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Perturbation appraisal of the dynamic buckling of an elastic model structure pressurized by a slowly varying load

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Abstract

In this paper, we examine the dynamic stability of a nonlinear dynamical system, with quadratic nonlinearity, pressurized by a strictly slowly varying time dependent load applied just after the initial time. Regular perturbation method in asymptotic expansions of the variables is used .The dynamic buckling load is determined nontrivially and is related to the static buckling load. Such a procedure by-passes the labour of repeating the entire asymptotic procedure for different imperfection parameters.

1.0 Introduction

Slowly varying nonlinear dynamical systems have long been investigated since its humble beginning by Kuzmak [1] .Over the intervening years , many expository contributions on the subject matter have been added to swell up the repository of knowledge on the subject . These include investigations by Luke [2] , Bourland and Haberman [3], Kervorkian [4] and Li and Kervorkian [5] , among others .However, none of the cited investigations , and perhaps ,very few of the uncited ones have specifically addressed the topic in a dynamic buckling setting. In this paper, we are therefore confronted with a nonlinear dynamical problem, with quadratic nonlinearity, that is trapped by a slowly varying explicitly time dependent loading history .Our objective is to determine the dynamic buckling load λ_D for which the structure becomes dynamically unstable.

2.0 Formulation

Budiansky and Hutchinson [6-8] were the first to investigate the dynamic stability of elastic model structures, one of which was the elastic quadratic structure. Later, Danielson [9] made a significant improvement on the Budiansky/ Hutchinson model by incorporating an additional mass M_0 and a spring ,with spring constant K_0 , to stimulate pre-buckling motion (see Figure 2.1) .

Except for the additional mass M_0 and spring with spring constant K_0 ,the rest of Figure 2.1 is the original version of the Budiansky/Hutchinson model. Danielson obtained the following equations of dynamic equilibrium which we have further refined by the inclusion of an arbitrary explicitly time dependent slowly varying load $\overline{F}(T)$

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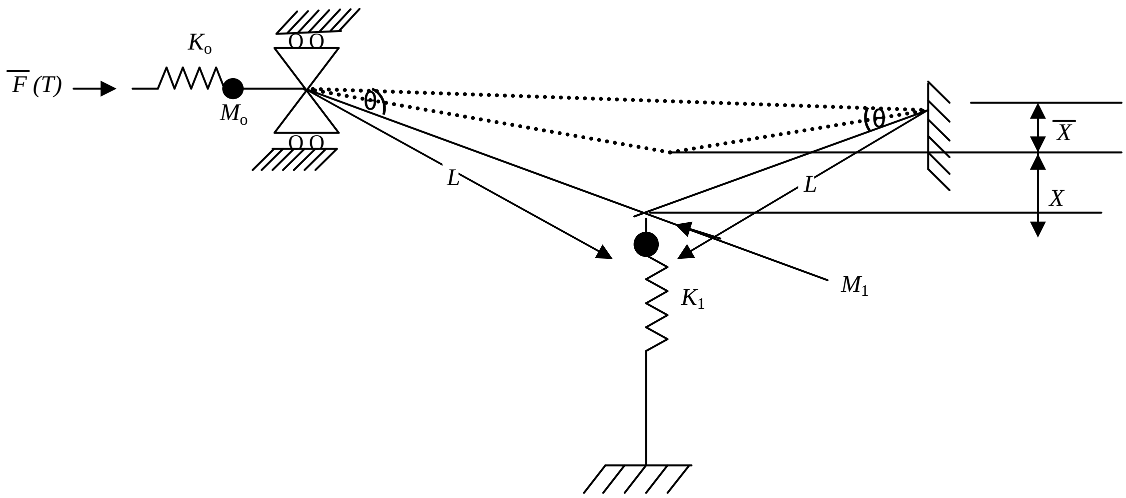

Figure 2.1: A simple quadratic – elastic model structure

$$\frac{1}{\omega_0^2}\frac{d^2\xi_0}{dT^2}+\xi_0-\frac{K_0}{\lambda_C}\xi_1\left(\xi_1+2\bar{\xi}\right)=\lambda\bar{F}\left(T\right) \tag{2.1}$$

$$\frac{1}{\omega_1^2}\frac{d^2\xi_1}{dT^2}+\xi_1\left(1-\xi_0\right)-\alpha\xi_1^2+\frac{K_0}{\lambda_C}\xi_1\left(\xi_1+\bar{\xi}\right)\left(\xi_1+2\bar{\xi}\right)=\bar{\xi}\xi_0 \tag{2.2}$$

$$\lambda_C=\frac{K_1}{2}\ ,\ \omega_0=\left(\frac{K_0}{M_0}\right)^{\frac{1}{2}}\ ,\omega_1=\left(\frac{K_1}{M_1}\right)^{\frac{1}{2}}$$

where ω_0 and ω_1 are the circular frequencies of the pre-buckling mode $\xi_0(T)$ and buckling mode $\xi_1(T)$ respectively. Both $\xi_0(T)$ and $\xi_1(T)$ are additional displacements from the equilibrium position while $\bar{\xi}$ is the nondimensional imperfection amplitude deemed small relative to unity .Similarly, λ is a nondimensional load amplitude that has been nondimensionalized with respect to the classical buckling load λ_c and satisfying the condition $0<\lambda<\lambda_C\le 1$, $\alpha>0$ is a constant otherwise called the imperfection-sensitivity parameter while T is the time variable. We let $\hat{t}=\omega_0 T$ and get the following equations

$$\frac{d^2\xi_0}{d\hat{t}^2}+\xi_0-\frac{K_0}{\lambda_C}\xi_1\left(\xi+2\bar{\xi}\right)=f\left(\bar{\xi}\ \hat{t}\right) \tag{2.3}$$

$$\frac{d^2\xi_1}{d\hat{t}^2}+Q^2\xi_1\left(1-\xi_0\right)-Q^2\alpha\xi_1^2+\frac{K_0}{\lambda_C}Q^2\left(\xi_1^3+2\bar{\xi}\xi_1^2+2\xi_1\bar{\xi}^2\right)=Q^2\bar{\xi}\xi_0 \tag{2.4}$$

where $$\xi_0(0) = \frac{d\xi_0(0)}{d\hat{t}} = \xi_1(0) = \frac{d\xi_1(0)}{d\hat{t}} = 0 \qquad (2.5)$$

and $f(\overline{\xi}\,\hat{t}) = \overline{F}\left(\frac{\hat{t}}{\omega_0}\right)$, $Q = \left(\frac{\omega_1}{\omega_0}\right)$, $0 < Q < 1$.Here, we consider the nondimensional load function $f(\overline{\xi}\,\hat{t})$ to be continuous and dynamically slowly varying over the natural period of the vibration of the structure and to have right hand derivatives of all orders at $\hat{t} = 0$ and also satisfies the following conditions

$$f(0) = 1\ ,\quad \left| f(\overline{\xi}\,\hat{t}) \right| \le 1 \quad \text{for} \quad \hat{t} > 0 \qquad (2.6)$$

Except for conditions (2.6), $f(\overline{\xi}\,\hat{t})$ is strictly arbitrary. Dynamic buckling problems with explicitly time dependent loadings were similarly analyzed by Svalbonas and Kalnins [10], Aksogan and Sofiyev [11] and Ette [12], among a few others. We define the dynamic buckling load λ_D as the largest value of λ for which the solution of the problem (2.3)-(2.6) has a bounded solution for all time $\hat{t} > 0$. As in [1-3] ,and in Ette [12],the usual procedure for determining λ_D is the maximization

$$\frac{d\lambda}{d\xi_a} = 0\ , \xi_a = \xi_{0\,\max} + \xi_{1\,\max} \qquad (2.7)$$

Where $\xi_{0\max}$ and $\xi_{1\max}$ are the maximum values of ξ_0 and ξ_1 respectively, as functions of the load parameter λ. In what follows, we shall first initiate a perturbation scheme for determining ξ_a.

3.0 Perturbation Solution

As in [9,10] , we shall assume that $0 < \frac{\omega_1}{\omega_0} < 1$ so that the natural period of oscillation of the mass M_1 is much more longer that that of mass M_0 with the result that we can neglect the pre-buckling inertia $\frac{d^2\xi_0}{d\hat{t}^2}$.This simplification yields the following equations derivable from equations (2.3) and (2.4)

$$\xi_0(\hat{t}) = f(\overline{\xi}\,\hat{t}) + \frac{K_0}{\lambda_C}\xi_1(\xi_1 + 2\overline{\xi}) \qquad (3.1a)$$

$$\frac{d^2\xi_1}{d\,\hat{t}^2} + Q^2\xi_1(1-\lambda f) - Q^2\xi_1^2\left(\alpha + \frac{K_0\overline{\xi}}{\lambda_C}\right) = Q^2\overline{\xi}\lambda f \qquad (3.1b)$$

where (3.1b) is obtained by substituting for $\xi_0(\hat{t})$ in (2.4) .We must acknowledge that the highest nonlinearity in (3.1b) is quadratic .We now let

$$\tau = \overline{\xi}\,\hat{t}\ ,\ \frac{dt}{d\,\hat{t}} = \left(1 - \lambda f(\overline{\xi}\,\hat{t})\right)^{\frac{1}{2}} = \left(1 - \lambda f(\tau)\right)^{\frac{1}{2}} \qquad (3.2)$$

so that we now have the following: $$\frac{d\xi_k}{d\,\hat{t}} = (1-\lambda f)^{\frac{1}{2}}\xi_{k,\,t} + \overline{\xi}\xi_{k\,,\,\tau} \qquad (3.3a)$$

$$\frac{d^2\xi_k}{d\,\hat{t}^2}=(1-\lambda f)\,\xi_{k,\,\mathrm{tt}}+2\bar{\xi}\,(1-\lambda f)^{\frac{1}{2}}\,\xi_{k,\,\mathrm{t}\,\tau}+\bar{\xi}^2\xi_{k\,,\,\tau\,\tau}-\frac{\lambda f'\bar{\xi}\,(1-\lambda f)^{-\frac{1}{2}}\,\xi_{k\,,\,\mathrm{t}}}{2}\;;\;\mathrm{k}=0{,}1 \qquad (3.3b)$$

where $\frac{d(\ )}{d\tau}=(\ )'$ and a subscript following a comma indicates partial differentiation. We let

$$\xi_1\left(\hat{t}\right)=\sum_{i=1}^{\infty}\zeta^{(i)}\left(t,\tau\right)\,\bar{\xi}^{\,\mathrm{i}} \qquad (3.4)$$

and now substitute (3.3a,b) and (3.4) into (3.1b), and after , equate equations of powers of $\bar{\xi}$ to get the following equations:

$$L\zeta^{(1)}\equiv\zeta^{(1)}_{1\,,\,\mathrm{t\,t}}++Q^2\zeta^{(1)}_1=Q^2B(\tau)\;;\;\left(\mathrm{B}=\frac{\lambda\ \mathrm{f}}{(1\text{-}\lambda\ \mathrm{f})}\right) \qquad (3.5)$$

$$L\zeta^{(2)}=-2(1-\lambda f)^{-\frac{1}{2}}\,\zeta^{(1)}_{1\,,\,\mathrm{t}\,\tau}+\frac{\lambda\ \mathrm{f}'(1\text{-}\lambda\ \mathrm{f})^{-\frac{3}{2}}}{2}\,\zeta^{(1)}_{1\,,\,\mathrm{t}}+\frac{\alpha\ Q^2\left(\zeta^{(1)}_1\right)^2}{(1\text{-}\lambda\ \mathrm{f})} \qquad (3.6)$$

$$L\zeta^{(3)}=-2(1-\lambda f)^{-\frac{1}{2}}\,\zeta^{(2)}_{1\,,\,\mathrm{t}\,\tau}+\frac{\lambda\ \mathrm{f}'(1\text{-}\lambda\ \mathrm{f})^{-\frac{3}{2}}}{2}\,\zeta^{(2)}_{1\,,\,\mathrm{t}}+\frac{2\alpha\ Q^2\zeta^{(1)}_1\zeta^{(2)}_1}{(1\text{-}\lambda\ \mathrm{f})}+\frac{2Q^2K_0\left(\zeta^{(1)}_1\right)^2}{\lambda_C\,(1\text{-}\lambda\ \mathrm{f})} \qquad (3.7)$$

The initial conditions are evaluated as follows:

$$\zeta^{(i)}(0,0)=0\ \ ,\ \zeta^{(1)}_{,\mathrm{t}}(0,0)=0\ \ ;\ \zeta^{(\mathrm{p})}_{,\mathrm{t}}(0,0)+(1-\lambda)^{-\frac{1}{2}}\,\zeta^{(\mathrm{k})}_{,\tau}(0,0)=0\;;\ \mathrm{k}=\mathrm{p}\text{-}1\;;\ \mathrm{p}=2,3,4,\mathrm{L} \qquad (3.8)$$

On solving (3.5) with appropriate initial conditions as in (3.8), we have

$$\zeta^{(1)}(t,\tau)=\alpha_1(\tau)\cos Q\,\mathrm{t}+\beta_1(\tau)\sin Q\,\mathrm{t}+Q^2\mathrm{B}\,;\ \alpha_1(0)=-Q^2B_0\ \ ,\mathrm{B}_0=B(0)=\frac{\lambda}{1-\lambda}\ \ ;\beta_1(0)=0 \qquad (3.9)$$

We now substitute (3.9) into (3.6), and to ensure a uniformly valid solution in t , equate to zero the coefficients of $\cos Q\,\mathrm{t}$ and $\sin Q\,\mathrm{t}$ and get the following respective equations

$$\beta_1'-\frac{\lambda f'\beta_1}{4(1-\lambda f)}=\frac{\alpha\alpha_1 B(1-\lambda f)^{\frac{1}{2}}}{Q}\quad;\quad\alpha_1'-\frac{\lambda f'\alpha_1}{4(1-\lambda f)}=-\frac{\alpha\beta_1 B(1-\lambda f)^{\frac{1}{2}}}{Q} \qquad (3.10a)$$

If we multiply the first and second of equations (3.10a) by β_1 and α_1 respectively , simplify the two , and integrate the resultant equations, we get the relationship between α_1 and β_1 as

$$\alpha_1^2+\beta_1^2=B_0^2\left(\frac{1-\lambda}{1-\lambda f}\right)^{\frac{1}{2}} \qquad (3.10b)$$

The actual solution of the equations in (3.10a) is accomplished by letting

$$\alpha_1(\tau)=\eta\ \mathrm{e}^{\,\theta\,(\tau)},\beta_1(\tau)=\gamma\ e^{\,\theta\,(\tau)} \qquad (3.10c)$$

On substituting (3.10c) into equations in (3.10a) , using $\alpha_1(0)$ and $\beta_1(0)$ as in (16a) and also using the fact that $\alpha_1'(0)=-\frac{B_0^2\,f'(0)}{4}$ and $\beta_1'(0)=\frac{\alpha B_0^2(1-\lambda)^{\frac{1}{2}}}{Q}$, each obtained by evaluating (3.10a) at $\tau=0$, we get

$$\alpha_1(\tau)=\frac{1}{(1-\lambda f)^{\frac{1}{4}}}\{A_1\cos\,\Omega\,(\tau)+B_1\sin\,\Omega\,(\tau)\}\;,\;\beta_1(\tau)=-B_0(1-\lambda f)^{\frac{1}{4}}\sin\,\Omega\,(\tau) \tag{3.10d}$$

$$A_1=-\frac{\lambda}{(1-\lambda)^{\frac{3}{4}}}\;,\;\mathrm{B}_1=-\frac{Q(1+\lambda)\lambda^2 f'(0)}{4\alpha(1-\lambda)^{\frac{5}{4}}}\;\;,\Omega\,(\tau)=\int_0^\tau\frac{\alpha\lambda\;\mathrm{f(s)ds}}{Q\left(1-\lambda f(s)\;\right)^{\frac{1}{2}}} \tag{3.10e}$$

The remaining equation in the substitution into (3.6) is now

$$L\zeta^{(2)}=\frac{\alpha\;\left(\alpha_1^2+\beta_1^2\right)}{2(1-\lambda f)}+\frac{\alpha\;\left(\alpha_1^2-\beta_1^2\right)\cos 2Q\;t}{2(1-\lambda f)}+\frac{\alpha\alpha_1\beta_1\sin 2Q\;\mathrm{t}}{(1-\lambda f)}+\frac{B^2}{(1-\lambda f)}; \tag{3.11a}$$

$$\zeta^{(2)}(0,0)=0\;;\zeta_{,t}^{(2)}(0,0)\;\;+\zeta_{,\tau}^{(1)}(0,0)=0 \tag{3.11b}$$

On solving (3.11a,b), we get

$$\zeta^{(2)}(t,\tau)=\alpha_2(\tau)\;\cos\mathrm{Q}\,\mathrm{t}+\beta_2\sin\,\mathrm{Q}\mathrm{t}+\frac{\alpha\left(\alpha_1^2+\beta_1^2\right)}{2Q^2(1-\lambda f)}-\frac{\alpha\left(\alpha_1^2-\beta_1^2\right)}{6Q^2(1-\lambda f)}-\frac{\alpha\,\alpha_1\beta_1\;\sin 2\mathrm{Q}\,\mathrm{t}}{3Q^2(1-\lambda f)}+\frac{Q^2B^2}{(1-\lambda f)} \tag{3.12a}$$

where $$\alpha_2(0)=-\frac{4B_0^2\;\;\alpha}{3(1-\lambda)},\,\beta_2(0)=-\frac{B_0 f'(0)(4-\lambda)}{4Q(1-\lambda)}\;\;;\mathrm{B}'(0)=\frac{B_0 f'(0)}{(1-\lambda)} \tag{3.12b}$$

Thus we now write $$\xi_1(t)=\bar{\xi}\zeta^{(1)}+\bar{\xi}^2\zeta^{(2)}+\mathrm{L} \tag{3.13a}$$

$$=\bar{\xi}(\alpha_1\cos Qt+\beta_1\sin\,\mathrm{Qt}+\mathrm{B})+\bar{\xi}^2\left[\alpha_2\cos\mathrm{Qt}+\beta_2\sin Q\mathrm{t}-\frac{\alpha\left(\alpha_1^2+\beta_1^2\right)\cos 2Q\,\mathrm{t}}{6\,(1-\lambda f)}-\frac{\alpha\,\alpha_1\beta_1\,\sin\mathrm{Qt}}{3(1-\lambda f)}+\frac{\alpha B^2}{(1-\lambda f)}\right]+\mathrm{L} \tag{3.13b}$$

The condition for maximum displacement , $\xi_{1\max}$, is

$$\xi_{1,\,\mathrm{t}}(t_a,\tau_a)+\bar{\xi}\xi_{1\,,\tau}(t_a,\tau_a)=0 \tag{13.14a}$$

where t_a and τ_a are the critical values of t and τ respectively each having the following asymptotic series expansions

$$t_a=t_0+\bar{\xi}t_1+\bar{\xi}^2t_2+\mathrm{L}\quad;\;\;\tau_a=\bar{\xi}t_a=\bar{\xi}\left\{t_0+\bar{\xi}t_1+\bar{\xi}^2t_2+\mathrm{L}\;\right\} \tag{13.14b}$$

Since, on using (21b), the maximum displacement $\xi_{1\max}$ will eventually take the form

$$\xi_{1\,\max}=\bar{\xi}\left[B(0)-\alpha_1(0)\right]+\bar{\xi}^2\left[B'(0)-\alpha_1'(0)\;\;t_0+\zeta^{(2)}(0,0)\right]+\mathrm{L} \tag{3.15}$$

we shall therefore evaluate only t_0 which is easily obtained , using (3.14a), from the equation $\zeta_{,\mathrm{t}}^{(1)}(0,0)=0$. This yields

$$t_0=\frac{\pi}{Q} \tag{3.16}$$

where we have taken the least nontrivial value of t_0 . On substituting for t_0 into (3.15) , we have

$$\xi_{1\,\max}=2B_0\bar{\xi}+\bar{\xi}^2\left[\frac{8B_0^2\alpha}{3(1-\lambda)}+\frac{B_0\pi f'(0)(4+\lambda\;)}{4Q(1-\lambda)}\right]+\mathrm{L} \tag{3.17}$$

Using (3.1a) and (3.13a),we have $$\xi_0(t)=\lambda f\left(\bar{\xi}\;\;\mathrm{t}\right)+\frac{\bar{\xi}^2K_0}{\lambda_C}\left(\zeta^{(1)\,2}+2\zeta^{(1)}\right)+\mathrm{L} \tag{3.18}$$

The condition for the maximum $\xi_{0\max}$ of $\xi_0(t)$ is $$\xi_{0,\,\mathrm{t}}(\%_a,\%_a)+\bar{\xi}\xi_{0\,,\tau}(\%_a,\%_a)=0 \tag{3.19a}$$

where $\tilde{t}_a$ and $\tilde{\tau}_a$ are the respective values of t and τ and each has the following series expansions $\tilde{t}_a = \tilde{t}_0 + \bar{\xi}\tilde{t}_1 + \bar{\xi}^2\tilde{t}_2 + \ldots \; ; \tau_a = \bar{\xi}\tilde{t}_a = \bar{\xi}\left\{\tilde{t}_0 + \bar{\xi}\tilde{t}_1 + \bar{\xi}^2\tilde{t}_2 + \ldots\right\}$ (3.19b)

Since, on using (3.19b), $\xi_{0\max}$ will eventually have the following expansion

$$\xi_{0\max} = \left[\lambda + \lambda\bar{\xi} f'(0)\, \tilde{t}_0 + \xi^2\left\{\frac{\tilde{t}_0\lambda f''(0)}{2} + \frac{K_0}{\lambda_C}\left(\zeta^{(1)\,2}\left(\tilde{t}_0, 0\right) + 2\zeta^{(1)}\left(\tilde{t}_0, 0\right)\right)\right\}\right] + \ldots \quad (3.20)$$

We shall here determine only $\tilde{t}_0$, which is evaluated ,using equations (3.19a,b),from the equation

$$\xi_{0\,,\,t}\left(\tilde{t}_0, 0\right) = 0 \quad (3.21a)$$

This yields

$$\tilde{t}_0 = \frac{\pi}{Q} = t_0 \quad (3.21b)$$

We now substitute for $\tilde{t}_0$ into (3.20) and get

$$\xi_{0\max} = \lambda\left(1 + \frac{\bar{\xi}\pi f'(0)}{Q}\right) + \bar{\xi}^2\left[\frac{4K_0}{\lambda_C}\left(B_0^2 + B_0\right) + \frac{\lambda\pi^2 f''(0)}{2Q^2}\right] + \ldots \quad (3.22)$$

Using the second part of (2.7), the net maximum displacement ξ_a is obtained using (3.17) and (3.22) as

$$\xi_m = \xi_a - \lambda = C_1\xi + C_2\bar{\xi}^2 + \ldots \quad (3.23a)$$

$$C_1 = 2B_0\tilde{A}_{11}(\lambda) + \frac{8B_0^2\alpha\,\tilde{A}_{22}(\lambda)}{3(1-\lambda)} \;, \tilde{A}_{11}(\lambda) = \left[1 + \frac{\pi f'(0)(1-\lambda)}{2Q}\right] \quad (3.23b)$$

$$\tilde{A}_{22}(\lambda) = \left[1 + \frac{3\pi}{32\lambda\alpha} f'(0)(4+\lambda)(1-\lambda) + \frac{3K_0}{2\alpha\lambda\lambda_C}\left\{1 + \frac{(1-\lambda)}{\lambda}\right\}(1-\lambda) + \frac{3\pi^2}{16\alpha\lambda Q} f''(0)(1-\lambda)^3\right] \quad 3.23c)$$

To determine the dynamic buckling load λ_D, we first [12] reverse the series (3.24a) in the form

$$\bar{\xi} = d_1\xi_m + d_2\xi_m^2 + \ldots \quad (3.24a)$$

If we substitute for ξ_m into (3.34a) from (3.23a) and equate the coefficients of $\bar{\xi}$ and $\bar{\xi}^2$, we get

$$d_1 = \frac{1}{C_1} \;, \mathrm{d}_2 = -\frac{C_2}{C_1^3} \quad (3.24b)$$

We now use an equivalent form of the first part of equation (2.7), which takes the form $\dfrac{d\lambda}{d\xi_m} = 0$

and this yields $\xi_{mD} \equiv \xi_m(\lambda_D) = \dfrac{d_1(\lambda_D)}{2d_2(\lambda_D)} = \dfrac{C_1^2(\lambda_D)}{2C_2(\lambda_D)}$ (3.25a)

If we evaluate (3.24a) at λ_D, using (3.25a) we get $\bar{\xi} = \dfrac{C_1(\lambda_D)}{4C_2(\lambda_D)}$ (3.25b)

On substituting into (3.25b) for $C_1(\lambda_D)$ and $C_2(\lambda_D)$, we get

$$(1-\lambda_D)^2 = \frac{16\lambda_D\bar{\xi}\alpha}{3}\left[\frac{1 + A_{22}(\lambda_D)}{1 + A_{11}(\lambda_D)}\right] \quad (3.26a)$$

$$A_{22}(\lambda_D)=\left[\frac{3\pi}{32\lambda_D\alpha}f'(0)\{\lambda_D+4\}\,(1-\lambda_D)\;+\frac{3K_0}{2\alpha\lambda_D\lambda_C}\left\{1+\frac{(1-\lambda_D)}{\lambda_D}\right\}(1-\lambda_D)+\frac{3\pi^2}{16\alpha\lambda_D}\left(\frac{\omega_0}{\omega_1}\right)f''(0)(1-\lambda_D)^3\right] \quad (3.26b)$$

$$A_{11}(\lambda_D)=\frac{1}{2}\pi\, f'(0)\;(1-\lambda_D)\;\left(\frac{\omega_0}{\omega_1}\right) \quad (3.26c)$$

4.0 Analysis of result and conclusion

We easily observe that the results (3.26a-c) depend on the first and second derivatives of $f(\bar{\xi}\,\hat{t})$, evaluated at the initial time $\hat{t}=0$ and also depend on the ratio $\frac{\omega_0}{\omega_1}$ of the circular frequencies of the pre-buckling mode to that of the buckling mode. Using the maximization $\frac{d\lambda}{d\xi_1}=0$, the static buckling load λ_S is easily obtained by neglecting the pre-buckling inertia $\frac{d^2\xi_0}{d\hat{t}^2}$ and setting $f(\bar{\xi}\,\hat{t})\equiv 1$. The result in this case gives

$$(1-\lambda_S)^2=4\alpha\lambda_S\bar{\xi}\left(1+\frac{K_0\bar{\xi}}{\lambda_C\alpha}\right) \quad (4.1)$$

If we eliminate the imperfection amplitude $\bar{\xi}$ from (4.1) using (3.26a), we get

$$\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^2=\frac{4}{3}\left(\frac{\lambda_D}{\lambda_S}\right)\left\{\frac{1+A_{22}(\lambda_D)}{1+A_{11}(\lambda_D)}\right\}\left[1+\frac{3K_0(1-\lambda_D)^2\{1+A_{11}(\lambda_D)\}}{16\alpha^2\lambda_C\lambda_D\{1+A_{22}(\lambda_D)\}}\right]^{-1} \quad (4.2)$$

We observe from (4.2) that given a certain value of the static buckling load λ_S, we can evaluate the corresponding value of the dynamic buckling load λ_D without necessarily solving the problem for different imperfection parameters .The result (4.2) is valid provided

$|A_{11}(\lambda_D)|<1, |A_{22}(\lambda_D)|<1$ and $\left|\frac{3K_0(1-\lambda_D)^2\{1+A_{11}(\lambda_D)\}}{16\alpha^2\lambda_C\lambda_D\{1+A_{22}(\lambda_D)\}}\right|<1$.

If $f'(0)=f''(0)=\Lambda=f^{(n)}(0)=0\;\forall n\geq 1$, then the loading is that of a step load characterized by $f(0)=1$ and the result in this case is obtained from (3.26a) as

$$(1-\lambda_D)^2=\frac{16\lambda_D\bar{\xi}\alpha}{3}\left[1+\frac{3K_0}{2\alpha\lambda_D\lambda_C}\left\{1+\frac{1-\lambda_D}{\lambda_D}\right\}(1-\lambda_D)\right] \quad (4.3)$$

On eliminating the imperfection parameter from (4.1) using (4.3) we get

$$\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^2=\frac{4A_{33}(\lambda_D)}{3}\left(\frac{\lambda_D}{\lambda_S}\right)\left[1+\frac{3\,(1-\lambda_D)^2}{16\alpha^2\lambda_D\lambda_C A_{33}(\lambda_D)}\right]^{-1} \quad (4.4a)$$

where

$$A_{33}(\lambda_D)=\left[1+\frac{3K_0}{2\alpha\lambda_D\lambda_C}\left\{1+\frac{1-\lambda_D}{\lambda_D}\right\}(1-\lambda_D)\right] \quad (4.4b)$$

where (4.4a,b) are valid provided

$$\left|\frac{3(1-\lambda_D)^2}{16\alpha^2\lambda_D\lambda_C A_{33}(\lambda_D)}\right| < 1, \text{ and } \left|\frac{3K_0}{2\lambda_D\lambda_C}\left\{1+\frac{1-\lambda_D}{\lambda_D}\right\}(1-\lambda_D)\right| < 1.$$

Alternatively, the result (4.4a) can be obtained by setting $f'(0) = f''(0) = 0$ in (4.2). Danielson [9] used Mathieu-type of instability and obtained the results for the step loading case as

$$\left(\frac{\lambda_D}{\lambda_S}\right) = \frac{3}{4}\left(\frac{1-\lambda_D}{1-\lambda_S}\right)^2 \quad , \text{ for } \quad 0 < \left(\frac{\omega_1}{\omega_0}\right) < \frac{1}{2} \tag{4.5a}$$

$$\left(\frac{\lambda_D}{\lambda_S}\right) = \frac{\frac{1}{6}\left\{4-\left(\frac{\omega_0}{\omega_1}\right)^2\right\}}{\lambda_S + \frac{10}{9}\left(\frac{\omega_1}{\omega_0}\right)^2(1-\lambda_S)^2} \quad , \text{for} \quad \frac{1}{2} < \left(\frac{\omega_1}{\omega_0}\right) < 1 \tag{4.5b}$$

We expect the step loading result (4.4a) to be a better representative of the dynamic buckling process compared to (4.5a,b) because, as noted by Budiansky [8,page 100], Mathieu-type of instability used by Danielson is normally associated with many cycles of oscillation as opposed to just one shot or cycle of oscillation that normally triggers off dynamic buckling.

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