ is closed, bounded and convex, by Riesz's theorem [9], it is relatively compact under the transformation T. The Schauder theorem implies that T has a fixed point $(x,u) \in Q(r)$. This fixed point (x,u) of T is a solution pair of the set of integral equations (3.4), (3.5). hence, the system (2.1) is Euclidean null controllable.

4.0 Conclusion

The paper contains sufficient conditions for the null controllability of nonlinear neutral Volterra integrodifferential systems with infinite delay. These conditions are given with respect to the uniform asymptotic stability of the free linear base system and the controllability of the linear controllable base system, with the assumption that the perturbation function f satisfies some smoothness and growth conditions.

References

[1] Balachandran and J. P. Dauer, Null controllability of nonlinear infinite delay systems with time varying multiple delays in control, *Appl. Math. Lett.*, 9 (1996), 115 – 121.

[2] Balachandran and A. Leelamani, Null controllability of neutral evolution integrodifferential systems with infinite delay, *Math. Prob. Engineering*, 2006 (2006), 1 – 18.

[3] N. Chukwu, *Stability and Time-Optimal Control of Hereditary systems*, Academic Press, New York, 1992.

[4] N. Eke, Null controllability for linear control systems, *J. Inst. Math. Comp. Sci. (Math. Ser.)*, 3 (1990), 149 – 154.

[5] J. K. Hale and S. M. Verduyn Lunel, *Introduction to Functional Differential Equations*, Applied Mathematical Sciences, 99, Springer-Verlag, New York, 1993.

[6] E. Hernandez, A remark on neutral partial differential equations, *Cadernos De Mathematica*, 4 (2003), 311 – 318.

[7] V. A. Iheagwam and J. U. Onwuatu, Relative controllability and null controllability of linear delay systems with distributed delays in the state and control, *J. Nigerian Asso. Math. Physics*, 9 (2005) 221 – 238.

[8] D. Iyai, Euclidean null controllability of infinite neutral differential systems, *ANZIAM J.*, 48 (2006), 285 – 293.

[9] L. V. Kantarovich and G. P. Akilov, *Functional Ana*lysis, Pergamon Press, Oxford, 1982.

[10] J. U. Onwuatu, Null controllability of nonlinear infinite neutral system, *Kybernetika*, 29 (1993), 325 – 336.

[11] S. C. Sinha, Null controllability of nonlinear infinite delay systems with restrained controls, *Int. J. Control*, 42 (1985), 735 – 741.

[12] A. Umana, Relative null controllability of linear systems with multiple delays in state and control, *J. Nigerian Asso. Math. Physics*, 10 (2006), 517 – 522.

[13] A. Umana and C. A. Nse, Null controllability of nonlinear integrodifferential systems with delays, *Advances in Modelling (Automatic Control)*, 61 (2006), 73 – 84.

[14] J. Wu, Globally stable periodic solutions of linear neutral Volterra integrodifferential equations, *J. Math. Anal. Appl.*, 130 (1988), 474 – 483.