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Numerical computation of the optimal control model of higher-order nondispersive wave with the extended conjugate gradient algorithm

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Abstract

The paper implemented the optimal control problem of higher-order nondispersive wave. The Extended Conjugate Gradient Method [1], was used to compute the optimal values of the control and state variables of the model while the analytical expressions of the state and control variables generated the analytical values. The corresponding values of the penalty function were also computed for each pair of the state and control variables.

Keywords

Optimal control, control variable, state variable, direction of descent, state and control gradients.

1.0 Introduction

The optimal control problem of higher-order nondispersive wave was formulated in [4] and was given as

$$\min J(z,u) = \int_0^1\int_0^1\int_0^1 (z^2(x,y,t) + u^2(x,y,t))$$

subject to

$$\frac{\partial^2 z}{\partial t^2} = -c_0^2 \nabla^2 z(x,y,t) + u(x,y,t)dxdydt \tag{1.1}$$

whereas the unconstrained formulation of equation(1.1) was also given as

$$J(z,u,\mu) = \int_0^1 \int_0^1\int_0^1 \{(z^2 + u^2) + \mu \left\| u - z_{tt} - c_0^2 z_{xx} - c_0^2 z_{yy} \right\|^2 \}dxdydt \tag{1.2}$$

In [5] the explicit expressions for the control variable u(x,y,t) and the state variable, $z(x,y,t)$, were obtained and given *as*

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$$u(x,y,t)=\{(\sum u_{it}(0)\sin i\pi x\sin i\pi y-(2(\sum u_{itt}(0)\sin i\pi x\sin i\pi y\ -((p+12k)/6p)\sum u_i(0)\sin i\pi x\sin i\pi y)$$
$$((2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2))-(\sum u_{itt}(0)\sin i\pi x\sin i\pi y-((p+12k)/6p)^2$$
$$\sum u_i(0)\sin i\pi x\sin i\pi y)/[(2(2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2))[(4p(-p-4(k(p+k))$$
$$-(p+12k)/6p)]-[((2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2)^2)-((p+12k)/6p)^2-$$
$$(p\sqrt{3}(p+k)/4(p^2-2k)^2+3p^2))^2]\}\exp((p+2k)/6p)t+$$
$$2(\sum u_{itt}(0)\sin i\pi x\sin i\pi y\ -((p+12k)/6p)\sum u_i(0)\sin i\pi x\sin i\pi y)(2p(-p-4k(p+k)))/$$
$$4(p^2-2k)^2+3p^2))-(\sum u_{itt}(0)\sin i\pi x\sin i\pi y-((p+12k)/6p)^2\sum u_i(0)\sin i\pi x\sin i\pi y)/$$
$$2(2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2)[(4p(-p-4(k(p+k))-(p+12k)/6p)]$$
$$-[((2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2)^2)-((p+12k)/6p)^2-(p\sqrt{3}(p+k)/$$
$$4(p^2-2k)^2+3p^2))^2]\exp(2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2))t$$
$$\cos(p\sqrt{3}(p+2k)/4(p^2-2k)^2+3p^2))t+$$
$$(\sum u_{itt}(0)\sin i\pi x\sin i\pi y-((p+12k)/6p)^2\sum u_i(0)\sin i\pi x\sin i\pi y))[4p(-p-4k(p+k)/$$
$$4(p^2-2k)^2+3p^2)-((p+2k)/6p)]-(\sum u_{it}(0)\sin i\pi x\sin i\pi y-((p+2k)/6p)\sum u_i(0)\sin i\pi x\sin i\pi y))$$
$$[((-p-4k((p+k))/4(p^2-2k)^2+3p^2)^2)-((p+2k)/6p)^2-((p\sqrt{3}(p+k)/4(p^2-2k)^2+3p^2)^2))/$$
$$(p\sqrt{3}(p+2k)/4(p^2-2k)^2+3p^2))((2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2)[(8p(-p-$$
$$4(k(p+k))-(p+12k)/6p)]-[((2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2)^2)-$$
$$((p+12k)/6p)^2-(p\sqrt{3}(p+k)/4(p^2-2k)^2+3p^2))^2]\exp(2p(-p-4k(p+k)))t/$$
$$4(p^2-2k)^2+3p^2))\sin(p\sqrt{3}(p+2k)/4(p^2-2k)^2+3p^2))t) \tag{1.3}$$

and

$$z(x,yt)=((p+2k)/6p)\{(\sum u_{it}(0)\sin i\pi x\sin i\pi y-(2(\sum u_{itt}(0)\sin i\pi x\sin i\pi y\ -$$
$$((p+12k)/6p)\sum u_i(0)\sin i\pi x\sin i\pi y)((2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2))$$
$$-(\sum u_{itt}(0)\sin i\pi x\sin i\pi y-((p+12k)/6p)^2\sum u_i(0)\sin i\pi x\sin i\pi y)/$$
$$[(2(2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2))[(4p(-p-4(k(p+k))-(p+12k)/6p)]$$
$$-[((2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2)^2)-((p+12k)/6p)^2-$$
$$(p\sqrt{3}(p+k)/4(p^2-2k)^2+3p^2))^2]\}\exp((p+2k)/6p)t+$$

$$2(\sum u_{itt}(0)\sin i\pi x\sin i\pi y\ -((p+12k)/6p)\sum u_i(0)\sin i\pi x\sin i\pi y)(2p(-p-4k(p+k)))/$$
$$4(p^2-2k)^2+3p^2))-(\sum u_{itt}(0)\sin i\pi x\sin i\pi y-((p+12k)/6p)^2\sum u_i(0)\sin i\pi x\sin i\pi y)/$$
$$2(2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2)[(4p(-p-4(k(p+k))-(p+12k)/6p)]$$

$$[((2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2)^2)-((p+12k)/6p)^2-(p\sqrt{3}(p+k)/$$
$$4(p^2-2k)^2+3p^2))^2]\exp(2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2))$$
$$t[((2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2))\cos(p\sqrt{3}(p+2k)/4(p^2-2k)^2+3p^2))t$$
$$-(p\sqrt{3}(p+2k)/4(p^2-2k)^2+3p^2))\ \sin(p\sqrt{3}(p+2k)/4(p^2-2k)^2+3p^2))t]+$$
$$(\sum u_{itt}(0)\sin i\pi x\sin i\pi y-((p+12k)/6p)^2\sum u_i(0)\sin i\pi x\sin i\pi y))[4p(-p-4k(p+k)/$$
$$4(p^2-2k)^2+3p^2)-((p+2k)/6p)]-(\sum u_{it}(0)\sin i\pi x\sin i\pi y-((p+2k)/6p)\sum u_i(0)\sin i\pi x\sin i\pi y))$$
$$[((-p-4k((p+k))/4(p^2-2k)^2+3p^2)^2)-((p+2k)/6p)^2-((p\sqrt{3}(p+k)/$$
$$4(p^2-2k)^2+3p^2)^2))/\quad(p\sqrt{3}(p+2k)/4(p^2-2k)^2+3p^2))((2p(-p-4k(p+k)))/$$
$$4(p^2-2k)^2+3p^2)[(8p(-p-4(k(p+k))-(p+12k)/6p)]-$$ w
$$[((2p(-p-4k(p+k)))/4(p^2-2k)^2+3p^2)^2)-((p+12k)/6p)^2-(p\sqrt{3}(p+k)/$$
$$4(p^2-2k)^2+3p^2))^2]\exp(2p(-p-4k(p+k)))t/4(p^2-2k)^2+3p^2))$$
$$[(2p(-p-4k(p+k)))t/4(p^2-2k)^2+3p^2))\sin(p\sqrt{3}(p+2k)/4(p^2-2k)^2+3p^2))t)$$
$$+\ (p\sqrt{3}(p+2k)/4(p^2-2k)^2+3p^2))t)\cos(p\sqrt{3}(p+2k)/4(p^2-2k)^2+3p^2))t)] \qquad (1.4)$$

here $P=(108+12\sqrt{(-12k^3+81)}^{1/3})$ and $k=-2c_0n^2\pi^2$.

To obtain the optimal values of these variables we employed the Extended Conjugate Gradient Algorithm that was developed by Ibiejugba et al in 1986. This algorithm is presented below:

***Algorithm* 1.1**

Step1: Guess x_0, u_0

Step 2: Compute g_0

For $i=0$

Step 3: $p_0=-g_0$

Step 4: $x_{i+1}=x_i+\alpha_i Ap_i, u_{i+1}=u_i+\alpha_i p_i$; where $\alpha_i=\dfrac{<g_i,g_i>}{<p_i,Ap_i>}$

For $i=1, 2,$

Step 5: $g_{x,i+1}=g_{x,i}+\alpha_i Ap_{x,i},\ g_{u,i+1}=g_{u,i}+\alpha_i Ap_{u,i}, p_{x,i+1}=p_{x,i}+\beta_i p_{x,i}, p_{x,i+1}=p_{x,i}+\beta_i p_{x,i};$

where $\beta_i=\dfrac{<g_{i+1},g_{i+1}>}{<g_i,g_i>}$ (1.5)

2.0 Computational fundamentals

Implementing equation(1.2) using the Algorithm1.1, we replaced the operator A in the algorithm with operator R as was also done by [2] and [3] that characterized the operator as B. The components of operator R are R_{11}, R_{12}, R_{21} and R_{22}. These components were constructed in [4] and they are summarized below:

$$
\begin{aligned}
R_{11} = {} & \frac{4}{\mu C_0^2 y} z_y(x,y,0) - \frac{1}{t}\int_0^t z(x,y,l)\,dl - \frac{3}{sty}\int_0^y\int_0^t z(x,k,l)\,dl\,dk - \frac{1}{y}\int_0^y z(x,k,t)\,dk \\
& - \frac{4}{\mu y}\int_0^y z(x,k,0)\,dk - \frac{x^3}{2ytC_0^2}\int_0^x\int_0^y z_{ttt}(x,k,0)\,dk\,dh - \frac{x^3}{2tC_0^2}\int_0^x z(x,y,0)\,dh \\
& + \frac{x^3}{yC_0^2 t^3}\int_0^x\int_0^y z_t(h,k,t)\,dk\,dh + \frac{x^3}{C_0^2 t^3}\int_0^x z_t(h,y,t)\,dh + \frac{4x}{\mu C_0^2 y}\int_0^x\int_0^y z(h,k,0)\,dk\,dh \\
& + \frac{x^3}{2tC_0^2}\int_0^x z_t(h,y,0)\,dh - \frac{6x}{\mu C_0^2 t^2 y}\int_0^x\int_0^y\int_0^t z(h,k,l)\,dl\,dk\,dh \\
& - \frac{3x}{\mu ytC_0^2}\int_0^x\int_0^y z(h,k,t)\,dk\,dh - \frac{12x}{\mu C_0^2 t^2}\int_0^x\int_0^t z(h,y,l)\,dl\,dh \\
& - \frac{3x}{\mu C_0^2 t}\int_0^x z(h,y,t)\,dh + \frac{3kx}{\mu C_0^2 t}\int_0^x z_{tt}(h,y,0)\,dh \\
& -\left(-\frac{12x}{\mu C_0^2 t^2} + \frac{4x}{\mu y C_0^2}\right)\int_0^x\int_0^y z_t(h,k,0)\,dk\,dh + \frac{6x}{yC_0^2 t^2}\int_0^x\int_0^y z_{tt}(h,k,0)\,dk\,dh \\
& + \frac{6x}{C_0^2 t^2}\int_0^x z_{tt}(h,y,0)\,dh - \left(-\frac{x^3}{\mu y C_0^2 t^2} - \frac{12x}{\mu C_0^2 t^2} + \frac{kx^3}{yC_0^2 t^2} + \right. \\
& \left. \frac{6x}{yC_0^2 t^2} + \frac{x^3}{yC_0^2 t^2}\right)\int_0^x\int_0^y z(h,k,0)\,dk\,dh - \left(-\frac{12x}{\mu C_0^2 t^2} - \frac{x^3}{\mu C_0^2 t^3} + \frac{kx^3}{C_0^2 t^3} + \right. \\
& \left. \frac{x^3}{C_0^2 t^2} + \frac{6x}{C_0^2 t^3} - \frac{6x}{C_0^2 t^2}\right)\int_0^x z(h,y,0)\,dh - 2z(x,y,0) + \frac{4x}{\mu C_0^2}\int_0^x z(h,y,0)\,dh - \frac{kx^3}{\mu y C_0^2 t^3}\int_0^x z(h,y,t)\,dh \\
& - \frac{kx^3}{\mu C_0^2 y t^3}\int_0^x z_t(h,y,t)\,dh - \frac{4x}{\mu C_0^2}\int_0^x z_t(h,y,0)\,dh
\end{aligned} \tag{2.1}
$$

$$
R_{21} = -\mu z_{tt} + \mu c_0 z_{xx} + \mu c_0 z_{yy} \tag{2.2}
$$

$$R_{22} = 4u(x,y,t) - \frac{3}{t}\int_0^t u(x,y,t)dt \;+\; \int_0^y u(x,y,t)\,dy$$
$$+\frac{2}{t}\int_0^y u(x,y,0)\,dy + \int_0^t u_t(x,y,0)\,dy + t0\int^y u_{tt}(x,y,0,)\,dy \quad (2.3)$$
$$+\frac{t^2}{2}\int_0^y u_{tt}(x,y,t)\,dy - 3ty\int_0^y\int_0^t u(x,y,t)\,dtdy$$
$$-y\int_0^y u(x,y,t)\,dy \;+\; 4y\int_0^y u(x,y,0)\,dy$$

and
$$R_{22} = u_2(1+\mu) \quad (2.4)$$

The components of the gradient g_i, are obtained by differentiating equation (1.2) with respect to *z*, the state variable and u, the control variable for the state and control components of the gradient respectively. Thus we have

$$g_{z,i} = J_{z,i}(z,u,\mu) = 2\int_0^1\int_0^1\int_0^1 z_i(x,y,t)\;dxdydt \quad (2.5)$$

and
$$g_{u,i} = J_{u,i}(z,u,\mu) = 2\int_0^1\int_0^1\int_0^1((1-\mu)u_i - \mu[z_{itt} - C_0 z_{ixx} - C_0 z_{iyy}])dxdydt \quad (2.6)$$

We also note that the direction of descent, P_i has two components $P_{i,z}$ and $P_{i,u}$. These components were computed in [3] and given as

$$P_{z,i} = P_z(z_i,u_i,\mu) = \int_0^x\int_0^y\int_0^t J_z(z_i,u_i,\mu)\,dxdydt \quad (2.7)$$

and
$$P_{u,i} = P_u(z_i,u_i,\mu) = \int_0^x\int_0^y\int_0^t J_u(z_i,u_i,\mu)dxdydt \quad (2.8)$$

Next is to compute RP_i that replaces AP_i in Algorithm 1.1. The function RP_i is the inner product of the direction of descent P_i, and the operator, *R*. This function is one of the major functions involved in the computation of the first steplength, α described in the Algorithm 1.1 above. Since *R* is of order 2, this product gives

$$RP_i = R_{11}P_{i,z} + R_{12}P_{i,u}, R_{21}P_{i,z} + R_{2,2}P_{i,u} \quad (2.9)$$

The inner product of g_i with itself is used to compute the second steplength, β in the second part of the algorithm. That is

$$\beta = \frac{< g_{i+1}, g_{i+1} >}{< g_{i,} g_i >} \quad (2.10)$$

The code was run on MATLAB version 6.5. The constants *ui*0, *uit*0 and *uitt*0 are the variable names of the constants in equation(1.3) and equation(1.4). The values of these constants were taken as *ui*0 = *uit*0 = *uitt*0 = 0.005 as was also the case in [6]. We took the initial values of x, y and *t* as 0.1 for each of the variables *x*, *y* and *t* respectively in order to avoid division by zero in each iteration. The mesh size of *x*, *y* and *t* is taken as *h* = 0.001 for each of the directions. Finally, *N*, the total number of iterations, was taken as 100. The outputs are summarized in the Tables 2.1 and 2.2.

Table 2.1 The analytical state and control variables
*ui*0 = *uit*0 = *uitt*0 = 0.005, *N* = 100

N	State	Control	Penalty Constant
10	3.4923×10^{-13}	2.03760×10^{-11}	0.01685
20	3.4774×10^{-13}	1.78976×10^{-11}	0.01905
30	3.4626×10^{-13}	1.5711×10^{-11}	0.02156

40	3.5374×10^{-13}	1.3795×10^{-11}	0.02500
50	3.5223×10^{-13}	1.2114×10^{-11}	0.02025
60	3.5073×10^{-13}	1.3796×10^{-11}	0.02479
70	3.4923×10^{-13}	3.2610×10^{-11}	0.01059
80	3.4774×10^{-13}	4.6400×10^{-11}	0.00743
90	3.4924×10^{-13}	1.46118×10^{-11}	0.02234
100	3.4924×10^{-13}	2.7848×10^{-11}	0.01238

Table 2.2: The optimal state and optimal control from ECGM algorithm

N	Optimal State	Optimal Control	Penalty Constant
10	2.514663×10^{-5}	1.505872×10^{-7}	0.9940
20	2.522323×10^{-5}	5.54362×10^{-7}	0.9784
30	2.531926×10^{-5}	1.060551×10^{-6}	0.9958
40	2.539506×10^{-5}	1.460085×10^{-6}	0.9942
50	2.54162×10^{-5}	1.571508×10^{-7}	0.9938
60	2.537016×10^{-5}	1.328871×10^{-7}	0.9947
70	2.527521×10^{-5}	8.2836×10^{-7}	0.9682
80	2.517521×10^{-5}	3.012034×10^{-7}	0.98817
90	2.512036×10^{-5}	1.207691×10^{-7}	0.9952
100	2.514146×10^{-5}	1.233005×10^{-7}	0.99512

3.0 Discussion of results

The analytical expressions for both the control and the state variables for higher-order nondispersive wave are given in equation (1.3) and equation (1.4) respectively. These two functions have three major components each carrying the sine function, $sin(\pi x)sin(\pi y)$. The amplitudes of two of these three functions indicate that they are nondispersive while the remaining component is dispersive. It is possible that the effect of the two nondispersive components can overshadow the dispersive component, because we are dealing with a small horizon $[t_0, t]$ where t is finite. In such interval the effect of the dispersion is negligible, [7].

Table 2.2 shows the optimal values of the state and the control variables. On the other hand the output in Table 2.1 shows the corresponding analytical values of the state and the control variables. Comparing these values as summarized in Table 2.2 and Table 2.1 show that they are similar. One can see from these two tables the performance of the Extended Conjugate Gradient Algorithm. This indicates a useful characteristic of the Extended Conjugate Gradient Algorithm especially as used for the control of higher-order nondispersive wave.

4.0 Conclusion

The research shows that the signals represented by the state variable and the control variable have very small amplitudes. This is suggesting that these signals might be present in the electro- magnetic spectrum. The phenomenon will make the research to have potential applications in areas such as communication and computer networking. In these two areas, information is transported via a wave system. Nondispersive wave can be very useful in communication and computer networking because they can retain the salient features of the original signals with a little distortion. They also have constant phase speed,[7], therefore they are often more stable than the dispersive wave system.

The optimal values in Table 2.2 were obtained from the application of the Extended Conjugate Gradient Algorithm. The algorithm appears to have tried in making the amplitudes more stable than those computed analytically as shown in Table 2.1. This explains the almost constant value of the penalty function in Table 2.2.

References

[1] Ibiejugba M. A., Rubio J. E., and Orisamobi R. J.(1986): A penalty optimization techniques for a class of regulator problems. ABACUS (Journal of Mathematical Association of Nigeria) 17(1): 19 – 50.

[2] Otunta O. Francis (1992): An extended Conjugate Gradient Algorithm for discrete Optimal Control problem, International Conference on Scientific Computing, Benin. Pp:(183 - 187).

[3] Reju, Sunday A., Matthew A. Ibiejugba and David J. Evans (1999): Computational Results of the optimal control of Diffusion Equation, International Journal of Computer Mathematics, Vol. 75, pp (247- 258).

[4] Ukwosa I. S. and Reju S. A. (2007): The control operator for the optimal control model of higher order nondispersive wave. Journal for the Nigerian Association of Mathematical Physics. Vol. 11, pp: 199-212.

[5] Ukwosa I. S. and Reju S. A. (2008): A new analytical solution to the optimal control problem for higher- order nondispersive wave. Journal for the Nigerian Association of Mathematical Physics. Vol. 11, pp: 253-260.

[6] Waziri, Victor Onomza and Reju, Sunday A. (2000): The control operator for the one-Dimensional Energized Wave Equation, AU Journal of Technology, 9(4): 243 – 247.

[7] Whitham G. B. (1974): Linear and Non-linear wave: John Wiley and Sons, New York pp :8-9