Journal of the Nigerian Association of Mathematical Physics
Volume **15 (November, 2009), pp 85 - 90**

Computational procedures for implementing the optimal control problem of higher-order nondispersive wave using the extended conjugate gradient algorithm

[1]Ifeanyi S. Ukwosa, [2]Sunday A. Reju and [3]C. Isreal-Cookey
[1,3]Department of Mathematics and Computer Science
Rivers State University of Science and Technology, Nkpolu
Port Harcourt, Rivers State, Nigeria.
[2]National Open University of Nigeria
Ahmadu Bello Way, Victoria Island, Lagos, Nigeria

Abstract

The Extended Conjugate Gradient Method, ECGM,* [1] *was used to compute the control and state gradients of the unconstrained optimal control problem for higher-order nondispersive wave. Also computed are the descent directions for both the control and the state variables. These functions are the most important ingredients for implementing the Extended Conjugate Gradient Algorithm..

Keywords
Optimal control, state variable, control variable, descent direction, state and control gradients.

1.0 Introduction

The optimal control problem of higher-order nondispersive wave was formulated in [5] and given as

$$\min J(z,u,\mu) = \min \iint \int [z^2(x,y,t) + u^2(x,y,t)]dxdydt$$

such that
$$\frac{\partial^2 z(x,y,t)}{\partial t^2} = -c^2 \frac{\partial^2}{\partial x^2} z(x,y,t) + u(x,y,t) \qquad (1.1)$$

$$z(0,y,t) = z(\lambda,y,t) = 0, z(x,y,0) = z_0(x,y), z_t(x,y,0) = z_1(x,y)$$

Equation (1.1) is a constrained optimal control problem. For the Extended Conjugate Gradient Method [1] to be applied we need a function, $\mu > 0$ called the penalty function, in order to convert equation (1.1) to an unconstrained optimal control problem. Thus we have

$$J(z,u,\mu) = \int_0^1 \int_0^1 \int_0^1 \{(z^2 + u^2) + \mu \left\| u - z_{tt} - c_0^2 z_{xx} - c_0^2 z_{yy} \right\|^2 \} dxdydt \qquad (1.2)$$

Corresponding author:
[1]e-mail address: nwalohurho@yahoo.com
[1]Telephone: 08068482824
[2]e-mail: sunnyareju@yahoo.com

where $z = z(x,y,t)$, $u = u(x,y,t)$ and $\mu = \mu(x,y,t)$. Using a two-dimensional operator, R, say we can express equation (1.2) in the form

$$\min < v, Rv >_H \;=\; \min J(z,u,\mu) \tag{1.3}$$

where $v = v(z,u)$ and H is a Hilbert space of continuous functions that are square integrable and of equivalence classes [4]. With this property endowed on the functions z and u through their association in v equation (1.3) can be expanded in a bilinear form. Thus we have

$$J(z,u,\mu) = \iiint \left\{ z_1 \bar{z}_2 + z_{1tt} z_{2tt} + z_{1xx} z_{2xx} + z_{1yy} \bar{z}_{2xx} + u_1 z_{2xx} \right\} dtdxdy \tag{1.4}$$

where $z_2 = \bar{z}_2$, $z_{2tt} = (\mu z_{2tt} - \mu c_0 z_{2xx} - \mu c_0 z_{2yy} - \mu u_2)$,

$$\bar{z}_{2xx} = \mu z_{2tt} - \mu c_0 z_{2xx} - \mu c_0 z_{2yy} - \mu u_2 \quad \text{and}$$

$$z_{2xx} = (-\mu c_0 z_{2tt} + \mu C_0^2 z_{2xx} + \mu c_0^2 z_{2yy} + \mu c_0 u_2)$$

The Extended Conjugate Gradient Algorithm constructed by [1] was given as

***Algorithm* 1.1**

Step 1: Guess x_0, u_0

Step 2: Compute g_0

For $i = 0$

Step 3: $p_0 = -g_0$

Step 4: $x_{i+1} = x_i + \alpha_i A p_i$

$u_{i+1} = u_i + \alpha_i p_i$;

where $\alpha_i = \dfrac{< g_i, g_i >}{< p_i, Ap_i >}$

For $i = 1, 2, \ldots$

Step 5: $g_{x,i+1} = g_{x,i} + \alpha_i A p_{x,i}$

$g_{u,i+1} = g_{u,i} + \alpha_i A p_{u,i}$

$p_{x,i+1} = p_{x,i} + \beta_i p_{x,i}$

$p_{x,i+1} = p_{x,i} + \beta_i p_{x,i}$; where $\beta_i = \dfrac{< g_{i+1}, g_{i+1} >}{< g_i, g_i >}$

For us to use Algorithm 1.1 to solve equation (1.2) our operator R now replaces the operator A in the algorithm as demanded by equation (1.3). The components of operator R was constructed in [5] and they were given as

$$R_{11} = \frac{4}{\mu C_0^2 y} z_y(x,y,0) - \frac{1}{t}\int_0^t z(x,y,l)\,dl - \frac{3}{sty}\int_0^y\int_0^t z(x,k,l)\,dl\,dk - \frac{1}{y}\int_0^y z(x,k,t)\,dk$$

$$- \frac{4}{\mu y}\int_0^y z(x,k,0)\,dk - \frac{x^3}{2ytC_0^2}\int_0^x\int_0^y z_{ttt}(x,k,0)\,dk\,dh - \frac{x^3}{2tC_0^2}\int_0^x z(x,y,0)\,dh$$

$$+ \frac{x^3}{yC_0^2t^3}\int_0^x\int_0^y z_t(h,k,t)\,dk\,dh + \frac{x^3}{C_0^2t^3}\int_0^x z_t(h,y,t)\,dh + \frac{4x}{\mu C_0^2 y}\int_0^x\int_0^y z(h,k,0)\,dk\,dh$$

$$+ \frac{x^3}{2tC_0^2}\int_0^x z_t(h,y,0)\,dh - \frac{6x}{\mu C_0^2t^2y}\int_0^x\int_0^y\int_0^t z(h,k,l)\,dl\,dk\,dh$$

$$- \frac{3x}{\mu ytC_0^2}\int_0^x\int_0^y z(h,k,t)\,dk\,dh - \frac{12x}{\mu C_0^2t^2}\int_0^x\int_0^t z(h,y,l)\,dl\,dh$$

$$- \frac{3x}{\mu C_0^2 t}\int_0^x z(h,y,t)\,dh + \frac{3kx}{\mu C_0^2 t}\int_0^x z_{tt}(h,y,0)\,dh$$

$$-\left(-\frac{12x}{\mu C_0^2t^2} + \frac{4x}{\mu yC_0^2}\right)\int_0^x\int_0^y z_t(h,k,0)\,dk\,dh + \frac{6x}{yC_0^2t^2}\int_0^x\int_0^y z_{tt}(h,k,0)\,dk\,dh$$

$$+ \frac{6x}{C_0^2t^2}\int_0^x z_{tt}(h,y,0)\,dh - \left(-\frac{x^3}{\mu yC_0^2t^2} - \frac{12x}{\mu C_0^2t^2} + \frac{kx^3}{yC_0^2t^2} + \right. \tag{1.5}$$

$$\left.\frac{6x}{yC_0^2t^2} + \frac{x^3}{yC_0^2t^2}\right)\int_0^x\int_0^y z(h,k,0)\,dk\,dh - \left(-\frac{12x}{\mu C_0^2t^2} - \frac{x^3}{\mu C_0^2t^3} + \frac{kx^3}{C_0^2t^3} + \frac{x^3}{C_0^2t^2} + \frac{6x}{C_0^2t^3}\right.$$

$$\left.- \frac{6x}{C_0^2t^2}\right)\int_0^x z(h,y,0)\,dh - 2z(x,y,0) + \frac{4x}{\mu C_0^2}\int_0^x z(h,y,0)\,dh - \frac{kx^3}{\mu yC_0^2t^3}\int_0^x z(h,y,t)\,dh$$

$$- \frac{kx^3}{\mu C_0^2yt^3}\int_0^x z_t(h,y,t)\,dh - \frac{4x}{\mu C_0^2}\int_0^x z_t(h,y,0)\,dh$$

$$R_{21} = -\mu z_{tt} + \mu c_0 z_{xx} + \mu c_0 z_{yy} \tag{1.6}$$

$$R_{22} = 4u(x,y,t) - \frac{3}{t}\int_0^t u(x,y,t)dt + \int_0^y u(x,y,t)\,dy$$

$$+ \frac{2}{t}\int_0^y u(x,y,0)\,dy + \int_0^t u_t(x,y,0)\,dy + t0\int^y u_{tt}(x,y,0,)\,dy$$

$$+ \frac{t^2}{2}\int_0^y u_{tt}(x,y,t)\,dy - 3ty\int_0^y\int_0^t u(x,y,t)\,dt\,dy \tag{1.7}$$

$$- y\int_0^y u(x,y,t)\,dy + 4y\int_0^y u(x,y,0)\,dy$$

and

$$R_{22} = u_2(1+\mu) \tag{1.8}$$

2.0 Computation of the step-length and descent direction`

The step-length represented as α, in Algorithm 1.1 is the most important function to compute from the point of view of the ECGM and incidentally it plays an essential role in the implementation of Algorithm 1.1. The state component of the gradient, g_i is obtained by differentiating equation (1.2) with respect to the state variable, *z*, according to [1], [2], [3] and [6]. Therefore we have

$$g_{z,i} = J_{z,i}(z,u,\mu) = 2\int_0^1\int_0^1\int_0^1 z_i(x,y,t)\ dxdydt \tag{2.1}$$

Similarly, the control component of gradient g_i, is obtained by differentiating equation (1.2) with respect to u, the control variable. Thus we have

$$g_{u,i} = J_{u,i}(z,u,\mu) = 2\int_0^1\int_0^1\int_0^1((1-\mu)u_i - \mu[z_{itt} - C_0 z_{ixx} - C_0 z_{iyy}])dxdydt \tag{2.2}$$

The descent direction, P_i, was defined by [3] as

$$P_{z,i} = P_z(z_i,u_i,\mu) = \int_0^x\int_0^y\int_0^t J_z(z_i,u_i,\mu)\,dxdydt \tag{2.3}$$

Therefore using equation(2.1) in equation (2.3) we have

$$P_{z,i} = P_z(z_i,u_i,\mu) = = 2\int_0^1\int_0^1\int_0^t\int_0^x\int_0^y\int_0^t z_i(x,y,t)\,dxdydtdxdy\ dt \tag{2.4}$$

The second component of P_i, is the direction of the control variable, $P_{u,i}$. This was also defined by [3] as

$$P_{u,i} = P_u(z_i,u_i,\mu) = \int_0^x\int_0^y\int_0^t J_u(z_i,u_i,\mu)dxdydt \tag{2.5}$$

Therefore, using equation (2.2) in equation (2.5), we have

$$P_{u,i} = P_u(z,u,\mu) = \int_0^1\int_0^1\int_0^1\int_0^x\int_0^y\int_0^t\Big((1-\mu)u_i - \mu(z_{itt} - C_0 z_{ixx} - C_0 z_{iyy})\Big)dxdydtdxdydt \tag{2.6}$$

On the other hand, the inner product of the operator, *R* and the direction vector, $P_i = \{P_{z,i}, P_{u,i}\}$ is

$$RP_i = R_{11}P_{z,i} + R_{21}P_{u,i}\ ,\ \ R_{12}P_{z,i} + R_{22}P_{u,i} \tag{2.7}$$

Using equation (2.4), equation (2.6) and the components of *R* [equation (1.5), equation (1.6), equation (1.7) and equation (1.8)], in equation (2.7) and simplifying as far as possible , we have

$$\begin{aligned}
RP_i = {} & \frac{4}{\mu C_0^2 y}\frac{\partial}{\partial y}P_{z,0}(x,y,0) - \frac{1}{t}\int_0^t P_{z,i}(x,y,t)dl - \frac{3}{xyt}\int_0^y\int_0^t P_{z,i}(x,k,t)dkdl \\
& - \frac{1}{y}\int_0^y P_{z,i}(x,k,t)dk - \frac{4}{\mu y}\int_0^y P_{z,0}(x,k,0)dk - \frac{x^3}{2ytc_0^2}\int_0^x\int_0^y \frac{\partial}{\partial t^3}P_{z,0}(h,k,0)dhdk \\
& \frac{-x^3}{2tc_0^2}\int_0^x P_{z,0}(h,y,0)dh + \frac{t}{yt^5C_0^2}x^3\int_0^x\int_0^y\frac{\partial}{\partial t}P_{z,i}(h,k,t)dhdk \\
& + \frac{x^3}{t^3C_0^2}\int_0^x\frac{\partial}{\partial t}P_{z,i}(h,y,t)dh + \frac{4x}{\mu yC_0^2}\int_0^x\int_0^y P_{z,0}(h,k,0)\,dkdh \\
& + \frac{X^3}{2tC_0^2}\int_0^x\frac{\partial}{\partial t}P_{2,0}(h,y,0)\,dh - \frac{6x}{\mu yt^2C_0^2}\int_0^x\int_0^y\int_0^t P_{2,i}(h,k,l)\,dhdkdl \\
& - \frac{3x}{\mu ytC_0^2}\int_0^x\int_0^y P_{2,i}(h,k,t)\,dhdk - \frac{12}{\mu t^2C_0^2}\int_0^x\int_0^t P_{2,i}(h,y,l)\,dhdl
\end{aligned}$$

$$-\frac{3x}{\mu t C_0^2}\int_0^x P_{2,i}(h,y,t)\,dh+\frac{3ky}{\mu t C_0^2}\int_0^x \frac{\partial^2}{\partial t^2}P_{2,0}(h,y,0)\,dh$$

$$-(\left(\frac{4x}{\mu y C_0^2}\right)-\frac{12x}{\mu t C_0^2})\int_0^x\int_0^y \frac{\partial}{\partial t}P_{2,0}(h,k,o)\,dhdk+\frac{6x}{yt^2c_0^2}\int\int\frac{\partial^2}{\partial t^2}P_{z,i}(h,k,0)dkdh$$

$$+\frac{6x}{t^2c_0^2}\int_0^x\frac{\partial^2}{\partial t^2}P_{z,i}(h,k,0)dh-\frac{1}{c_0^2}|(\frac{x^3(k+1)+6x}{yt^2}-\frac{x^3}{\mu yt^2}-\frac{12x}{\mu t^2})\int_0^x\int_0^y P_{z,i}(h,k,0)dkdh$$

$$-\frac{1}{c_0^2}(\frac{x^3-6x}{t^2}+\frac{kx^3+6x}{t^2}-\frac{12x}{\mu t^2}-\frac{x^3}{\mu t^3})\int_0^x P_{z,0}(h,k,0)dh$$

$$-2P_{z,0}(x,y,0)+\frac{4x}{\mu c_0^2}\int^x P_{z,0}(h,k,0)dh+\frac{kx^3}{\mu yt^3c_0^2}\int^x P_{z,i}(h,k,t)dh$$

$$\frac{kx^3}{\mu yt^3c_0^2}\int^x\frac{\partial}{\partial t}P_{z,i}(h,k,t)dh-\frac{4x}{\mu c_0^2}\int^x\frac{\partial}{\partial t}P_{z,i}(h,y,t)dh+(1+\mu)P_{z,i}(x,y,t)$$

$$-\frac{\partial^2}{\partial t^2}P_{u,i}(x,y,t)+\mu c_0^2(\frac{\partial^2}{\partial t^2}P_{u,i}(x,y,t)+\frac{\partial^2}{\partial y^2}P_{u,i}(x,y,t)+4P_{u,i}(x,y,t)$$

$$\frac{3}{t}\int_0^y P_{u,i}(x,k,t)dk+\int_0^y\frac{\partial}{\partial t}P_{u,i}(x,k,0)dk+t\int_0^y\frac{\partial^2}{\partial t^2}P_{u,i}(x,k,0)dk$$

$$+\int_0^y P_{u,i}(x,y,t)dy+\frac{2}{t}\int_0^y P_{u,0}(x,y,0)dy+\frac{t^2}{2}\int_0^y\frac{\partial^2}{\partial t^2}P_{u,i}(x,y,t)dy \tag{2.8}$$

$$-3ty\int_0^y\int_0^t P_{u,i}(x,y,t)dtdy-y\int_0^y P_{u,i}(x,y,t)dy+4y\int_0^y P_{u,0}(x,y,0)dy$$

3.0 Determination of the penalty function

We shall employ the symmetric nature of the operator R [1], to construct an expression for the penalty cost function, $\mu(x,y,t)$. From equation (1.9) we have that:

$$R_{21}=(1+\mu) \tag{3.1}$$

Also, from equation (1.7) we have:

$$R_{12}=\frac{-\mu z_{2tt}+\mu c_0^2(z_{2xx}+z_{2yy})}{z_2} \tag{3.2}$$

Therefore combining equation (3.1) and equation (3.2) yields

$$1+\mu=\frac{-\mu z_{2tt}+\mu c_0^2(z_{2xx}+z_{2yy})}{z}$$

$$\Rightarrow \qquad \frac{1+\mu}{\mu} = \frac{-z_{2tt} + c_0^2(z_{2xx} + z_{2yy})}{z}$$

$$\frac{1}{\mu} + 1 = \frac{-z_{2tt} + c_0^2(z_{2xx} + z_{2tt})}{z}$$

$$\therefore \frac{1}{\mu} = \frac{-z_{2tt} + c_0^2(z_{2xx} + z_{2yy})}{z}$$

$$\Rightarrow \frac{1}{\mu} = \frac{-z_{2tt} + c_0^2(z_{2xx} + z_{2yy}) - z_2}{z_2}$$

Therefore, $$\mu = \frac{z_2}{-z_{2tt} + c_0^2(z_{2xx} + z_{2yy}) - z_2} \qquad (3.3)$$

But from equation (1.4), we have $z_{2tt} = c_0^2\,(z_{2xx} + z_{2yy}) + u$ therefore $z_{2tt} - u = C_0(z_{2xx} + z_{2yy})$

Hence, equation (3.3) becomes

$$\mu = \frac{z_2}{-z_{2tt} + (z_{2tt} - u) - z_2}$$

$$\mu = \frac{-z_2}{z_2 + \mu}$$

Since by definition the penalty function μ, is always positive therefore

$$\mu = \left| \frac{z_2}{z_2 + u} \right| \qquad (3,4)$$

We employed bilinear expansion to obtain equation(1.4) therefore $z = z_1 = z_2$ and hence equation(3.4) becomes $$\mu = \left| \frac{z}{z+u} \right| \qquad (3.5)$$

4.0 Conclusion

With the result of equation (2.8) the Extended Conjugate Gradient Algorithm, ECGM can be implemented for the unconstrained optimal control problem of higher-order nondispersive wave, equation (1.2). More so, the experience gained here makes the computation of the second step length, β trivial, because the computation of β depends on the gradient, g_i , equation (2.1) and equation (2.2). Also the exact value of the penalty function $\mu(x,y,t)$ for each value of the state and the corresponding value of the control variables can be obtained from equation 3.5).

References

[1] Ibiejugba M. A., Rubio J. E., and Orisamobi R. J.(1986): A penalty optimization techniques for a class of regulator problems. ABACUS (Journal of Mathematical Association of Nigeria) 17(1): 19 – 50.

[2] Otunta O. Francis (1992): An extended Conjugate Gradient Algorithm for discrete Optimal Control problem, International Conference on Scientific Computing, Benin. Pp:(183 - 187).

[3] Reju, Sunday A., Matthew A. Ibiejugba and David J. Evans (2001): Optimal Control of the wave propagation

problem with the Extended Conjugate Gradient Method, International Journal of Computer Mathematics, Vol. 77, pp (425 439).

[4] Reju, Sunday A., Matthew A. Ibiejugba and David J. Evans (2000): On the control of Navier-Stokes equation with the Extended Conjugate Gradient Method. International Journal of Computer Mathematics, Vol. 76, pp (75-91).

[5] Ukwosa I. S. and Reju S. A.(2007): The control operator for the optimal control model of higher order non-dispersive wave. Journal of the Nigerian Association of Mathematical Physics, vol.11, pp:199-212.

[6] Waziri, Victor Onomza and Reju, Sunday A. (2000): The control operator for the one-Dimensional Energized Wave Equation, AU Journal of Technology, 9(4): 243 - 247