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Torsional vibration of thin-walled elastic beams with doubly-symmetric cross-sections traversed by concentrated masses

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Abstract

In this paper, the problem of analyzing the torsional vibration of thin-walled elastic beams, with open cross-sections that are doubly symmetric and traversed by moving concentrated masses at constant speeds is addressed. The mathematical model adopted accounts for both the gravitational and inertial effects of the moving loads, thus making the problem a moving-force moving-mass problem. Variable coefficients with strong singularities are therefore present in the characterizing differential equation. By means of Green's function of the associated moving-force problem, the complete moving-force moving–mass problem is transformed into an integro-differential equation. An iteration scheme for solving the integro-differential equation has been proposed and shown to converge to a unique continuous function of space and time, the only solution to the equation.

Keywords
Torsional vibration; thin-walled elastic beams; doubly-symmetric cross-section; Green's function; integro-differential equation

1.0 Introduction

On the strength of its importance in bridge engineering, the flexural and or torsional vibration of beams due to moving loads has remained a subject of intense research. Early contribution by research scientists and engineers, especially [1, 2, 3] focused principally on the flexural vibration of beams traversed by moving concentrated masses. In order to achieve a simplified model, the inertial effects of the moving loads were ignored by these early contributors. However, attempts were later made to address both the gravitational and inertial effects of the moving concentrated masses. The research work conducted and reported on this subject area by [8] is a good example of the attempt to account for both effects of moving concentrated masses.

In the paper by [7], the problem of assessing the response of elastic beams to moving concentrated masses termed in that paper as a "moving-mass moving-force problem" was rigorously analysed. The paper treated the approach given by [9] as a good first approximation to the problem and proceeded to give a scheme for generating a better improved solution to the moving-force moving-mass problem.

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Fryba [3] offers an excellent comprehensive up-to-date account of the various contributions in building up the literature in this subject area.

However, it is pertinent to observe that the torsional or coupled flexural-torsional vibration of beams carrying moving concentrated masses has not been given the same level of attention by researchers, as given to the flexural vibration. Yet the torsional vibration, especially of thin-walled beams may be the dominant effect of moving concentrated masses on the supporting beams. For instance, an eccentric moving load from an overhead travelling crane, moving on a crane gantry girder, can result in dynamic twisting moments setting the girder into torsional vibration.

A recent publication by [5] represents one of the earliest attempts at opening up this area for further investigation. In that paper the authors analysed the flexural-torsional vibration of simply supported beams under moving loads. They considered only the gravitational effects of the moving loads. When one considers that the structural systems belonging to this class, almost invariably, are of thin-walled sections, then the inertia of the moving masses cannot be safely ignored. Be that as it may, that paper presents a commendable pioneering effort. The authors remarked that the subject of the torsional vibrations of beams in the context of moving loads remained largely unexplored.

In a paper on a related subject, [2] examined the coupled axial-torsional vibrations of thin-walled Z-section beam although not in the context of moving loads.

The objective of the present paper is, therefore, to analyze, in some great detail, the response of an elastic beam, with an open section that is doubly-symmetric, to torsional vibration arising from moving concentrated masses. Evidently, the system represents a moving-force moving-mass problem. The differential equation of motion contains variable and singular coefficients. The essence of the presentation is to devise analytical tools for handling rigorous analysis of the system despite singularities in the coefficients of the characterizing differential equation.

First, Green's function for the approximate moving-force problem is developed. Utilizing that Green's function, the complete moving-force moving-mass problem is transformed from a differential equation into an integro-differential equation. A scheme for iterative solution procedure is proposed. It is shown that the scheme converges to a continuous function of space and time which is not only a solution to the integro-differential equation but the only solution to that equation. Thus, existence and uniqueness of the solution to the integro-differential equation are thereby established.

The strength of this presentation derives from the following;

(*i*) The need to stimulate research work in the important subject area of torsional vibration of beams carrying moving masses; and

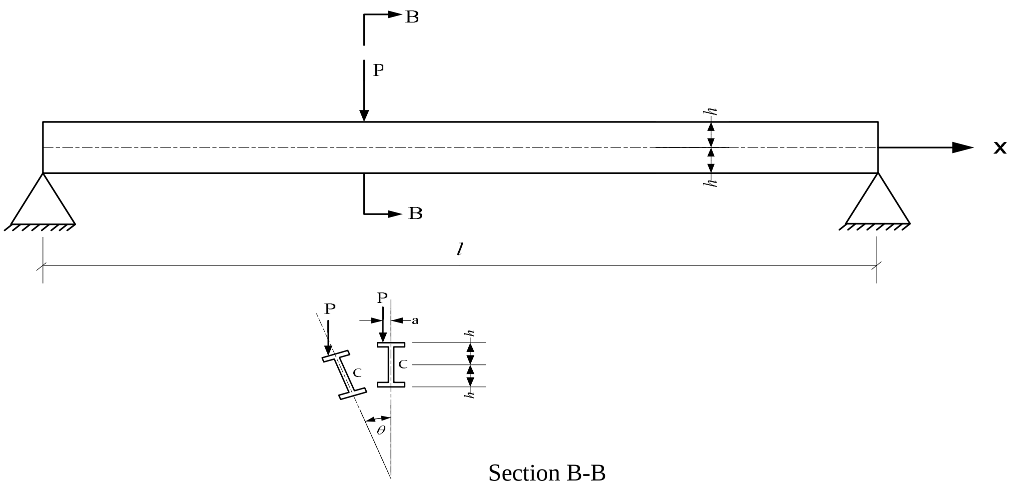
(*ii*) That despite the challenges posed by the presence of variable and singular coefficients in the describing differential equation, a rigorous analysis of the system can be made.

2.0 Problem formulation

The partial differential equations of motion, describing the coupled flexural-torsional vibration of a beam, have been lucidly set up in [9]. For a beam that is doubly symmetric, the flexural and torsional vibrations are completely uncoupled. The system under consideration here (see Figure 2.1) falls into this latter category. The elastic torsional vibration of the system due to moving-force moving-mass type of excitation is governed by:

$$\alpha\frac{\partial^4\theta}{\partial x^4}-\beta\frac{\partial^2\theta}{\partial x^2}+\phi(x,t)\frac{\partial^2\theta}{\partial t^2}=Pa\,\delta(x-vt)\ . \qquad (2.1)$$

We consider, here, simple boundaries and for simplicity of presentation, without any loss of generality, homogeneous initial conditions. Thus, the boundary and initial conditions are, respectively, given by;



Section B-B

Figure 2.1: A beam of doubly-symmetric cross-section undergoing torsional vibration

$$\theta(0,t)=\theta(1,t)=\theta''(0,t)=\theta''(1,t)=0, \tag{2.2}$$

and
$$\theta(x,0)=\frac{\partial\theta(x,0)}{\partial t}=0 \tag{2.3}$$

Equations (2.1) through (2.3) constitute the problem for which the solution is sought. We will be referring to the problem, in all further developments, simply as Problem P1.

The notations used in Problem P1, which are not obvious from Figure 2.1, are defined as follows: $\theta(x,t)$ is the angular displacement, α is the warping rigidity of the section while β is its torsional rigidity, g is the acceleration due to gravity, v is the constant velocity of traverse of the mass and δ is the Dirac distribution. $\phi(x,t)$ is given by;

$$\phi(x,t)=\rho J+\frac{P}{g}h^2\delta(x-vt), \tag{2.4}$$

where, ρ is the density of the beam material, J is the centroidal polar moment of inertia of the section and t is time.

3.0 Problem analysis and solution

3.1 The associated moving-force problem

The associated moving-force problem is obtained from problem P1 by ignoring the inertia of the moving mass, thus resulting in the following problem:

$$D\equiv\left(\alpha\frac{\partial^4}{\partial x^4}-\beta\frac{\partial^2}{\partial x^2}+\mu\frac{\partial^2}{\partial t^2}\right), \tag{3.1}$$

$$D[\eta(x,t)]=Pa\delta(x-vt) \tag{3.2}$$

$$\eta(0,t)=\eta(\lambda,t)=\eta''(0,t)=\eta''(\lambda,t)=0 \tag{3.3}$$

$$\eta(x,0)=\dot{\eta}(x,0)=0. \tag{3.4}$$

In equations (3.2) through (3.4), $\eta(x,t)$ is the moving force response and a dot on top of any quantity indicates differentiation with respect to time.

The notation μ , replaces ρJ , for convenience. Let it be assumed, meanwhile, that D is an invertible operator. Thus,

$$\exists D^{-1} : \eta(x,t) = D^{-1}\{Pa\delta(x - vt)\} \; . \tag{3.5}$$

By the existence of D^{-1}, it is implied that, there exists Green's function G attached to D which allows the solution $u(x,t)$ of the general operator equation

$$D[u(x,t)] = q(x,t) \, , \tag{3.6}$$

to be written in the form

$$u(x,t) = \int_S \int_T G(x,\xi,t,\tau)q(\xi,\tau)d\xi d\tau \quad . \tag{3.7}$$

In (3.7), the boundary and initial conditions (2.2) and (2.3) respectively, are duly satisfied. S and T indicate the spatial and temporal domain of the problem respectively.

To establish the existence of G as well as to obtain explicit expression for G, we consider the general operator eq (3.6). In the context of the appropriate boundary and initial conditions of the problem as laid out in equations (2.2) and (2.3) respectively, the Laplace transformation of eq (3.6) yields:

$$\alpha\overline{u}^{////}(x,s) - \beta\overline{u}''(x,s) + \mu\overline{u}(x,s) = \overline{q}(x,t) \, , \tag{3.8}$$

where primes denote differentiations with respect to x. The Laplace images of $u \quad and \quad q$ are defined as follows:

$$\overline{u}(x,s) = \int_0^\infty e^{-st}u(x,t); \text{ and } \overline{q}(x,s)\int_0^\infty e^{-st}q(x,t)dt. \tag{3.9}$$

We can express $\overline{u}$ and $\overline{q}$ in their bilinear expansion forms, as infinite series, given by:

$$\overline{u}(x,s) = \sum_{j=1}^{\infty} A_j(s) f_j(x); \tag{3.10}$$

$$\overline{q}(x,s) = \sum_{j=1}^{\infty} B_j(s) f_j(x) \tag{3.11}$$

The coordinate functions $f_j(x)$ in eqs (3.10) and (3.11) are the eigenfunctions of the following auxiliary problem:

$$\alpha f_j^{////}(x) - \beta f_j''(x) = \mu\omega_j^2 f_j(x) \, , \tag{3.12}$$

$$f_j(0) = f_j(\lambda) = f_j''(0) = f_j''(\lambda) = 0 \tag{3.13}$$

***Proposition* 3.1**

The series in (3.10) and (3.11) are absolutely and uniformly convergent.

Proof

Let F be a differential operator given by;

$$F \equiv \alpha \frac{d^4}{dx^4} - \beta \frac{d^2}{dx^2} \quad . \tag{3.14}$$

Consider $u, w \in C^4[0,\lambda]$ such that:

$$u(0) = u(l) = u''(0) = u''(l) = 0, \text{and } w(0) = w(l) = w''(0) = w''(l) = 0. \tag{3.15}$$

In the sense of $L_2(0,\lambda)$ inner product,

$$(\mathrm{Fu, w}) = \int_0^1 \left\{ \alpha \frac{\mathrm{d}^4 u}{dx^4} - \beta \frac{\mathrm{d}^2 u}{dx^2} \right\} w(x) dx = \int_0^1 \left\{ \alpha \frac{\mathrm{d}^4 w}{dw^4} - \beta \frac{\mathrm{d}^2 w}{dx^2} \right\} u(x) dx = (Fw, u) \tag{3.16}$$

Thus F is symmetric and self adjoint. Furthermore,

$$(\mathrm{Fu, u}) = \int_0^\lambda \left\{ \alpha \frac{\mathrm{d}^4 u}{dx^4} - \beta \frac{\mathrm{d}^2 u}{dx^2} \right\} w(x) dx = \int_0^\lambda \left\{ \alpha \left(\frac{\mathrm{d}^2 u}{dw^2} \right)^2 + \beta \left(\frac{\mathrm{d}w}{dx} \right)^2 \right\} dx. \tag{3.17}$$

But by Friedrich's inequality {see Reddy [6]}

$$\int_0^1 \mathrm{u}^2 dx \le c_1 \int_0^1 \left(\frac{\mathrm{du}}{dx} \right)^2 dx, \text{where } c_1 > 0 \,. \tag{3.18}$$

Following from (3.18) and, due to the homogeneous boundary conditions, we also have;

$$\int_0^\lambda \left(\frac{\mathrm{du}}{dx} \right)^2 dx \le c_1 \int_0^\lambda \left(\frac{\mathrm{d}^2 \mathrm{u}}{dx^2} \right)^2 \mathrm{dx} \tag{3.19}$$

Now, introducing (3.18) and (3.19) into (3.17) leads to:

$$(\mathrm{Fu, u}) \ge \left(\frac{\alpha}{c_1^2} + \frac{\beta}{c_1} \right) \int_0^\lambda u^2 dx = \left(\frac{\alpha}{c_1^2} + \frac{\beta}{c_1} \right) \|\mathrm{u}\|^2_{L_2} dx \,. \tag{3.20}$$

Thus, F is a positive-definite operator. Since F, which is the differential operator that generates $f_j(x)$, has been shown to be symmetric, self-adjoint and positive-definite, the series in (3.10) and (3.11) are absolutely and uniformly convergent, in accordance with Hilbert-Schmidt theorem, {see Bolotin [1]}. In addition, the coordinate functions, $f_j(x)$, can be appropriately named to satisfy the following orthonormality conditions:

$$\int_0^l f_j(x) f_k dx = \delta_{jk} = \begin{cases} 1, if \;\; j = k, \\ 0, otherwise. \end{cases} \tag{3.21}$$

In view of eqs (3.10) and (3.11), eq (3.7) yields

$$\sum_{j=1}^{\infty} A_j(s) \left\{ \alpha f_j^{\prime\prime\prime\prime} - \beta f_j^{\prime\prime} + \mu s^2 f_j \right\} = \sum_{j=1}^{\infty} B_j(s) f_j(x). \tag{3.22}$$

When we introduce eq (3.12) into the left hand side of (3.22), we obtain

$$\sum_{j=1}^{\infty} A_j(s) \mu \left\{ s^2 + \omega_j^2 \right\} f_j(x) = \sum_{j=1}^{\infty} B_j(s) f_j(x) \tag{3.23}$$

Due to the orthonormality conditions (3.21), eq (3.23) yields

$$A_j(s) = \frac{B_j(s)}{\mu\left(s^2 + \omega_j^2 \right)}. \tag{3.24}$$

Similarly, from (3.11) and (3.21) we obtained

$$B_j(s) = \int_0^1 \bar{q}(x, s) f_j(x) dx, \tag{3.25}$$

which can be used in (3.24) to yield $$A_j(s)=\frac{\int_0^{\lambda}\bar{q}(x,s)f_j(x)dx,}{\mu\left(s^2+\omega_j^2\right)}\qquad(3.26)$$

When (3.26) is introduced into (3.10), we obtain

$$\bar{u}(x,s)=\sum_{j=1}^{\infty}\frac{1}{\mu\left(s^2+\omega_j^2\right)}\int_0^{\lambda}f_j(x)f_j(\xi)\bar{q}(\xi,s)d\xi\quad .\qquad(3.27)$$

Upon inverting the Laplace transforms in (3.27), using the convolution theorem, one obtains:

$$u(x,t)=\sum_{j=1}^{\infty}\frac{1}{\mu\omega_j}\int_0^{\lambda}\int_o^{t}f_j(x)f_j(\xi)\sin\omega_j(t-\tau)q(\xi,\tau)d\xi d\tau,\qquad(3.28)$$

as the solution, at least in the formal sense, to problem (3.12).,

In order to establish that the formal representation of the solution, as given in (3.28), is indeed the actual solution, the series in (3.28) has to be proved to be absolutely and uniformly convergent.

Let the integration with respect to τ on the right hand side of (3.28) be carried out by parts, once, to obtain:

$$u(x,t)=\sum_{j=1}^{\infty}\frac{1}{\mu\omega_j^{\;2}}\int_0^{\lambda}\{q(\xi,t)-\cos\omega_j tq(\xi,o)-\int_0^{t}\cos\omega_j(t-\tau).\frac{\partial q}{\partial\tau}d\tau\}f_j(x)f_j(\xi)d\xi.\qquad(3.29)$$

We, first, consider the third series on the right hand side of (3.29). Noting that ω_j^2 are all positive,

$$\left|\sum_{j=1}^{\infty}\frac{1}{\mu\omega_j^2}\int_0^{l}\int_0^{t}\cos\omega_j(t-\tau)\frac{\partial q}{\partial\tau}f_y(x)f_y(\xi)d\xi d\tau\right|\le\sum_{j=1}^{\infty}\frac{1}{\mu\omega_j^2}\int_0^{l}\int_0^{t}|\cos\omega_j(t-\tau)|\,|f_j(x)|\,|f_j(\xi)|\left|\frac{\partial q}{\partial\tau}\right|d\xi d\tau\qquad(3.30)$$

For a stable system, the eigenfunctions are uniformly bounded, that is,

$$\forall x\in[o,1],\exists\gamma:\gamma<\infty,\text{and }\left|f_j(x)\right|<\gamma\ for\ j=1,2,3...\qquad(3.31)$$

Besides,

$$\max\left|\cos\omega_j(t-\tau)\right|=1,\text{for all }\tau:0\le\tau\le t.\qquad(3.32)$$

Suppose the improper integral $\int_0^{\infty}\left|\frac{\partial q}{\partial\tau}\right|d\tau$ is bounded, that is,

$$\int_0^{\infty}\left|\frac{\partial q}{\partial\tau}\right|d\tau=\varepsilon<\infty.\qquad(3.33)$$

Introducing the relation (3.31), (3.32) and (3.33) into (3.30) results in:

$$\left|\sum_{j=1}^{\infty}\frac{1}{\mu\omega_j^2}\int_0^{\lambda}\int_0^{t}\cos\omega_j(t-\tau)\frac{\partial q}{\partial\tau}f_j(x)f_j(\xi)d\xi d\tau\right|\le\sum_{j=1}^{\infty}\frac{\lambda\gamma^2\varepsilon}{\mu\omega_j^2}.\qquad(3.34)$$

The auxiliary problem (3.12) actually corresponds to the free vibration of the system. It follows then that it admits Green's function $G^*(x,\xi)$ given by

$$G^*(x,\xi)=\sum_{j=1}^{\infty}\frac{f_j(x)f_j(\xi)}{\mu\omega_j^2},\qquad(3.35)$$

a symmetric and bounded quantity with the series on the right being absolutely and uniformly convergent. In view of the boundedness of G^* and the orthonormality properties of $f_j(x)$, we have:

$$\int_0^{\lambda}\sum_{j=1}^{\infty}\frac{f_j(x)f_j(x)}{\mu\omega_j^2}dx=\sum_{j=1}^{\infty}\frac{1}{\mu\omega_j^2}=\int_0^{\lambda}G^*(x,x)dx=\sigma<\infty\,. \tag{3.36}$$

As it can be seen from (3.36) the series, $\sum_{j=1}^{\infty}\frac{1}{\mu\omega_j^2}$ is uniformly convergent. Therefore inequality (3.34) becomes

$$\left|\sum_{j=1}^{\infty}\frac{1}{\mu\omega_j^2}\int_0^{l}\int_0^{t}\cos\omega_j(t-\tau)\frac{\partial q}{\partial\tau}f_j(x)f_j(\xi)d\xi d\tau\right|\le\lambda\gamma^2\sigma\varepsilon\,\cdot \tag{3.37}$$

We can now examine the first two series on the right hand side of (3.29). From all practical considerations, we are justified to assume that q is uniformly bounded, that is;

$$\exists q^*:\left|q(x,t)\right|<q^*<\infty,\forall t\ge 0\,. \tag{3.38}$$

Then, it follows that

$$\left|\sum_{j=1}^{\infty}\frac{1}{\mu w_j^2}\int_0^{t}\{q(\xi,t)-\cos\omega_j tq(\xi,0)\}f_j(x)\,f_y(\xi)\,d\xi\right|\le\sum_{j=1}^{\infty}\frac{1}{\mu\omega_j^{\,2}}\int_0^{1}\{\left|q(\xi,t)\right|$$
$$\left|f_j(x)\right|\left|f_j(\xi)\right|+\left|\cos\omega_j t\right|\left|q(\xi,t)\right|\left|f_j(x)\right|\left|f_j(\xi)\right|\}d\xi\le 2\sigma 1\,q^*\gamma. \tag{3.39}$$

Relations (3.37) and (3.39) establish the absolute and uniform convergent of the series in (3.29) or (3.28). The conclusion, then, is that (3.28) or equivalently, (3.29) is the actual solution to problem (3.12).

The proceeding proof of convergence justifies interchange of summation and integration in (3.28) or (3.29). Thus, we can re-write (3.28) as

$$u(x,t)=\int_0^{\lambda}\int_0^{t}G(x,\xi,t,\tau)q(\xi,\tau)d\xi d\tau\,, \tag{3.40}$$

where G is Green's function of the operator *D*, given by

$$G(x,\xi,t,\tau)=\sum_{j=1}^{\infty}\frac{1}{\mu\omega_j}f_j(x)f_j(\xi)\sin\omega_j(t-\tau) \tag{3.41}$$

Consequently, the moving-force response of the system, as given by (3.5), is as follows:

$$\eta(x,t)=\int_0^{\lambda}\int_0^{t}G(x,\xi,t,\tau).Pa\delta(\xi-v\tau)d\xi d\tau\,. \tag{3.42}$$

In view of the properties of the Dirac distribution, the right hand side of (3.42) can be integrated with respect to ξ to yield:

$$\eta(x,t)=Pa\int_0^{t}G(x,v\tau;t,\tau)dt=Pa\int_o^{t}\sum_{j=1}^{\infty}\frac{1}{\mu\omega_j}f_j(x)f_j(v\tau)\sin\omega_j(t-\tau)d\tau\,. \tag{3.43}$$

If one examines the auxiliary problem (3.12), it is clear that it admits the solution

$$f_j(x) = \sqrt{\frac{2}{\lambda}} \sin\left(\frac{j\pi x}{\lambda}\right). \tag{3.44}$$

And, the eigenvalues are given by:
$$\omega_j^2 = \frac{\alpha\left(\frac{j\pi}{\lambda}\right)^4 + \beta\left(\frac{j\pi}{\lambda}\right)^2}{\mu}. \tag{3.45}$$

Therefore,

$$G(x,\xi,t,\tau) = \frac{2}{\mu\lambda}\sum_{j=1}^{\infty} \frac{\sin\left(\frac{j\pi x}{\lambda}\right)\sin\left(\frac{j\pi\xi}{\lambda}\right)\sin\omega_j(t-\tau)}{\omega_j}, \tag{3.46}$$

and

$$\eta(x,t) = \frac{2Pa}{\lambda\mu}\sum_{j=1}^{\infty} \frac{\sin\left(\frac{j\pi x}{\lambda}\right)\left\{\left(\frac{j\pi v}{\lambda}\right)\sin\omega_j t - \omega_j \sin\left(\frac{j\pi vt}{\lambda}\right)\right\}}{\omega_j\left(\frac{j^2\pi^2 v^2}{\lambda^2} - \omega_j^{\,2}\right)}. \tag{3.47}$$

3.2 The moving-force moving-mass problem

This is the problem P(1), given by (2.1) through (2.3). Equivalently, (2.1) can be re-stated as:

$$D[\theta(x,t)] = Pa\delta(x-vt) - \frac{Ph^2}{g}\frac{\partial^2\theta}{\partial t^2}\delta(x-vt)\ . \tag{3.48}$$

Pre-multiplying each term of eq. (3.48) by the operator D^{-1} results in the following:

$$D^{-1}D[\theta(x,t)] = D^{-1}[Pa\delta(x-vt)] + D^{-1}[-\frac{Ph^2}{g}\frac{\delta^2\theta}{\delta t^2}\delta(x-vt)]\,. \tag{3.49}$$

But, by definition, $D^{-1}D$ is an identity operator. In addition; $D^{-1}[Pa\delta(x-vt)]$ is the moving-force response, $\eta(x,t)$, according to (3.5). Moreover, D^{-1} is an integral transformation, as given in (3.7). Consequently, (3.49) reduces to:

$$\theta(x,t) = \eta(x,t) - \int_0^{\lambda}\int_0^{t} G(x,\xi,t,\tau)\frac{Ph^2}{g}\frac{\partial^2\theta(\xi,\tau)}{\partial\tau^2}\delta(\xi - v\tau)d\xi d\tau\ . \tag{3.50}$$

Using a fundamental property of the Dirac distribution, (3.50) can be simplified to read:

$$\theta(x,t) = \eta(x,t) - \frac{Ph^2}{g}\int_0^{t} G(x,v\tau;t,\tau)\frac{d^2\theta(v\tau,\tau)}{d\tau^2}d\tau, \quad . \tag{3.51}$$

where $\frac{\partial^2\theta(v\tau,\tau)}{\partial\tau^2} = \left[\frac{\partial^2\theta(\xi,\tau)}{\partial\tau^2}\right]_{\xi=v\tau}$. Thus, the differential equation of the moving-force moving-mass problem has been transformed into an integro-differential equation, given by (3.51).

4.0 Solution of the integro-differential equation

Equation (3.51) is not readily amenable to exact or closed-from solution. However, we can seek the solution through the successive approximations procedure. This approach constructs a sequence of approximations, $\theta_n(x,t)$, based on the scheme below, which progressively improves the accuracy of the solution. It is technically logical to initialize the scheme with a simple function which satisfies the boundary and initial conditions of the problem. In view of this, it is proposed that:

$$\theta_0(x,t) = \sin\frac{\pi x}{1}(1-\cos\pi t)$$

$$\theta_1(x,t) = \eta(x,t) - \lambda\int_0^t G(x,v\tau,.t,\tau)\frac{\partial^2\theta_0(v\tau,\tau)}{\partial\tau^2}d\tau$$

$$\theta_2(x,t) = \eta(x,t) - \lambda\int_0^t G(x,v\tau,t,\tau)\frac{\partial^2\theta_1(v\tau,\tau)}{\partial\tau^2}d\tau \qquad (4.1)$$

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$$.\theta_n(x,t) = \eta(x,t) - \lambda\int_o^t G(x,v\tau,t,\tau)\frac{\partial^2\theta_{n-1}(v\tau,\tau)}{\partial\tau^2}d\tau.$$

***Proposition* 4.1**

As $n \to \infty$, the sequence of functions given in (4.1) converges to a continuous function of *x* and *t*.

Proof

Let us consider a generic term of the sequence given by:

$$\theta_{n+1} = \eta(x,t) - \lambda\int_0^t G(x,v\tau,t,\tau)\frac{\partial^2\theta_n(v\tau,\tau)d\tau}{\partial\tau^2}. \qquad (4.2)$$

But, following from (3.46), we easily deduce that

$$\left|G(x,\xi,t,\tau)\right| = \left|\frac{2}{\mu\lambda}\sum_{j=1}^{\infty}\frac{\sin\left(\frac{j\pi x}{\lambda}\right)\sin\left(\frac{j\pi\xi}{\lambda}\right)\sin\omega_j(t-\tau)}{\omega_j}\right| \qquad (4.3)$$

Examining (3.45), noting that

$$\frac{\beta}{\mu}\left(\frac{j\pi}{\lambda}\right)^2 > 0, \qquad 4.4)$$

it follows that:

$$\omega_j^2 = \frac{\alpha}{\mu}\left(\frac{j\pi}{\lambda}\right)^4 + \frac{\beta}{\mu}\left(\frac{j\pi}{\lambda}\right)^2 > \frac{\alpha}{\mu}\left(\frac{j\pi}{\lambda}\right)^4. \qquad (4.5)$$

If we combine (4.4) and (4.5) we obtain:

$$\left|G(x,\xi,t,\tau)\right| = \frac{2}{\mu\lambda}\sum_{j-1}^{\infty}\frac{1}{\left|\frac{\alpha}{\mu}(\frac{j\pi}{\lambda})^4+\frac{\beta}{\mu}(\frac{j\pi}{\lambda})^2\right|} \quad \leq \quad \frac{2\lambda}{\pi^2\sqrt{\alpha\mu}}\sum_{j=1}^{\infty}\frac{1}{j^2}. \tag{4.6}$$

The quantity, $\sum_{j=1}^{\infty}\frac{1}{j^2}$, is clearly the Riemann Zeta, $\zeta(2)$, function whose majorant is $\frac{\pi^2}{6}$.

Therefore, relation (4.6) reduces to

$$\left|G(x,v\tau,t,\tau)\right| \leq \frac{\lambda}{3\sqrt{\alpha\mu}} \quad . \tag{4.7}$$

Hence,

$$\int_0^t \left|G(x,v\tau,t,\tau)\right|^2 d\tau \leq \frac{\lambda^2}{9\alpha\mu}, \tag{4.8}$$

where it has been observed that:

$$0 \leq v\tau \leq \lambda. \tag{4.9}$$

Moreover, from the chosen $\theta_{0,}$, as given in (4.1),

$$\int_0^t \left|\frac{\partial^2\theta(v\tau,\tau)}{\partial\tau^2}\right|^2 d\tau = \int_0^t \left|\pi^2\sin\left(\frac{\pi v\tau}{\lambda}\right)\cos\pi\tau\right| \leq \frac{\lambda\pi^4}{v}. \tag{4.10}$$

The elastic properties of the vibrating system allow the assumption that the differential operator $\frac{\partial^2}{\partial t^2}$, is Lipshitz continuous. In that case, $\exists B : 0 < B < \infty$.

$$\left|\frac{\partial^2\theta_m}{\partial\tau^2} - \frac{\partial^2\theta_n}{\partial\tau^2}\right| \leq B\left|\theta_m - \theta_n\right|, \tag{4.11}$$

where B is a constant. Now, from(4.3) follows that:

$$(\theta_{n+1} - \theta_n)^2 = \left(\lambda\int_0^t\left\{\frac{\partial^2\theta_n}{\partial\tau^2} - \frac{\partial^2\theta_{n-1}}{\partial\tau^2}\right\}Gd\tau\right)^2. \tag{4.12}$$

Applying the Schwartz inequality to (4.12) leads to the following:

$$\left(\theta_{n+1} - \theta_n\right)^2 \leq \lambda^2\int_0^t \left|\frac{\partial^2\theta_{n-1}}{\partial\tau^2} - \frac{\partial^2\theta_{n-1}}{\partial\tau^2}\right|^2 d\tau\int_0^t\left|G\right|^2 d\tau \leq \frac{\lambda^2B^2\lambda^2}{9\alpha\mu v}\int_0^t\left|\theta_n - \theta_{n-1}\right|^2 d\tau, \tag{4.13}$$

where we have made use of (4.8) and (4.11) to arrive at the last inequality of (4.13). By a similar application of the Schwartz inequality, one can easily obtain:

$$\left(\theta_1 - \theta_0\right)^2 \leq \lambda^2\int_0^t\left|\frac{\partial^2\theta_0}{\partial\tau^2}\right|^2 d\tau\int_0^t\left|G\right|^2 d\tau \leq \frac{\lambda^2\pi^4\lambda^3}{9\mu\alpha v}, \tag{4.14}$$

where (4.8) and (4.10) were applied to obtain the last inequality of (4.14). As a consequence of (4.13) and (4.14), the following sequence ensures:

$$(\theta_2 - \theta_1)^2 \leq \frac{\lambda^2B^2\lambda^3}{9\alpha\mu v}\int_0^t\left|\theta_1 - \theta_0\right|^2 d\tau \leq \left(\frac{\lambda^2B^2\lambda^3}{9\alpha\mu v}\right)\left(\frac{\lambda^2\pi^4\lambda^3}{9\alpha\mu v}\right)\int_0^t d\tau,$$

$$\left(\theta_3-\theta_2\right)^2 \le \frac{\lambda^2 B^2 1^3}{9\alpha\mu\nu}\left(\frac{\lambda^2\pi^4 1^3}{9\alpha\mu\nu}\right)^2 \int_0^t dy \int_0^t d\tau,$$

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$$\left(\theta_{n+1}-\theta_n\right)^2 \le \quad \frac{\lambda^2 B^2 1^3}{9\alpha\mu\nu}\left(\frac{\lambda^2\pi^4 1^3}{9\alpha\mu\nu}\right)^n \left(\int_0^t dz \,\mathrm{L} \int_0^t dy \int_0^t d\tau \right) \quad . \tag{4.15}$$

The last quantity in brackets on the right hand side of (4.15) is a product of n integrals. Noting the following:

$$\int_0^t dz ... \int_0^t dy \int_0^t d\tau = \frac{t^n}{n!} \quad , \tag{4.16}$$

and
$$\max\left[\frac{t^n}{n!}\right] = \frac{1^n}{n! v^n}, \tag{4.17}$$

then the last member of the sequence (4.15) reduces to

$$(\theta_{n+1}-\theta_n)^2 \le \left(\frac{\lambda^2 B^2 \lambda^3}{9\alpha\mu\nu}\right)\left(\frac{\lambda^2 \pi^2 \lambda^3}{9\alpha\mu\nu}\right)^n \cdot \frac{\lambda^n}{n! v^n}. \tag{4.18}$$

Thus,
$$\left|\theta_{n+1}-\theta_n\right| \le \quad \frac{\Gamma\Omega^n}{\sqrt{n!}} \quad , \tag{4.19}$$

where,
$$\Gamma = \left(\frac{\lambda^2 B^2 \lambda^3}{9\alpha\mu\nu}\right), \tag{4.20}$$

and,
$$\Omega = \left(\frac{\lambda^2\pi^2 1^3}{9\alpha\mu\nu}\right)^{\frac{1}{2}} \tag{4.21}$$

Let us, now, consider the following infinite series:

$$\sigma(x,t) = \theta_0(x,t) + \{\theta_1(x,t)-\theta_0(x,t)\} + \{\theta_2(x,t)-\theta_1(x,t)\} + ... + \{\theta_{k+1}(x,t)-\theta_k(x,t)\} ... \tag{4.22}$$

The n+1st partial sum of the series (4.22) is evidently, $\theta_n(x,t)$. In view of this fact as well as the inequality (4.19), we then have:

$$\theta_n(x,t) \le \Gamma \sum_{k=0}^{n} \frac{\Omega^k}{\sqrt{k!}}. \tag{4.23}$$

It follows from the inequality (4.23) that the majorant of $\theta_n(x,t)$ is given by:

$$\hat{\theta}_n(x,t) = \Gamma \sum_{k=1}^{n} \frac{\Omega^k}{\sqrt{k!}} \quad , \tag{4.24}$$

which is always a converging series for finite values of Γ ,and Ω . Therefore,

$$\lim_{n\to\infty} \theta_n(x,t) = \sigma(x,t), \tag{4.25}$$

with the series converging uniformly.

***Proposition* 4.2**

The limit-function $\sigma(x,t)$ is a solution of the integro-differential equation (3.51).

Proof

We set, $$\sigma(x,t)=\theta_n(x,t)+T_n(x,t), \tag{4.26}$$

where, $T_n(x,t)$ is the remainder, as a result of the truncation of the infinite series. Therefore, by the inequality (4.23) we can put:

$$|T_n(x,t)| \quad \leq \quad \Gamma\sum_{k=n+1}^{\infty}\frac{\Omega^k}{\sqrt{k!}} \quad . \tag{4.27}$$

Since the majorant series is convergent, it follows that

$$\lim_{n\to 0}T_n^2(x,t)=0 \tag{4.28}$$

But from (4.1) and (4.26), we obtain:

$$\sigma(x,t)-\eta(x,t)+\lambda\int_o^t G(x,v\tau,t,\tau)\frac{\partial^2\sigma}{\partial\tau^2}d\tau = T_n(x,t)+\lambda\int_0^t \{\frac{\partial^2\sigma}{\partial\tau^2}-\frac{\partial^2\theta_{n-1}}{\partial\tau^2}\}Gd\tau\,. \tag{4.29}$$

Therefore,

$$\left\{\sigma(x,t)-\eta(x,t)+\lambda\int_0^t G\frac{\partial^2\sigma}{\partial\tau^2}\right\}^2 = \left\{T_n(x,t)+\lambda\int_0^t\left(\frac{\partial^2\sigma}{\partial\tau^2}-\frac{\partial^2\theta_{n-1}}{\partial\tau^2}\right)Gd\tau\right\}^2 . \tag{4.30}$$

By the algebra of real numbers, we know that, for $p, q\in\Re$

$$(p+q)^2 \leq 2p^2+2q^2\,. \tag{4.31}$$

Applying (4.31) to (4.30) results in

$$\left\{\sigma(x,t)-\eta(x,t)+\lambda\int_0^t G\frac{\partial^2\sigma}{\partial\tau^2}\right\}^2 \leq 2T_n^{\,2}+2\left\{\lambda\int_0^t\left(\frac{\partial^2\sigma}{\partial\tau^2}-\frac{\partial^2\theta_{n-1}}{\partial\tau^2}\right)Gd\tau\right\}^2$$

$$\leq 2T_n^{\,2}+2\lambda^2\int_0^t\left|\frac{\partial^2\sigma}{\partial\tau^2}-\frac{\partial^2\theta_{n-1}}{\partial\tau^2}\right|^2 d\tau\int_0^t|G|^2d\tau\,, \tag{4.32}$$

where the last inequality of (4.32) derives from the application of the Schwartz inequality.

In view of the inequality (4.8) and (4.11), the relation (4.32) can be simplified to read:

$$\left\{\sigma(x,t)-\eta(x,t)+\lambda\int_0^t G\frac{\partial^2\sigma}{\partial\tau^2}\right\}^2 \leq 2T_n^2+\frac{2\lambda^2 Bl^3}{9\alpha\mu v}\int_0^t|\sigma-\theta_{n-1}|^2\,d\tau\,. \tag{4.33}$$

Let us define the quantity $T_{n-1}(x,t)$ as

$$T_{n-1}(x,t)=\sigma_{n-1}(x,t)-\theta_{n-1}(x,t). \tag{4.34}$$

Then, $$\int_0^t|\sigma-\theta_{n-1}|^2\,d\tau \leq \quad t.\sup T_{n-1}^{\,2} \leq \frac{1}{v}\sup T_{n-1}^{\,2}\,. \tag{4.35}$$

Thus, inequality (4.33) reduces to

$$\sigma(x,t)-\eta(x,t)+\lambda\int_0^t G\frac{\partial^2\sigma}{\partial\tau^2}d\tau)^2 \leq 2T_n^2+\frac{2\lambda^2 B\lambda^4}{9\alpha\mu v^2}\sup T_{n-1}^2 \tag{4.36}$$

Observing that, as $n\to\infty$, the quantity on the right hand side of (4.36) vanishes and that the quantity on the left, must be non-negative, only the equality of (4.36) holds. Therefore

$$\sigma(x,t)=\eta(x,t)-\lambda\int_0^t G(x,v\tau,t,\tau)\frac{\partial^2\sigma(v\tau,\tau)}{\partial\tau^2}d\tau . \tag{4.37}$$

The proof is complete.

***Proposition* 4.3**

$\sigma(x,t)$ is the only continuous function which satisfies the integro-differential equation.

Proof

Let there exist another continuous function, $g(x,t)$, distinct from $\sigma(x,t)$, that satisfies the integro-differential equation (3.51). Let us assume that

$$\left|g(x,t)-\theta_0(x,t)\right|\le g^*<\infty , \tag{4.38}$$

where g^*, is a constant. Then, as a solution of the integro-differential (3.51), we have:

$$g(x,t)=\eta(x,t)-\lambda\int_0^t G(x,v\tau,t,\tau)\frac{\partial^2 g(v\tau,\tau)}{\partial\tau^2}d\tau \tag{4.39}$$

From (4.1) and (4.39), we obtain

$$g(x,t)-\theta_n(x,t)=\lambda\int_0^t G(x,v\tau,t,\tau)\left\{\frac{\partial^2\theta_{n-1}}{\partial\tau^2}-\frac{\partial^2 g(v\tau,\tau)}{\partial\tau^2}\right\}d\tau \tag{4.40}$$

Again, applying the Schwartz inequality to (4.40) results in:

$$\left|g(x,t)-\theta_n(x,t)\right|^2\le\lambda^2\int_0^t|G|^2\int_0^t\left|\frac{\partial^2 g}{\partial\tau^2}-\frac{\partial^2\theta_{n-1}}{\partial\tau^2}\right|^2 d\tau \quad \le\frac{\lambda^2 Bl^3}{9\alpha\mu v}\int_0^t\left|g-\theta_{n-1}\right|^2 d\tau \tag{4.41}$$

Putting n = 1 in (4.41) yields:

$$\left|g(x,t)-\theta_1(x,t)\right|^2\le\frac{\lambda^2 B\lambda^3 g^{*2}}{9\alpha\mu v^2}t . \tag{4.42}$$

Similarly, allowing n to take the value of 2, we obtain:

$$\left|g(x,t)-\theta_2(x,t)\right|^2\le\frac{\lambda^2 B\lambda^3 g^{*2}}{9\alpha\mu v}\int_0^t\left|g-\theta_1\right|^2 d\tau\le g*^2\left(\frac{\lambda^2 B\lambda^3}{9\alpha\mu v}\right)^2\frac{t^2}{2!} . \tag{4.43}$$

Thus, in general,

$$\left|g(x,t)-\theta_n(x,t)\right|^2\le g^{*2}\left(\frac{\lambda^2 Bl^3}{9\alpha\mu v}\right)^n\frac{t^n}{n!}\le\frac{g^{*2}}{n!}\left(\frac{\lambda^2 Bl^4}{9\alpha\mu v}\right)^n=\frac{g^{*2}\Omega^{2n}}{n!} . \tag{4.44}$$

Hence,

$$\left|g(x,t)-\theta_n(x,t)\right|\le\frac{g^*\Omega^n}{\sqrt{n!}} . \tag{4.45}$$

Along the line of our earlier discussions leading to (4.25), we have

$$\lim_{n\to\infty}\left|g(x,t)-\theta_n(x,t)\right|\le\lim_{n\to\infty}\frac{g^*\Omega^n}{\sqrt{n!}}=0 . \tag{4.46}$$

Therefore,

$$g(x,t)=\lim_{n\to\infty}\theta_n(x,t)=\sigma(x,t) . \tag{4.47}$$

The proof is complete.

5.0 Numerical illustration

From the foregoing developments, we have the following:

$$G(x,\xi,t,\tau)=\frac{2}{\mu\lambda}\sum_{j=1}^{\infty}\frac{\sin\left(\frac{j\pi x}{\lambda}\right)\sin\left(\frac{j\pi\xi}{\lambda}\right)\sin\omega_j(t-\tau)}{\omega_j}; \tag{5.1}$$

$$\eta(x,t)=\frac{2Pa}{\lambda\mu}\sum_{j=1}^{\infty}\frac{\sin\left(\frac{j\pi x}{\lambda}\right)\left\{\left(\frac{j\pi v}{\lambda}\right)\sin\omega_j t-\omega_j\sin\left(\frac{j\pi vt}{\lambda}\right)\right\}}{\omega_j\left(\frac{j^2\pi^2v^2}{\lambda^2}-\omega_j{}^2\right)}. \tag{5.2}$$

Hence, if $\theta_0=\sin\frac{\pi x}{\lambda}\left(1-\cos\frac{\pi v}{l}t\right)$, Then

$$\theta_1=\frac{2Pa}{\lambda\mu}\sum_{j=1}^{\infty}\frac{\sin\left(\frac{j\pi x}{\lambda}\right)\left\{\left(\frac{j\pi x}{\lambda}\right)\sin\omega_j t-\omega_j\sin\left(\frac{j\pi vt}{\lambda}\right)\right\}}{\omega_j\left(\frac{j^2\pi^2v^2}{\lambda^2}-\omega_j{}^2\right)}+$$

$$+\frac{\pi^2v^2Ph^2}{2\mu\lambda g}\sum_{j=1}^{\infty}\sin\left(\frac{j\pi x}{\lambda}\right)\left\{\frac{\cos\frac{(j-2)\pi vt}{\lambda}-\cos\omega_j t}{\omega_j^2-\frac{(j-2)^2\pi^2v^2}{\lambda^2}}-\frac{\cos\frac{(j+2)\pi vt}{\lambda}-\cos\omega_j t}{\omega_j^2-\frac{(j+2)^2\pi^2v^2}{\lambda^2}}\right\}. \tag{5.3}$$

Continuing this procedure, higher order approximations, $\theta_2,\theta_3,\dots$, can be similarly generated.

Graphs of factored moving-force and moving-force-moving-mass responses, at mid span of the beam as functions of time have been plotted as shown in Figures 5.1 and 5.2 respectively for the following system's parameters:

h = 0.295m, ρ = 7846 kg/m^3, $J=1.885\times10^{-6}\,m^4$, $\mu=\rho J=0.0148kgm$,

$\alpha=4.65\times10^6\,Nm^4$, $\beta=GJ=1.45\times10^5\,Nm^2$, $\lambda=2.5m$, $v=3m/s$, $a=0.15m$.

The factored responses are defined as follows: $$\eta^*=\frac{\mu\lambda\eta}{2Pa}, \tag{5.4}$$

and ` $$\theta_1^*=\frac{\mu\lambda\theta_1}{2Pa} \tag{5.5}$$

The figures are based on the first terms of the series only in each case.

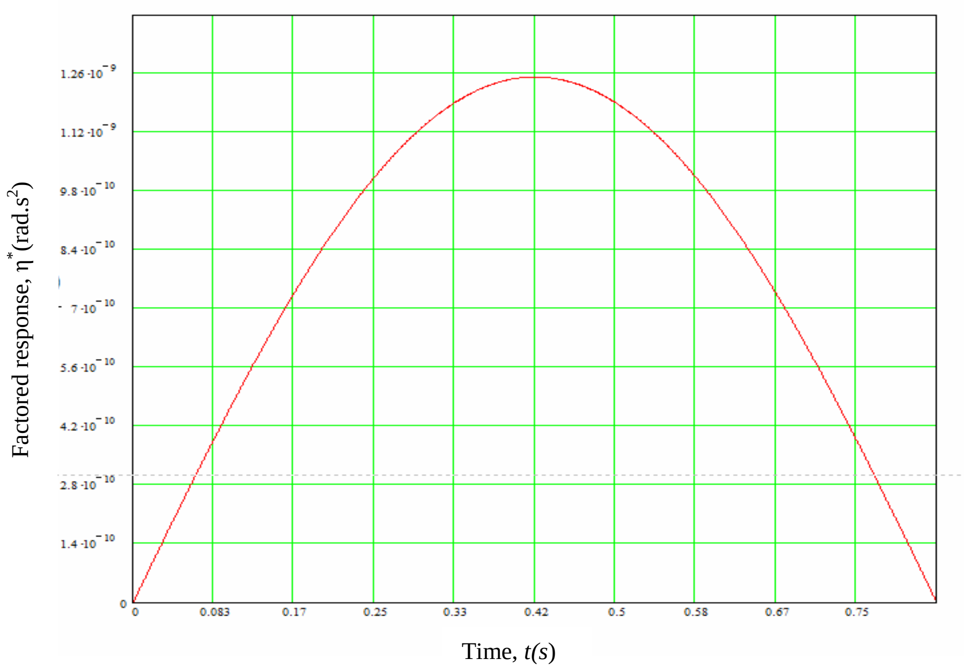

Figure 5.1: Graph of the factored mid-span response versus time for the associated moving-force problem

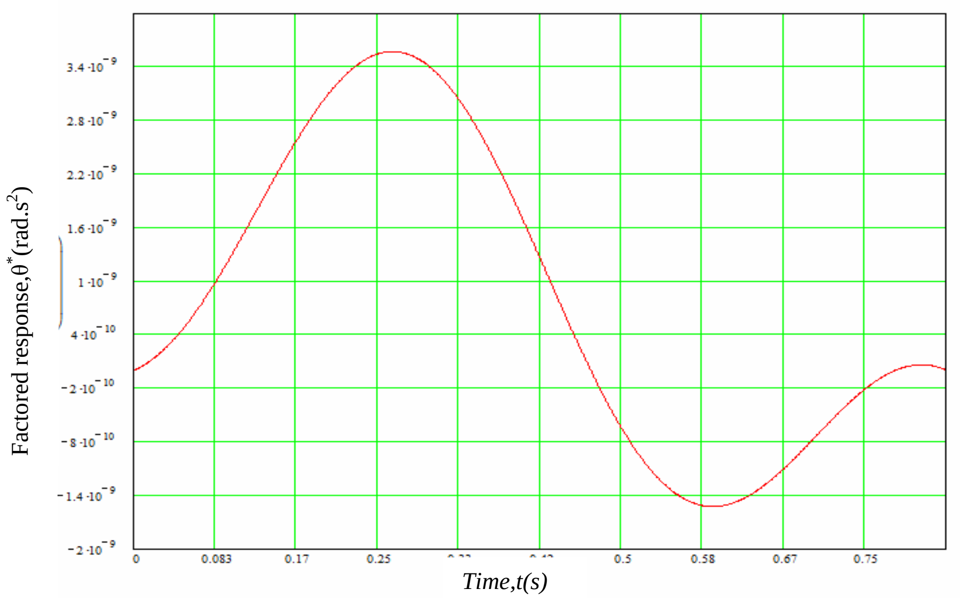

Figure 5.2: Graph of the factored mid-span response versus time for the moving-force moving-mass problem

6.0 Discussion

From Figures 5.2 and 5.3, the difference between the moving-force response and the moving-force-moving-mass response is considerable. First, on the criterion of the greatest magnitude of the angular deformation, we cannot rely on the moving force response in designing the system as it undervalues the greatest deformation. Second, the zeros of the motion are different. These underscore the desirability of assessing the moving-force moving-mass response as against the moving force response alone.

7.0 Conclusion

A rigorous analysis of the torsional vibration of an elastic beam with open doubly-symmetric cross-section subjected to moving concentrated masses has been presented. By using Green's function of the associated moving-force problem, the complete moving-force moving-mass problem was converted to an integro-differential equation solvable by an iterative scheme which has been shown to converge to a unique function of space and time as the only solution to the integro-differential equation. A comparison of the moving-force response and the moving-force-moving-mass response shows that the former can only be treated, at best, as a first approximation to the actual response.

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